GTU

As per revised 2018 syllabus

First Year Engineering

MATHEMATICS-2

McGraw Hill Education

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Mathematics-2

Gujarat Technological University 2018

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CHAPTER 1

Vector Calculus

Chapter Outline

1.1 INTRODUCTION

Vector calculus deals with the differentiation and integration of vector functions. We will learn about derivative of a vector function, gradient, divergence and curl in vector differential calculus. In vector integral calculus, we will learn about line integral, surface integral, volume integral and three theorems, namely Green's theorem, divergence theorem and Stokes' theorem. It plays an important role in the differential geometry and in the study of partial differential equations. It is useful in the study of rigid dynamics, fluid dynamics, heat transfer, electromagnetism, theory of relativity, etc.

1.2 VECTOR FUNCTION OF A SINGLE SCALAR VARIABLE

If, in some interval (a, b) or $[a, b]$, for every value of a scalar variable t, there corresponds a value of $\bar{r}$ then $\bar{r}$ is called a vector function of the scalar variable 't' and is denoted by $\bar{r} = \bar{f}(t)$.

1.2.1 Decomposition of a Vector Function

If $\hat{i}, \hat{j}, \hat{k}$ be three unit vectors along the three mutually perpendicular fixed directions (x, y, and z axes) then $\bar{r} = \bar{f}(t)$ can be decomposed as

$$\bar{r} = \bar{f}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$$

where $f_1(t), f_2(t)$ and $f_3(t)$ are scalar functions of t. This relation can also be denoted by $\bar{f} = (f_1, f_2, f_3)$

$$|\bar{f}(t)| = \sqrt{[f_1(t)]^2 + [f_2(t)]^2 + [f_3(t)]^2}$$

1.3 PARAMETERIZATION OF CURVES AND SURFACES

The ability to parameterize arbitrary curves and surfaces is an important part in multivariable calculus. In parameterization, curves and surfaces are treated as vector valued functions which can be analyzed using vector based algebraic operations. A parameterization for a curve is a set of functions depending only on a parameter t, along with the bounds for the parameter. When a curve is parameterized by taking values of t from some interval $[a, b]$, the position vector $\bar{r}(t)$ of any point t on the curve can be written as

$$\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

A parameterization for a surface is a set of functions depending on two parameters, usually u and v, along with the bounds for the parameter. When a surface is parameterized by taking points (u, v), out of some two dimensional space, the position vector $\bar{r}(u, v)$ can be written as

$$\bar{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

Example 1

Give parametric representations for each of the following curves:

(i) *The line* $9x - 5y = 7$

(ii) *The parabola* $y = (x - 2)^2$

(iii) *The circle* $x^2 + y^2 = 4$

Solution

(i) For the line $9x - 5y = 7$,

Let $x = t$

Then $y = \dfrac{9t-7}{5}$

The parametric representation is

$$\bar{r}(t) = t\,i + \left(\frac{9t-7}{5}\right)j$$

(ii) For the parabola $y = (x-2)^2$,

Let $x - 2t = t$

Then $x = t + 2$ and $y = t^2$

The parametric representation is

$$\bar{r}(t) = (t+2)i + t^2 j$$

(iii) For the circle $x^2 + y^2 = 4$,

Radius = 2

Then $x = 2\cos t$ and $y = 2\sin t$

The parametric representation is

$$\bar{r}(t) = (2\cos t)i + (2\sin t)j$$

Example 2

Give parametric representations for each of the following surfaces:

(a) *The elliptic paraboloid* $x = 5y^2 + 2z^2 - 10$.

(b) *The sphere* $x^2 + y^2 + z^2 = 30$.

(c) *The cylinder* $y^2 + z^2 = 25$.

Solution

(a) For the elliptic paraboloid $x = 5y^2 + 2z^2 - 10$,

Let $y = u$, $z = v$

Then $x = 5u^2 + 2v^2 - 10$

The parametric representation is

$$\bar{r}(u,v) = (5u^2 + 2v^2 - 10)\,\hat{i} + u\hat{j} + v\hat{k}$$

(b) For the sphere $x^2 + y^2 + z^2 = 30$,

Radius $\rho = \sqrt{30}$

In spherical coordinates,

$$x = \rho\sin\theta\cos\phi,\quad y = \rho\sin\theta\sin\phi,\quad z = \rho\cos\theta$$

The parametric representation is

$$\bar{r}(\theta,\phi)=\sqrt{30}\sin\theta\cos\phi\,\hat{i}+\sqrt{30}\sin\theta\sin\phi\,\hat{j}+\sqrt{30}\cos\theta\,\hat{k}$$

where $0\le\theta\le\pi$ and $0\le\phi\le 2\pi$

(c) For the cylinder $y^2+z^2=25$,
Radius $\rho=5$
In cylindrical coordinates,

$$x=x,\quad y=\rho\sin\theta,\quad z=\rho\cos\theta$$

The parametric representation is

$$\bar{r}(x,\theta)=x\,\hat{i}+5\sin\theta\,\hat{j}+5\cos\theta\,\hat{k}$$

where $0\le\theta\le 2\pi$

1.4 ARC LENGTH OF CURVES IN SPACE

When a curve is parameterized by taking values of t from some interval $[a, b]$, the position vector $\bar{r}(t)$ of any point t on the curve can be written as,

$$\bar{r}(t)=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}$$

The tangent vector $\bar{r}'(t)$ is

$$\bar{r}'(t)=x'(t)\hat{i}+y'(t)\hat{j}+z'(t)\hat{k}$$

$$|\bar{r}'(t)|=\sqrt{[x'(t)]^2+[y'(t)^2]+[z'(t)]^2}$$

Hence, the arc length of the parameterized curve is

$$s=\int_a^b|\bar{r}'(t)|\,dt$$

Example 1

Find the length of the curve $\bar{r}(t)=2t\,\hat{i}+3\sin 2t\,\hat{j}+3\cos 2t\,\hat{k}$ *on the interval* $0\le t\le 2\pi$.

Solution

$$\bar{r}'(t)=2i+6\cos 2t\,j-6\sin 2t\,k$$

$$|\bar{r}'(t)|=\sqrt{4+36\cos^2 2t+36\sin^2 2t}$$

$$=\sqrt{4+36}$$

$$=2\sqrt{10}$$

$$s=\int_a^b|\bar{r}'(t)|\,dt$$

$$=\int_0^{2\pi}2\sqrt{10}\,dt$$

$$=4\pi\sqrt{10}$$

Example 2

Find the length of the arc of the parameterized curve $\bar{r}(t)=\frac{2\sqrt{2}}{3}t^{\frac{3}{2}}\ i+\frac{t^2}{2}\ j+(t+3)k$ *between t = 0 and t = 2.*

Solution

$$\bar{r}'(t)=\sqrt{2}\,t^{\frac{1}{2}}\ i+tj+k$$

$$|\bar{r}'(t)|=\sqrt{2t+t^2+1}$$

$$=t+1$$

$$s=\int_a^b|\ \bar{r}'(t)|\ \mathrm{d}t$$

$$=\int_0^2(t+1)\ \mathrm{d}t$$

$$=\left[\frac{1}{2}t^2+t\right]_0^2$$

$$=2+2$$

$$=4$$

EXERCISE 1.1

1. Find the arc length of the following parameterized curves.

   (a) $\bar{r}(t)=\cos t\,\hat{i}+\sin t\,\hat{j}+t\,\hat{k}$ for $0\le t\le 6\pi$.

   $\left[\textbf{Ans.: } 6\sqrt{2}\pi\right]$

   (b) $\bar{r}(t)=(1+3t^2)\hat{i}+(4+2t^3)\,\hat{j}$ for $0\le t\le 1$.

   $\left[\textbf{Ans.: } 2(2\sqrt{2}-1)\right]$

2. Find the arc length of the curve $\bar{r}(t)=t^2\,\hat{i}+t^3\,\hat{j}$ between (1, 1) and (4, 8).

   $\left[\textbf{Ans.: } \frac{1}{27}(80\sqrt{10}-13\sqrt{13})\right]$

1.5 SCALAR AND VECTOR FIELDS

1.5.1 Field

If a function is defined in any region of space, for every point of the region then this region is known as field.

1.5.2 Scalar Field

A scalar function $\phi(x, y, z)$ is called scalar field defined in the region R, if it associates a scalar quantity with every point in the region R of space. The temperature distribution in a heated body, density of a body and potential due to gravity are the examples of scalar fields.

1.5.3 Vector Field

A vector function $\overline{F}(x, y, z)$ is called vector field defined in the region R, if it associates a vector quantity with every point in the region R of space. The velocity of a moving fluid, gravitational force are the examples of vector fields.

1.6 GRADIENT

The vector differential operator Del (or nabla) is denoted by ∇ and is defined as

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

The gradient of a scalar field ϕ is denoted by grad ϕ or $\nabla\phi$ and is defined as

$$\text{grad } \phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

which is a vector quantity.

$\phi(x, y, z)$ is a function of three independent variables and its total differential $d\phi$ is given as

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right)$$

$$= \nabla\phi \cdot d\overline{r} \qquad (1.1) \qquad \left[\because \overline{r} = x\hat{i} + y\hat{j} + z\hat{k}, \therefore d\overline{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz\right]$$

$$= |\nabla\phi|\,|d\overline{r}|\cos\theta$$

where θ is the angle between the vectors $\nabla\phi$ and $d\overline{r}$. If $d\overline{r}$ and $\nabla\phi$ are in the same direction then $\theta = 0$,

$$d\phi = |\nabla\phi|\,|d\overline{r}|$$

Since $\cos\theta = 1$ is the maximum value of $\cos\theta$, $d\phi$ is maximum at $\theta = 0$ and its maximum value is $|\nabla\phi|\,|d\overline{r}|$.

1.6.1 Normal

Let $\phi(x, y, z) = c$ represent a family of surfaces for different values of the constant c. Such a surface for which the value of the function is constant is called *level surface*.

Differentiating $\phi(x, y, z) = c$,

$$d\phi = 0$$

$$\nabla\phi \cdot d\bar{r} = 0 \qquad \text{[Using Eq (1.1)]}$$

Hence, $\nabla\phi$ and $d\bar{r}$ are perpendicular to each other. Since vector $d\bar{r}$ is in the direction of the tangent to the given surface, vector $\nabla\phi$ is perpendicular to the tangent to the surface and hence $\nabla\phi$ is in the direction of normal to the surface.

Thus, geometrically $\nabla\phi$ represents a vector normal to the surface $\phi(x, y, z) = c$.

1.6.2 Directional Derivative

Let $\phi(x, y, z)$ be a scalar field. Then $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$ are the directional derivative of ϕ *in the direction of the coordinate axes.*

Similarly, if $\bar{f}(x, y, z)$ be a vector field then $\frac{\partial\bar{f}}{\partial x}, \frac{\partial\bar{f}}{\partial y}, \frac{\partial\bar{f}}{\partial z}$ are the directional derivative of $\bar{f}$ *in the direction of the coordinate axes.*

The directional derivative of a scalar field $\phi(x, y, z)$ in the *direction of a line* whose direction cosines are l, m, n is $l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y} + n\frac{\partial\phi}{\partial z}$.

The directional derivative of a scalar field $\phi(x, y, z)$ in the *direction of vector* $\bar{a}$, is the component of $\nabla\phi$ in the direction of $\bar{a}$. If $\hat{a}$ is the unit vector in the direction of $\bar{a}$ then directional derivatives of ϕ in the direction of $\bar{a} = \nabla\phi \cdot \hat{a} = \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$.

Standard Results

(i) $\nabla(\phi \pm \psi) = \nabla\phi \pm \nabla\psi$

(ii) $\nabla(\phi\psi) = f(\nabla\psi) + (\nabla\phi)\psi$

(iii) $\nabla f(u) = \hat{i}\frac{\partial f(u)}{\partial x} + \hat{j}\frac{\partial f(u)}{\partial y} + \hat{k}\frac{\partial f(u)}{\partial z} = f'(u)\nabla u.$

Example 1

Find $\nabla\phi$ if $\phi = x^2 + y^2 + z^2$ at $(1, -1, 1)$.

Solution

$$\phi = x^2 + y^2 + z^2$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right) + \hat{j}\frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right) + \hat{k}\frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

At the point $(1, -1, 1)$, $\nabla\phi = 2\hat{i} + 2\hat{j} + 2\hat{k}$

Example 2

Evaluate ∇e^{r^2}, *where* $r^2 = x^2 + y^2 + z^2$.

Solution

$$r^2 = x^2 + y^2 + z^2$$

Differentiating partially w.r.t. x, y and z,

$$2r\frac{\partial r}{\partial x} = 2x, \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y, \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r\frac{\partial r}{\partial z} = 2z, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla e^{r^2} = \hat{i}\frac{\partial e^{r^2}}{\partial x} + \hat{j}\frac{\partial e^{r^2}}{\partial y} + \hat{k}\frac{\partial e^{r^2}}{\partial z}$$

$$= \hat{i}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial x} + \hat{j}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial y} + \hat{k}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \hat{i}(e^{r^2}\cdot 2r)\frac{x}{r} + \hat{j}(e^{r^2}\cdot 2r)\frac{y}{r} + \hat{k}(e^{r^2}\cdot 2r)\frac{z}{r}$$

$$= 2e^{r^2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

Example 3

Prove that $\nabla r^n = nr^{n-2}\overline{r}$, where $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k},\ r = |\overline{r}|$.

Solution

$$\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r^2 = |\overline{r}|^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla r^n = \hat{i}\frac{\partial r^n}{\partial x} + \hat{j}\frac{\partial r^n}{\partial y} + \hat{k}\frac{\partial r^n}{\partial z}$$

$$= \hat{i}\frac{\partial r^n}{\partial r}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r^n}{\partial r}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r^n}{\partial r}\frac{\partial r}{\partial z}$$

$$= \hat{i}\,nr^{n-1}\,\frac{x}{r} + \hat{j}\,nr^{n-1}\,\frac{y}{r} + \hat{k}\,nr^{n-1}\,\frac{z}{r}$$

$$= nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= nr^{n-2}\,\overline{r}$$

Example 4

Find the unit vector normal to the surface $x^2 + xy + z^2 = 4$ at the point $(1, -1, 2)$.

Solution

Given surface is $x^2 + xy + z^2 = 4$.

Let $$\phi = x^2 + xy + z^2$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^2 + xy + z^2) + \hat{j}\frac{\partial}{\partial y}(x^2 + xy + z^2) + \hat{k}\frac{\partial}{\partial z}(x^2 + xy + z^2)$$

$$= (2x + y)\hat{i} + x\hat{j} + 2z\hat{k}$$

At the point $(1, -1, 2)$, $\nabla\phi = [2(1) + (-1)]\hat{i} + (1)\hat{j} + [2(2)]\hat{k}$

$$= \hat{i} + \hat{j} + 4\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{\hat{i} + \hat{j} + 4\hat{k}}{\sqrt{1+1+16}}$$

$$= \frac{\hat{i} + \hat{j} + 4\hat{k}}{\sqrt{18}}$$

Example 5

Find the unit vector normal to the surface $x^2 + y^2 + z^2 = a^2$ at $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$.

Solution

$\nabla\phi$ is the vector which is normal to the surface $\phi(x, y, z) = c$.
Given surface is $x^2 + y^2 + z^2 = a^2$.

Let $\phi(x, y, z) = x^2 + y^2 + z^2$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right) + \hat{j}\frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right) + \hat{k}\frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right)$$

$$= (2x)\hat{i} + (2y)\hat{j} + (2z)\hat{k}$$

At the point $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$, $\nabla\phi = \frac{2a}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2a}{\sqrt{3}} \frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{\frac{4a^2}{3}+\frac{4a^2}{3}+\frac{4a^2}{3}}}$$

$$= \frac{2a(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}\cdot 2a}$$

$$= \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

Example 6

Find unit vector normal to the surface $x^2y + 2xz^2 = 8$ *at the po*[int] (1, 0, 2).

Solution

Given surface is $x^2y + 2xz^2 = 8$

Let $\phi(x, y, z) = x^2y + 2xz^2$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^2y+2xz^2) + \hat{j}\frac{\partial}{\partial y}(x^2y+2xz^2) + \hat{k}\frac{\partial}{\partial z}(x^2y+2xz^2)$$

$$= (2xy+2z^2)\hat{i} + (x^2)\hat{j} + (4xz)\hat{k}$$

At the point (1, 0, 2), $\nabla\phi = 8\hat{i} + \hat{j} + 8\hat{k}$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{8\hat{i}+\hat{j}+8\hat{k}}{\sqrt{64+1+64}}$$

$$= \frac{8\hat{i}+\hat{j}+8\hat{k}}{\sqrt{129}}$$

Example 7

Find a unit vector normal to the surface $x^2 + y^2 - z = 10$ *at the p*[oint] (1, 1, 1).

Solution

Given surface is $x^2 + y^2 - z = 10$.

Let $\phi = x^2 + y^2 - z$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}\left(x^2+y^2-z\right)+\hat{j}\frac{\partial}{\partial y}\left(x^2+y^2-z\right)+\hat{k}\frac{\partial}{\partial z}\left(x^2+y^2-z\right)$$
$$= 2x\hat{i}+2y\hat{j}-\hat{k}$$

At the point (1, 1, 1), $\nabla\phi = 2\hat{i}+2\hat{j}-\hat{k}$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$
$$= \frac{2\hat{i}+2\hat{j}-\hat{k}}{\sqrt{4+4+1}}$$
$$= \frac{2\hat{i}+2\hat{j}-\hat{k}}{3}$$

Example 8

Find the directional derivatives of $\phi = xy^2 + yz^2$ at the point (2, –1, 1) in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

Solution

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy^2+yz^2)+\hat{j}\frac{\partial}{\partial y}(xy^2+yz^2)+\hat{k}\frac{\partial}{\partial z}(xy^2+yz^2)$$
$$= y^2\hat{i} + (2xy+z^2)\,\hat{j} + (2yz)\,\hat{k}$$

At the point (2, –1, 1), $\nabla\phi = \hat{i} + (-4+1)\,\hat{j} + (-2)\,\hat{k}$
$$= \hat{i} - 3\hat{j} - 2\hat{k}$$

Directional derivative in the direction of the vector $(\bar{a} = \hat{i} + 2\hat{j} + 2\hat{k})$ is

$$\nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|} = \left(\hat{i}-3\hat{j}-2\hat{k}\right)\cdot\frac{\left(\hat{i}+2\hat{j}+2\hat{k}\right)}{\sqrt{1+4+4}}$$
$$= \frac{1-6-4}{3}$$
$$= -3$$

Example 9

Find the directional derivative of $\phi = \dfrac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$ at the point P (1, –1, 1) in the direction of $\bar{a} = \hat{i} + \hat{j} + \hat{k}$.

Solution

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}\left[\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}\right] + \hat{j}\frac{\partial}{\partial y}\left[\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}\right] + \hat{k}\frac{\partial}{\partial z}\left[\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}\right]$$

$$= \left[-\frac{2x}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right]\hat{i} + \left[-\frac{2y}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right]\hat{j} + \left[-\frac{2z}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right]\hat{k}$$

$$= -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

At the point (1, –1, 1), $\nabla\phi = \dfrac{-(\hat{i}-\hat{j}+\hat{k})}{(3)^{\frac{3}{2}}}$

Directional derivative in the direction of ($\bar{a} = \hat{i}+\hat{j}+\hat{k}$) is

$$\nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|} = \frac{-(\hat{i}-\hat{j}+\hat{k})}{(3)^{\frac{3}{2}}}\cdot\frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{1+1+1}}$$

$$= \frac{-1+1-1}{3^2}$$

$$= -\frac{1}{9}$$

Example 10

Find the directional derivative of $\phi = xy^2 + yz^3$ at (2, –1, 1) in the direction of the normal to the surface $x \log z - y^2 = -4$ at (–1, 2, 1).

Solution

Let $\psi = x\log z - y^2$.

$\nabla\psi$ is normal to the surface $x\log z - y^2 = -4$.

$$\nabla\psi = \hat{i}\frac{\partial}{\partial x}(x\log z - y^2) + \hat{j}\frac{\partial}{\partial y}(x\log z - y^2) + \hat{k}\frac{\partial}{\partial z}(x\log z - y^2)$$

$$= (\log z)\hat{i} + (-2y)\hat{j} + \left(\frac{x}{z}\right)\hat{k}$$

At the point (−1, 2, 1), $\nabla\psi = \hat{i}\,(\log 1) - 4\hat{j} - \hat{k}$

$$= -4\hat{j} - \hat{k}$$

which is normal to the surface $x \log z - y^2 = -4$ at (−1, 2, 1).

$$\phi = xy^2 + yz^3$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy^2 + yz^3) + \hat{j}\frac{\partial}{\partial y}(xy^2 + yz^3) + \hat{k}\frac{\partial}{\partial z}(xy^2 + yz^3)$$

$$= (y^2)\,\hat{i} + (2xy + z^3)\,\hat{j} + (3yz^2)\,\hat{k}$$

At the point (2, −1, 1), $\nabla\phi = \hat{i} + (-4 + 1)\hat{j} + (-3)\,\hat{k} = \hat{i} - 3\hat{j} - 3\hat{k}$

Directional derivative of ϕ in the direction of the vector ($\nabla\psi = -4\hat{j} - \hat{k}$) is

$$\nabla\phi \cdot \frac{\nabla\psi}{|\nabla\psi|} = (\hat{i} - 3\hat{j} - 3\hat{k}) \cdot \frac{(-4\hat{j} - \hat{k})}{\sqrt{16+1}}$$

$$= \frac{12+3}{\sqrt{17}}$$

$$= \frac{15}{\sqrt{17}}$$

Example 11

If the directional derivative of the function $\phi = xyz$ *at* (1, 1, 1) *in the direction of* $\alpha\hat{i} + \hat{j} + \hat{k}$ *is* $\sqrt{3}$, *find* α.

Solution

$$\phi = xyz$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xyz) + \hat{j}\frac{\partial}{\partial y}(xyz) + \hat{k}\frac{\partial}{\partial z}(xyz)$$

$$= yz\hat{i} + xz\hat{j} + xy\hat{k}$$

At the point (1, 1, 1), $\nabla\phi = \hat{i} + \hat{j} + \hat{k}$

Directional derivative in the direction of $\bar{a} = \alpha\hat{i} + \hat{j} + \hat{k}$ is $\sqrt{3}$.

$$\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|} = \sqrt{3}$$

$$(\hat{i} + \hat{j} + \hat{k}) \cdot \frac{(\alpha\hat{i} + \hat{j} + \hat{k})}{\sqrt{\alpha^2 + 1 + 1}} = \sqrt{3}$$

$$\frac{\alpha+1+1}{\sqrt{\alpha^2+2}}=\sqrt{3}$$

$$\alpha+2=\sqrt{3}\left(\sqrt{\alpha^2+2}\right)$$

$$(\alpha+2)^2=3(\alpha^2+2)$$

$$\alpha^2+4\alpha+4=3\alpha^2+6$$

$$2\alpha^2-4\alpha+2=0$$

$$\alpha^2-2\alpha+1=0$$

$$\alpha=1$$

Example 12

Find the rate of change of $\phi = xyz$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point (1, 1, 1).

Solution

Rate of change of ϕ in the given direction is the directional derivative of ϕ in that direction.

$$\nabla\phi=\hat{i}\frac{\partial}{\partial x}(xyz)+\hat{j}\frac{\partial}{\partial y}(xyz)+\hat{k}\frac{\partial}{\partial z}(xyz)$$

$$=(yz)\,\hat{i}+(xz)\,\hat{j}+(xy)\,\hat{k}$$

At the point (1, 1, 1), $\nabla\phi=\hat{i}+\hat{j}+\hat{k}$

Given surface is $x^2y + y^2x + yz^2 = 3$.

Let $\psi=x^2y+y^2x+yz^2$

$$\nabla\psi=\hat{i}\frac{\partial\psi}{\partial x}+\hat{j}\frac{\partial\psi}{\partial y}+\hat{k}\frac{\partial\psi}{\partial z}$$

$$=(2xy+y^2)\,\hat{i}+(x^2+2xy+z^2)\,\hat{j}+(2yz)\,\hat{k}$$

At the point (1, 1, 1), $\nabla\psi=3\hat{i}+4\hat{j}+2\hat{k}$

Directional derivative in the direction of normal to the given surface is

$$\nabla\phi\cdot\frac{\nabla\psi}{|\nabla\psi|}=(\hat{i}+\hat{j}+\hat{k})\cdot\frac{(3\hat{i}+4\hat{j}+2\hat{k})}{\sqrt{9+16+4}}$$

$$=\frac{3+4+2}{\sqrt{29}}$$

$$=\frac{9}{\sqrt{29}}$$

Hence, rate of change of $\phi = \frac{9}{\sqrt{29}}$

Example 13

Find the direction in which the directional derivative of $\phi = \frac{(x^2 - y^2)}{xy}$ *at* (1, 1) *is zero.*

Solution

$$\phi(x, y) = \frac{x}{y} - \frac{y}{x},$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}\left(\frac{x}{y} - \frac{y}{x}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{x}{y} - \frac{y}{x}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{x}{y} - \frac{y}{x}\right)$$

$$= \left(\frac{1}{y} + \frac{y}{x^2}\right)\hat{i} + \left(-\frac{x}{y^2} - \frac{1}{x}\right)\hat{j},$$

At the point (1, 1), $\nabla\phi = 2\hat{i} - 2\hat{j}$.

Let the directional derivative of ϕ is zero in the direction of vector $\bar{r} = x\hat{i} + y\hat{j}$.

$$\nabla\phi \cdot \frac{x\hat{i} + y\hat{i}}{\sqrt{x^2 + y^2}} = 0$$

$$(2\hat{i} - 2\hat{j}) \cdot (x\hat{i} + y\hat{j}) = 0$$

$$2x - 2y = 0, x = y$$

$$\therefore \bar{r} = x\hat{i} + x\hat{j}$$

Unit vector in this direction $= \frac{x(\hat{i} + \hat{j})}{x\sqrt{1+1}}$

$$= \frac{\hat{i} + \hat{j}}{\sqrt{2}}$$

Hence, directional derivative is zero in the direction of $\frac{\hat{i} + \hat{j}}{\sqrt{2}}$.

EXERCISE 1.2

1. Find ∇f if

(i) $f = \log(x^2 + y^2 + z^2)$

(ii) $f = (x^2 + y^2 + z^2)\, e^{-\sqrt{x^2 + y^2 + z^2}}$

$\left[\textbf{Ans.: } \text{(i) } \frac{2\bar{r}}{r^2} \text{ (ii) } (2 - r)e^{-r}\bar{r} \text{ where } \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}, r = |\bar{r}|\right]$

2. Find ∇f and $|\nabla f|$ if

   (i) $f = 2xz^4 - x^2y$ at $(2, -2, -1)$

   (ii) $f = 2xz^2 - 3xy - 4x$ at $(1, -1, 2)$

$$\left[\text{Ans.: (i) } 10\hat{i} - 4\hat{j} - 16\hat{k},\ 2\sqrt{93}\ \text{(ii) } 7\hat{i} - 3\hat{j} + 8\hat{k},\ \ 2\sqrt{29}\right]$$

3. If $\bar{A} = 2x^2\hat{i} - 3yz\hat{j} + xz^2\hat{k}$ and $f = 2z - x^3y$, find

   (i) $\bar{A} \cdot \nabla f$

   (ii) $\bar{A} \times \nabla f$ at $(1, -1, 1)$

[Ans.: (i) 5 (ii) $7\hat{i} - \hat{j} - 11\hat{k}$]

4. If $f = 3x^2y$, $y = xz^2 - 2y$, find $\nabla\ (\nabla f \cdot \nabla y\ )$.

[Ans.: $(6yz^2 - 12x)\hat{i} + 6xz^2\hat{j} + 12xyz\hat{k}$]

5. If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\bar{r}|$, prove that

   (i) $\nabla(\log r) = \dfrac{\bar{r}}{r^2}$

   (ii) $\nabla|\bar{r}|^3 = 3r\ \bar{r}$

   (iii) $\nabla f(r) = f'(r)\dfrac{\bar{r}}{r}$

6. Find a unit vector normal to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

$$\left[\text{Ans.: } \frac{1}{3}(\hat{i} - 2\hat{j} - 2\hat{k})\right]$$

7. Find the unit outward drawn normal to the surface $(x - 1)^2 + y^2 + (z + 2)^2 = 9$ at the point $(3, 1, -4)$

$$\left[\text{Ans.: } \frac{(2\hat{i} + \hat{j} - 2\hat{k})}{3}\right]$$

8. Find a unit vector normal to the surface $xy^3\ z^2 = 4$ at the point $(-1, -1, 2)$.

$$\left[\text{Ans.: } \frac{(\hat{i} + 3\hat{j} - \hat{k})}{\sqrt{11}}\right]$$

9. Find the directional derivative of $f = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$.

$$\left[\text{Ans.: } \frac{37}{\sqrt{3}}\right]$$

10. Find the directional derivative of $f = xy + yz + zx$ at (1, 2, 0) in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

$$\left[\text{Ans.: } \frac{10}{3}\right]$$

12. Find the directional derivative of $f = x^2y + y^2z + z^2x^2$ at (1, 2, 1) in the direction of the normal to the surface $x^2 + y^2 - z^2x = 1$ at (1, 1, 1).

$$\left[\text{Ans.: } \frac{4}{3}\right]$$

13. Find the directional derivative of $f = x^2y + yz^2$ at (2, −1, 1) in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at (1, 1, 1).

$$\left[\text{Ans.: } \frac{-13}{\sqrt{29}}\right]$$

14. Find the directional derivative of $f = x^2y + y^2z + z^2x$ at (2, 2, 2) in the direction of the normal to the surface $4x^2y + 2z^2 = 2$ at the point (2, −1, 3).

$$\left[\text{Ans.: } \frac{36}{\sqrt{41}}\right]$$

15. Find the rate of change of $f = xy + yz + zx$ at (1, −1, 2) in the direction of the normal to the surface $x^2 + y^2 = z + 4$.

$$\left[\text{Ans.: } \frac{14}{\sqrt{21}}\right]$$

1.7 DIVERGENCE

The divergence of a vector field $\overline{F}$ is denoted by div $\overline{F}$ or $\nabla \cdot \overline{F}$ and is defined as

$$\text{div } \overline{F} = \nabla \cdot \overline{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \overline{F}$$

If $$\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k},$$

then $$\nabla \cdot \overline{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

which is a scalar quantity.

Note: (i) $\nabla \cdot \overline{F} \neq \overline{F} \cdot \nabla$, because $\nabla \cdot \overline{F}$ is a scalar quantity whereas

$\overline{F} \cdot \nabla = F_1\frac{\partial}{\partial x} + F_2\frac{\partial}{\partial y} + F_3\frac{\partial}{\partial z}$ is a scalar differential operator.

(ii) $\nabla \cdot \overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$= \hat{i} \cdot \frac{\partial \overline{F}}{\partial x} + \hat{j} \cdot \frac{\partial \overline{F}}{\partial y} + \hat{k} \cdot \frac{\partial \overline{F}}{\partial z}$ where $\overline{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

1.7.1 Physical Interpretation of Divergence

Consider the case of a homogeneous and incompressible fluid flow. Consider a small rectangular parallelepiped of dimensions $\delta x, \delta y, \delta z$ parallel to x, y and z axes respectively.

Let $\bar{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$ be the velocity of the fluid at point $A\,(x, y, z)$.

The velocity component parallel to x-axis (normal to the face $PQRS$) at any point of the face $PQRS$

$= v_1 (x + \delta x, y, z)$

$= v_1 + \frac{\partial v_1}{\partial x} \delta x$ [Expanding by Taylor's series and ignoring higher powers of δx]

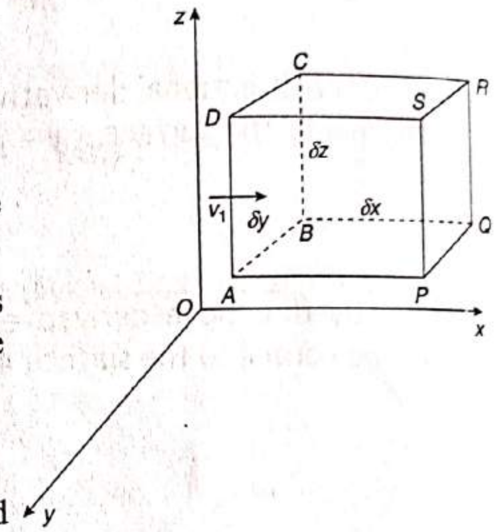

Fig. 1.1

Mass of the fluid flowing in across the face $ABCD$ per unit time

= Velocity component normal to the face $ABCD$ × Area of the face $ABCD$

$= v_1 \,(\delta y\, \delta z)$

Mass of the fluid flowing out across the face $PQRS$ per unit time

= Velocity component normal to the face $PQRS$ × area of the face $PQRS$

$$= \left(v_1 + \frac{\partial v_1}{\partial x} \delta x \right) \delta y \delta z$$

Gain of fluid in the parallelepiped per unit time in the direction of x-axis

$$= \left(v_1 + \frac{\partial v_1}{\partial x} \delta x \right) \delta y \delta z - v_1 \delta y \delta z$$

$$= \frac{\partial v_1}{\partial x} \delta x \delta y \delta z$$

Similarly, gain of fluid in the parallelepiped per unit time in the direction of y-axis

$$= \frac{\partial v_2}{\partial y} \delta x \, \delta y \, \delta z$$

and gain of fluid in the parallelepiped per unit time in the direction of z-axis

$$= \frac{\partial v_3}{\partial z} \delta x \, \delta y \, \delta z$$

Total gain of fluid in the parallelepiped per unit time

$$= \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) \delta x \, \delta y \, \delta z$$

But, $\delta x \, \delta y \, \delta z$ is the volume of the parallelepiped.

Hence, total gain of fluid per unit volume $= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$

$$= \text{div } \bar{v}$$

$$= \nabla \cdot \bar{v}$$

Thus, the divergence of the linear velocity of the fluid represents total gain (loss) of fluid per unit volume in a rectangular parallelepiped.

Note: A point in a vector field $\overline{F}$ is said to be a **source** if div $\overline{F}$ is positive, i.e., $\nabla \cdot \overline{F} > 0$ and is said to be a **sink** if div $\overline{F}$ is negative, i.e, $\nabla \cdot \overline{F} < 0$.

1.7.2 Solenoidal Vector Fields

A vector field $\overline{F}$ is said to be solenoidal if div $\overline{F} = 0$ at all points of the function. For such a vector, there is no loss or gain of fluid.

Example 1

If $\bar{A} = x^2 z\hat{i} - 2y^3 z^2 \hat{j} + xy^2 z\hat{k}$, *find* $\nabla \cdot \bar{A}$ *at the point* (1, –1, 1).

Solution

$$\nabla \cdot \overline{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}, \text{ where } \overline{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$$

$$\nabla \cdot \overline{A} = \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-2y^3 z^2) + \frac{\partial}{\partial z}(xy^2 z)$$

$$= 2xz - 6y^2 z^2 + xy^2$$

At the point (1, –1, 1), $\nabla \cdot \overline{A} = 2(1)(1) - 6(-1)^2(1)^2 + 1(-1)^2$

$$= 2 - 6 + 1$$

$$= -3$$

Example 2

Prove that div $\bar{r} = 3$.

Solution

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla \cdot \bar{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$= 3$$

Example 3

Show that $\bar{A} = 3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}$ is solenoidal.

Solution

$$\bar{A} = 3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}$$

$$\nabla \cdot \bar{A} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$$

$$= 0$$

Hence, $\bar{A}$ is solenoidal.

Example 4

Show that $\bar{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is solenoidal.

Solution

$$\bar{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$$

$$\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

$$= -2 + 2x - 2x + 2$$

$$= 0$$

Hence, $\bar{F}$ is solenoidal.

Example 5

Determine the constant b such that $\bar{A} = (bx + 4y^2z)\hat{i} + (x^3\sin z - 3y)\hat{j} - (e^x + 4\cos x^2y)\hat{k}$ is solenoidal.

Solution

If $\overline{A}$ is solenoidal then

$$\nabla \cdot \overline{A} = 0$$

$$\frac{\partial}{\partial x}(bx + 4y^2z) + \frac{\partial}{\partial y}(x^3 \sin z - 3y) + \frac{\partial}{\partial z}(-e^x - 4\cos x^2 y) = 0$$

$$b - 3 = 0$$

$$b = 3$$

Example 6

If $\overline{A} = (ax^2y + yz)\,\hat{i} + (xy^2 - xz^2)\,\hat{j} + (2xyz - 2x^2y^2)\,\hat{k}$ *is solenoidal, find the constant a.*

Solution

If $\overline{A}$ is solenoidal then

$$\nabla \cdot \overline{A} = 0$$

$$\nabla \cdot [(ax^2y + yz)\,\hat{i} + (xy^2 - xz^2)\,\hat{j} + (2xyz - 2x^2y^2)\,\hat{k}] = 0$$

$$\frac{\partial}{\partial x}(ax^2y + yz) + \frac{\partial}{\partial y}(xy^2 - xz^2) + \frac{\partial}{\partial z}(2xyz - 2x^2y^2) = 0$$

$$2axy + 2xy + 2xy = 0$$

$$2a = -4$$

$$a = -2$$

Example 7

Find a such that $(x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ *is solenoidal.*

Solution

Let $\overline{F} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$

Since $\overline{F}$ is solenoidal,

$$\nabla \cdot \overline{F} = 0$$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$$

$$1 + 1 + a = 0$$

$$a = -2$$

Example 8

Find the constant b such that $\overline{A} = (bx + 4y^2z)\hat{i} + (x^3 \sin z - 3y)\hat{j}$ $-(e^x + 4\cos x^2y)\hat{k}$ *is solenoidal.*

Solution

$$\overline{A} = (bx + 4y^2z)\hat{i} + (x^3 \sin z - 3y)\hat{j} - (e^x + 4\cos x^2 y)\hat{k}$$

Since $\overline{A}$ is solenoidal,

$$\nabla \cdot \overline{A} = 0$$

$$\frac{\partial}{\partial x}(bx + 4y^2z) + \frac{\partial}{\partial y}(x^3 \sin z - 3y) + \frac{\partial}{\partial z}\left[-(e^x + 4\cos x^2 y)\right] = 0$$

$$b - 3 + 0 = 0$$

$$b = 3$$

Example 9

If $\nabla\phi$ *is solenoidal, find* $\nabla^2\phi$.

Solution

If $\nabla\phi$ is solenoidal,

$$\nabla \cdot \nabla\phi = 0$$

$$\nabla^2\phi = 0$$

Example 10

Show that the vector field $\overline{A} = \dfrac{a(x\hat{i} + y\hat{j})}{\sqrt{x^2 + y^2}}$ *is a source field or sink field according as* $a > 0$ *or* $a < 0$.

Solution

Vector field $\overline{A}$ is a source field if $\nabla \cdot \overline{A} > 0$ and is a sink field if $\nabla. \overline{A} < 0$.

$$\nabla \cdot \overline{A} = \nabla \cdot \left(\frac{ax}{\sqrt{x^2 + y^2}}\hat{i} + \frac{ay}{\sqrt{x^2 + y^2}}\hat{j} \right)$$

$$= \frac{\partial}{\partial x}\left(\frac{ax}{\sqrt{x^2 + y^2}} \right) + \frac{\partial}{\partial y}\left(\frac{ay}{\sqrt{x^2 + y^2}} \right)$$

$$= a\left[\frac{1}{\sqrt{x^2+y^2}} - \frac{x\cdot 2x}{2(x^2+y^2)^{\frac{3}{2}}} + \frac{1}{\sqrt{x^2+y^2}} - \frac{y\cdot 2y}{2(x^2+y^2)^{\frac{3}{2}}}\right]$$

$$= a\left[\frac{2}{\sqrt{x^2+y^2}} - \frac{(x^2+y^2)}{(x^2+y^2)^{\frac{3}{2}}}\right]$$

$$= \frac{a}{\sqrt{x^2+y^2}}$$

$$\because \sqrt{x^2+y^2} > 0$$

$\therefore\ \nabla\cdot\overline{A} > 0$ if $a > 0$,

and $\nabla\cdot\overline{A} < 0$ if $a < 0$.

Hence, $\overline{A}$ is a source field if $a > 0$ and is a sink field if $a < 0$.

1.8 CURL

The curl of a vector field $\overline{F}$ is denoted by curl $\overline{F}$ or $\nabla\times\overline{F}$ and is defined as

$$\text{curl}\,\overline{F} = \nabla\times\overline{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

which is a vector quantity.

1.8.1 Physical Interpretation of Curl

Let $\overline{\omega}$ be the angular velocity of a rigid body moving about a fixed point. The linear velocity $\overline{v}$ of any particle of the body with position vector $\overline{r}$ w.r.t. to the fixed point is given by,

$$\overline{v} = \overline{\omega}\times\overline{r}$$

Let $\overline{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k},\ \overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\overline{v} = \overline{\omega}\times\overline{r}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\,(\omega_2 z - \omega_3 y) - \hat{j}\,(\omega_1 z - \omega_3 x) + \hat{k}\,(\omega_1 y - \omega_2 x)$$

$$\text{curl } \overline{v} = \nabla \times \overline{v}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \hat{i}\,(\omega_1 + \omega_1) - \hat{j}\,(-\omega_2 - \omega_2) + \hat{k}\,(\omega_3 + \omega_3)$$

$$= 2(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k})$$

$$= 2\,\overline{\omega}$$

$$\text{curl } \overline{v} = 2\,\overline{\omega}$$

Thus, the curl of the linear velocity of any particle of a rigid body is equal to twice the angular velocity of the body.

This shows that curl of a vector field is connected with rotational properties of the vector field and justifies the name rotation used for curl.

1.8.2 Irrotational Vector Fields

A vector field $\overline{F}$ is said to be *irrotational,* if curl $\overline{F} = 0$ at all points of the function. otherwise it is said to be rotational.

Note: If $\overline{F} = \nabla\phi$ then curl $\overline{F} = \nabla \times \overline{F} = \nabla \times \nabla\phi = 0.$

1.8.3 Conservative Fields and Scalar Potential Function

The vector field $\overline{F}$ is conservative if there is a scalar potential function ϕ such that $\overline{F} = \nabla\phi.$

1.8.4 Component Test for Conservative Fields

Let $\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector field where F_1, F_2 and F_3 have continuous first order partial derivatives and domain of $\overline{F}$ is connected and simply connected. Then vector field $\overline{F}$ is conservative if the following conditions are satisfied.

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$
$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$
$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$$

This test is called component test for conservative fields.

1.8.5 Exact Differential Forms

Any expression $F_1(x, y, z)\mathrm{d}x + F_2(x, y, z)\mathrm{d}y + F_3(x, y, z)\mathrm{d}z$ is a differential form. A differential form is exact on a domain D in space if

$$F_1\mathrm{d}x + F_2\mathrm{d}y + F_3\mathrm{d}z = \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y + \frac{\partial \phi}{\partial z}\mathrm{d}z = \mathrm{d}\phi$$

for some scalar function ϕ throughout domain D.

Example 1

If $\overline{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$, *find curl* $\overline{F}$.

Solution

$$\overline{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(z^3) - \frac{\partial}{\partial z}(y^3)\right] - \hat{j}\left[\frac{\partial}{\partial x}(z^3) - \frac{\partial}{\partial z}(x^3)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial y}(x^3)\right]$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0))$$

$$= 0$$

Example 2

Calculate curl of the vector $\overline{F} = xyz\hat{i} + 3x^2y\hat{j} + (xz^2 - y^2z)\hat{k}$ *at the point* (1, −1, 1).

Solution

$$\overline{F} = xyz\hat{i} + 3x^2y\hat{j} + (xz^2 - y^2z)\hat{k}$$

$$\nabla\times\bar{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xyz & 3x^2y & xz^2-y^2z\end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}(xz^2-y^2z)-\frac{\partial}{\partial z}(3x^2y)\right]-\hat{j}\left[\frac{\partial}{\partial x}(xz^2-y^2z)-\frac{\partial}{\partial z}(xyz)\right]+\hat{k}\left[\frac{\partial}{\partial x}(3x^2y)-\frac{\partial}{\partial y}(xyz)\right]$$

$$=\hat{i}(-2yz-0)-\hat{j}(z^2-xy)+\hat{k}(6xy-xz)$$

$$=-2yz\hat{i}-(z^2-xy)\hat{j}+(6xy-xz)\hat{k}$$

At the point $(1, -1, 1)$, $\nabla\times\bar{F}=[-2(-1)(1)]\hat{i}-[(1)-(1)(-1)]\hat{j}+[6(1)(-1)-(1)(1)]\hat{k}$

$$=2\hat{i}-2\hat{j}-7\hat{k}$$

Example 3

Prove that curl of a constant vector is zero.

Solution

Let $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ where a_1, a_2, a_3 are constants.

$$\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$\bar{a}\times\bar{r}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ x & y & z\end{vmatrix}$$

$$=\hat{i}(a_2z-a_3y)-\hat{j}(a_1z-a_3x)+\hat{k}(a_1y-a_2x)$$

$$\nabla\cdot(\bar{a}\times\bar{r})=\frac{\partial}{\partial x}(a_2z-a_3y)-\frac{\partial}{\partial y}(a_1z-a_3x)+\frac{\partial}{\partial z}(a_1y-a_2x)$$

$$=0-0+0$$

$$=0$$

Example 4

Find div F and curl F where $\bar{F}=grad(x^3+y^3+z^3-3xyz)$.

Solution

$$\bar{F}=\text{grad }(x^3+y^3+z^3-3xyz)$$

$$\bar{F}=\nabla(x^3+y^3+z^3-3xyz)$$

$$\bar{F}=(3x^2-3yz)\hat{i}+(3y^2-3xz)\hat{j}+(3z^2-3xy)\hat{k}$$

$$\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$
$$= 6x + 6y + 6z$$
$$= 6(x + y + z)$$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$
$$= \hat{i}\left[\frac{\partial}{\partial y}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3y^2 - 3xz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3x^2 - 3yz)\right]$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}(3y^2 - 3xz) - \frac{\partial}{\partial y}(3x^2 - 3yz)\right]$$
$$= \hat{i}(-3x + 3x) - \hat{j}(-3y + 3y) + \hat{k}(-3z + 3z)$$
$$= 0$$

Example 5

If $\bar{F} = xz^3\hat{i} - 2xy\hat{j} + xz\hat{k}$, *find div* $\bar{F}$ *and curl* $\bar{F}$ *at* (1, 2, 0).

Solution

$$\bar{F} = xz^3\hat{i} - 2xy\hat{j} + xz\hat{k}$$
$$\text{div } \bar{F} = \nabla \cdot \bar{F}$$
$$= \frac{\partial}{\partial x}(xz^3) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(xz)$$
$$= z^3 - 2x + x$$

At the point (1, 2, 0), $\nabla \cdot F = 0 - 2(1) + 1 = -1$

$$\text{curl } \bar{F} = \nabla \times \bar{F}$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2xy & xz \end{vmatrix}$$
$$= \hat{i}\left[\frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(-2xy)\right] - \hat{j}\left[\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial z}(xz^3)\right]$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(xz^3)\right]$$
$$= \hat{i}(0 - 0) - \hat{j}(z - 3xz^2) + \hat{k}(-2y - 0)$$
$$= -(z - 3xz^2)\hat{j} - 2y\hat{k}$$

At the point (1, 2, 0), $\nabla \times \bar{F} = (0-0)\hat{j} - 4\hat{k}$

$$= -4\hat{k}$$

Example 6

If ϕ is a scalar field, prove that curl (grad ϕ) = 0.

Solution

$$\text{grad } \phi = \nabla\phi$$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\text{curl (grad } \phi) = \nabla \times \nabla\phi$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right] - \hat{j}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right] + \hat{k}\left[\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right]$$

$$= 0$$

Example 7

Find the curl of $\bar{A} = e^{xyz}(\hat{i} + \hat{j} + \hat{k})$ at the point (1, 2, 3).

Solution

$$\text{curl of } \bar{A} = \nabla \times \bar{A}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xyz} & e^{xyz} & e^{xyz} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}e^{xyz} - \frac{\partial}{\partial z}e^{xyz}\right) - \hat{j}\left(\frac{\partial}{\partial x}e^{xyz} - \frac{\partial}{\partial z}e^{xyz}\right) + \hat{k}\left(\frac{\partial}{\partial x}e^{xyz} - \frac{\partial}{\partial y}e^{xyz}\right)$$

$$= (e^{xyz} \cdot xz - e^{xyz} \cdot xy)\,\hat{i} - (e^{xyz} \cdot yz - e^{xyz} \cdot xy)\,\hat{j} + (e^{xyz} \cdot yz - e^{xyz} \cdot xz)\,\hat{k}$$

At the point (1, 2, 3), curl $\bar{A} = e^6\,[\hat{i}\,(3-2) - \hat{j}\,(6-2) + \hat{k}\,(6-3)]$

$$= e^6\,(\hat{i} - 4\hat{j} + 3\hat{k})$$

Example 8

If $\overline{F} = 3\hat{i} + x\hat{j} + y\hat{k}$, *show that curl curl* $\overline{F} = 0$.

Solution

$$\overline{F} = 3\hat{i} + x\hat{j} + y\hat{k}$$

$$\text{curl } \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3 & x & y \end{vmatrix}$$

$$= \hat{i}(1-0) - \hat{j}(0-0) + \hat{k}(1-0)$$

$$= \hat{i} + \hat{k}$$

$$\text{curl curl } \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 0$$

Example 9

Find curl curl $\overline{A} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$ *at the point* (1, 0, 2).

Solution

$$\text{curl } \overline{A} = \nabla \times \overline{A}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2y)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2y)\right]$$

$$= (2z + 2x)\,\hat{i} - (0 - 0)\,\hat{j} + (-2z - x^2)\,\hat{k}$$

$$\text{curl}\left(\text{curl}\,\bar{A}\right) = \nabla \times (\nabla \times \bar{A})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2(z+x) & 0 & -(x^2+2z) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-x^2-2z) - \frac{\partial}{\partial z}(0)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-x^2-2z) - \frac{\partial}{\partial z}2(z+x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}2(z+x)\right]$$

$$= \hat{i}\,(0-0) - \hat{j}\,(-2x-2) + \hat{k}\,(0-0)$$

$$= (2x+2)\,\hat{j}$$

At the point (1, 0, 2), $\text{curl}\left(\text{curl}\,\bar{A}\right) = (2+2)\,\hat{j}$

$$= 4\,\hat{j}$$

Example 10

Prove that $\bar{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ is irrotational.

Solution

$$\bar{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - \hat{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$= \hat{i}[x-x] - \hat{j}[y-y] + \hat{k}[z-z]$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$$= 0$$

Hence, $\bar{F}$ is irrotational.

Example 11

Is the position vector $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ irrotational? Justify.

Solution

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla \times \bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$= 0$$

Hence, $\bar{r}$ is irrotational.

Example 12

Prove that vector $\bar{F} = (3x+2y+4z)\hat{i} + (2x+5y+4z)\hat{j} + (4x+4y-8z)\hat{k}$ *is both solenoidal and irrotational.*

Solution

$$\bar{F} = (3x+2y+4z)\hat{i} + (2x+5y+4z)\hat{j} + (4x+4y-8z)\hat{k}$$

$$\nabla \cdot \bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[(3x+2y+4z)\hat{i} + (2x+5y+4z)\hat{j} + (4x+4y-8z)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(3x+2y+4z) + \frac{\partial}{\partial y}(2x+5y+4z) + \frac{\partial}{\partial z}(4x+4y-8z)$$

$$= 3+5-8$$

$$= 0$$

Hence, $\bar{F}$ is solenoidal.

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x+2y+4z & 2x+5y+4z & 4x+4y-8z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(4x+4y-8z) - \frac{\partial}{\partial z}(2x+5y+4z)\right]$$

$$- \hat{j}\left[\frac{\partial}{\partial x}(4x+4y-8z) - \frac{\partial}{\partial z}(3x+2y+4z)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(2x+5y+4z) - \frac{\partial}{\partial y}(3x+2y+4z)\right]$$

$$= \hat{i}(4-4) - \hat{j}(4-4) + \hat{k}(2-2)$$

$$= 0$$

Hence, $\bar{F}$ is irrotational.

Example 13

Find the values of a, b, c so that the vector $\overline{F}=(x+y+az)\hat{i}+(bx+2y-z)\hat{j}+(-x+cy+2z)\hat{k}$ *may be irrotational.*

Solution

$$\overline{F}=(x+y+az)\hat{i}+(bx+2y-z)\hat{j}+(-x+cy+2z)\hat{k}$$

Since $\overline{F}$ is irrotational,

$$\nabla\times\overline{F}=0$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y+az & bx+2y-z & -x+cy+2z \end{vmatrix}=0$$

$$\hat{i}\left[\frac{\partial}{\partial y}(-x+cy+2z)-\frac{\partial}{\partial z}(bx+2y-z)\right]-\hat{j}\left[\frac{\partial}{\partial x}(-x+cy+2z)-\frac{\partial}{\partial z}(x+y+az)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(bx+2y-z)-\frac{\partial}{\partial y}(x+y+az)\right]=0$$

$$(c+1)\hat{i}-(-1-a)\hat{j}+(b-1)\hat{k}=0$$

$$(c+1)\hat{i}+(1+a)\hat{j}+(b-1)\hat{k}=0$$

Comparing coefficients of $\hat{i},\hat{j}$ and $\hat{k}$,

$$c+1=0$$
$$1+a=0$$
$$b-1=0$$

Solving these equations,

$$a=-1,\ b=1,\ c=-1$$

Example 14

Prove that $\overline{A}=2xyz^2\hat{i}+[x^2z^2+z\cos(yz)]\hat{j}+(2x^2yz+y\cos yz)\hat{k}$ *is a conservative vector field.*

Solution

Vector field $\overline{A}$ is conservative if $\nabla\times\overline{A}=0$.

$$\nabla\times\overline{F}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2+z\cos yz & 2x^2yz+y\cos yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2x^2yz + y\cos yz) - \frac{\partial}{\partial z}(x^2z^2 + z\cos yz)\right]$$

$$- \hat{j}\left[\frac{\partial}{\partial x}(2x^2yz + y\cos yz) - \frac{\partial}{\partial z}(2xyz^2)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x^2z^2 + z\cos yz) - \frac{\partial}{\partial y}(2xyz^2)\right]$$

$$= (2x^2z + \cos yz - yz\sin yz - 2x^2z - \cos yz + zy\sin yz)\,\hat{i}$$

$$- (4xyz - 4xyz)\,\hat{j} + (2xz^2 - 2xz^2)\,\hat{k}$$

$$= 0$$

Hence, $\overline{F}$ is conservative vector field.

Example 15

Determine the constants a and b such that curl of $(2xy + 3yz)\,\hat{i} + (x^2 + axz - 4z^2)\,\hat{j} + (3xy + 2byz)\,\hat{k}$ *is zero.*

Solution

Let $\overline{F} = (2xy + 3yz)\,\hat{i} + (x^2 + axz - 4z^2)\,\hat{j} + (3xy + 2byz)\,\hat{k}$

$$\text{curl } \overline{F} = 0$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + 3yz & x^2 + axz - 4z^2 & 3xy + 2byz \end{vmatrix} = 0$$

$$\hat{i}\left[\frac{\partial}{\partial y}(3xy + 2byz) - \frac{\partial}{\partial z}(x^2 + axz - 4z^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(3xy + 2byz) - \frac{\partial}{\partial z}(2xy + 3yz)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x^2 + axz - 4z^2) - \frac{\partial}{\partial y}(2xy + 3yz)\right] = 0$$

$$(3x + 2bz - ax + 8z)\,\hat{i} - (3y - 3y)\,\hat{j} + (2x + az - 2x - 3z)\,\hat{k} = 0$$

$$[(3 - a)x + 2(b + 4)z]\,\hat{i} - 0\,\hat{j} + (a - 3)z\,\hat{k} = 0$$

Comparing coefficients of $\hat{i}$ and $\hat{k}$,

$$(3 - a)x + 2(b + 4)z = 0$$

$$(a - 3)z = 0$$

Solving both the equations, $a = 3, b = -4$

Example 16

Show that $\overline{F} = (y^2 - z^2 + 3yz - 2x)\,\hat{i} + (3xz + 2xy)\,\hat{j} + (3xy - 2xz + 2z)\,\hat{k}$ *is both solenoidal and irrotational.* **[Winter 2014, Summer 2013]**

Solution

$$\nabla \cdot \overline{F} = \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

$$= -2 + 2x - 2x + 2$$

$$= 0$$

Hence, $\overline{F}$ is solenoidal.

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial x}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(y^2 - z^2 + 3yz - 2x)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(3xz + 2xy) - \frac{\partial}{\partial y}(y^2 - z^2 + 3yz - 2x)\right]$$

$$= (3x - 3x)\,\hat{i} - (3y - 2z + 2z - 3y)\,\hat{j} + (3z + 2y - 2y - 3z)\,\hat{k}$$

$$= 0$$

Hence, $\overline{F}$ is irrotational.

Example 17

A fluid motion is given by $\overline{F} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$. *Is this motion irrotational? If so, find the velocity potential.*

Solution

$$\overline{F} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$$

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial\partial}{\partial y}(x+y)-\frac{\partial}{\partial z}(z+x)\right]-\hat{j}\left[\frac{\partial}{\partial x}(x+y)-\frac{\partial}{\partial z}(y+z)\right]$$
$$+\hat{k}\left[\frac{\partial}{\partial x}(z+x)-\frac{\partial}{\partial y}(y+z)\right]$$
$$= \hat{i}(1-1)-\hat{j}(1-1)+\hat{k}(1-1)$$
$$= 0$$

Hence, $\bar{F}$ is irrotational.

$$\bar{F} = \nabla\phi$$

$$(y+z)i+(z+x)\hat{j}+(x+y)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$$

Equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$,

$$\frac{\partial\phi}{\partial x} = y+z \quad ...(1)$$

$$\frac{\partial\phi}{\partial y} = z+x \quad ...(2)$$

$$\frac{\partial\phi}{\partial z} = x+y \quad ...(3)$$

Integrating Eq. (1) w. r. t. x,

$$\phi = xy + xz + f_1(y, z)$$

Integrating Eq. (2) w. r. t. y,

$$\phi = yz + xy + f_2(x, z)$$

Integrating Eq. (3) w. r. t. z,

$$\phi = xz + yz + f_3(x, y)$$

$$\therefore \quad \phi = xy + yz + xz + c$$

Example 18

A vector field is given by $\bar{F} = (x^2 - y^2 - x)\hat{i} - (2xy + y)\hat{j}$. *Show that the field is irrotational and find its scalar potential.*

Solution

$$\bar{F} = (x^2 - y^2 - x)\hat{i} - (2xy + y)\hat{j}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 - x & -(2xy + y) & 0 \end{vmatrix}$$

$$= \hat{i}\left[0 - \frac{\partial}{\partial z}(-2xy - y)\right] - \hat{j}\left[0 - \frac{\partial}{\partial z}(x^2 - y^2 - x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(-2xy - y) - \frac{\partial}{\partial y}(x^2 - y^2 - x)\right]$$

$$= \hat{i}(0 - 0) - \hat{j}(0 - 0) + \hat{k}(-2y + 2y)$$

$$= 0$$

Hence, $\bar{F}$ is irrotational.

$$\bar{F} = \nabla \phi$$

$$(x^2 - y^2 - x)\hat{i} - (2xy + y)\hat{j} = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

Equating coefficients of $\hat{i}$ and $\hat{j}$,

$$\frac{\partial \phi}{\partial x} = x^2 - y^2 - x \quad \text{...(}$$

$$\frac{\partial \phi}{\partial y} = -(2xy + y) \quad \text{...(}$$

Integrating Eq. (1) w. r. t. x,

$$\phi = \frac{x^3}{3} - y^2 x - \frac{x^2}{2} + f_1(y, z)$$

$$= \frac{x^3}{3} - \frac{x^2}{2} - y^2 x + f_1(y, z)$$

Integrating Eq. (2) w. r. t. y,

$$\phi = -\frac{2xy^2}{2} - \frac{y^2}{2} + f_2(x, z)$$

$$= -xy^2 - \frac{y^2}{2} + f_2(x, z)$$

$$\therefore \phi = \frac{x^3}{3} - \frac{x^2}{2} - xy^2 - \frac{y^2}{2} + c$$

Example 19

Prove that $\bar{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ *is irrotationa vector and find the scalar potential such that* $\bar{F} = \nabla \phi$.

Solution

$$\bar{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(6xy + z^3)\right] + \hat{k}\left[\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3)\right]$$

$$= \hat{i}[(-1) - (-1)] - \hat{j}[3z^2 - 3z^2] + \hat{k}[6x - 6x]$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$$= 0$$

Hence, $\bar{F}$ is an irrotational vector.

$$\bar{F} = \nabla\phi$$

$$(6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Equating coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$,

$$\frac{\partial\phi}{\partial x} = 6xy + z^3 \quad ...(1)$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z \quad ...(2)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y \quad ...(3)$$

Integrating Eq. (1) w. r. t. x,

$$\phi = 6y\frac{x^2}{2} + z^3x + f_1(y, z)$$

$$= 3x^2y + xz^3 + f_1(y, z)$$

Integrating Eq. (2) w. r. t. y,

$$\phi = 3x^3y - zy + f_2(x, z)$$

Integrating Eq. (3) w. r. t. z,

$$\phi = xz^3 - yz + f_3(x, y)$$

$$\therefore \phi = 3x^2y + xz^3 - yz + c$$

EXERCISE 1.3

1. Find divergence and curl of $x^2 \cos z\, \hat{i} + y \log x\, \hat{j} - yz\, \hat{k}$.

$$\left[\text{Ans.: } 2x\cos z + \log x - y,\ \hat{i}z - \hat{j}x^2 \sin z + \hat{k}\frac{y}{x}\right]$$

2. If $\phi = 2x^3y^2z^4$, prove that div (grad ϕ) $= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$.

3. Find curl (curl $\bar{A}$), if $\bar{A} = x^2y\, \hat{i} - 2xz\, \hat{j} + 2yz\, \hat{k}$. [Ans.: $(2x + 2)\,\hat{j}$]

4. If $\bar{A} = x^2\, \hat{i} + xye^x \hat{j} + \sin z\, \hat{k}$, find $\nabla \cdot (\nabla \times \bar{A})$. [Ans.: 0]

5. If $\phi = \tan^{-1}\left(\frac{y}{x}\right)$, find div (grad ϕ). [Ans.: 0]

6. If $\phi = 2x^2 - 3y^2 + 4z^2$, find curl (grad ϕ). [Ans.: 0]

7. Prove that for every field $\bar{A}$, div (curl $\bar{A}$) = 0.

8. Prove that gradient field describing a motion is irrotational.

9. Prove that $\bar{A} = \hat{i}\, yz + \hat{j}\, xz + \hat{k}\, xy$ is irrotational and find a scalar function ϕ (x, y, z) such that $\bar{A}$ = grad ϕ.

[Ans.: $xyz + c$]

10. Prove that $\bar{A} = (6xy + z^3)\, \hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\, \hat{k}$ is irrotational. Find the function ϕ such that $\bar{A} = \nabla\phi$.

[Ans.: $\phi = 3x^2y + xz^3 - yz$]

11. Prove that the velocity given by $\bar{A} = (y + z)\, \hat{i} + (z + x)\, \hat{j} + (x + y)\hat{k}$ is irrotational and find its scalar potential. Is the motion possible for an incompressible fluid?

[Ans.: $\phi = yz + zx + xy$, motion is possible because $\nabla \cdot \bar{A} = 0$]

12. Prove that $\bar{A} = (z^2 + 2xy + 3y)\hat{i} + (3x + 2y + z)\hat{j} + (y + 2zx)\, \hat{k}$ is irrotational and find scalar potential ϕ such that $\bar{A} = \nabla\phi$ and ϕ (1, 1, 0) = 4.

[Ans.: $\phi = z^2x + x^2 + 3xy + y^2 + yz - 1$]

13. Prove that $\bar{A} = (z^2 + 2x + 3y)\, \hat{i} + (3x + 2y + z)\, \hat{j} + (y + 2zx)\, \hat{k}$ is conservative and find scalar potential ϕ such that $\bar{A} = \nabla\phi$.

[Ans.: $\phi = x^2 + y^2 + z^2x + 3xy + zy$]

14. Prove that $a = -1$ or $b = 0$, if $(xyz)b\, (x^a\hat{i} + y^a\hat{j} + z^a\hat{k})$ is an irrotational vector.

15. Find the constant a if $\bar{A} = (x + 3y^2)\, \hat{i} + (2y + 2z^2)\, \hat{j} + (x^2 + az)\, \hat{k}$ is solenoidal.

[Ans.: $a = -3$]

16. Find the constants a, b, c if $\bar{A} = (axy + bz^2)\, \hat{i} + (3x^2 - cz)\, \hat{j} + (3xz^2 - y)\, \hat{k}$ is irrotational.

[Ans.: $a = 6$, $b = 1$, $c = 1$]

1.9 LINE INTEGRALS

The line integral is a simple generalisation of a definite integral $\int_a^b f(x)\,dx$ which is integrated from $x = a$ (point A) to $x = b$ (point B) along the x-axis. In a line integral, the integration is done along a curve C in space.

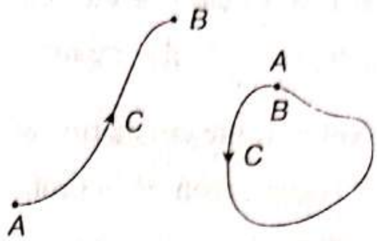

Fig. 1.2

Let $\overline{F}(\overline{r})$ be a vector field defined at every point of a curve C. If $\overline{r}$ is the position vector of a point $P\ (x, y, z)$ on the curve C then the line integral of $\overline{F}(\overline{r})$ over a curve C is defined by

$$\int_C \overline{F}(\overline{r})\cdot d\overline{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

where $$\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k} \text{ and } \overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

If the curve C is represented by a parametric representation

$$\overline{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k},$$

then the line integral along the curve C from $t = a$ to $t = b$ is

$$\int_C \overline{F}(\overline{r})\cdot d\overline{r} = \int_a^b \overline{F}\cdot\frac{d\overline{r}}{dt}\,dt$$

$$= \int_a^b \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)dt$$

If C is a closed curve then the symbol of the line integral $\int_C$ is replaced by $\oint_C$.

Note:

(i) A curve C is called the path of integration, the points $\overline{r}(a)$ and $\overline{r}(b)$ are called initial and terminal points respectively.

(ii) The direction from A to B along which t increases is called positive direction on C.

(iii) A curve C is called *closed* if its initial and final points are the same point. For example, a circle is a closed curve.

(iv) A curve C is *simple* if it doesn't cross itself. A circle is a simple curve while a figure 8 type curve is not simple.

(v) A curve C is piecewise smooth if it is made up of infinitely many smooth pieces connected end to end.

(vi) A domain or region D is *open* if it doesn't contain any of its boundary points.

(vii) A domain or region D is *connected* if any two points in the domain or region can be connected with a path that lies completely in D.

(viii) A domain or region D is *simply-connected* if it is connected and it contains no holes.

1.9.1 Work Done by a Force

If vector field $\overline{F}$ is the force acting on a particle moving along the arc AB of the curve C, then the line integral $\int_A^B \overline{F}\cdot d\overline{r}$ represents the work done in displacing (moving) the particle from the point A to the point B.

1.9.2 Circulation

If vector field $\overline{F}$ is the velocity of a fluid particle and C is a closed curve, then the line integral $\oint_C \overline{F} \cdot d\overline{r}$ represents the circulation of $\overline{F}$ around the curve C.

Note: If the circulation of vector field $\overline{F}$ around every closed curve C in the region R is zero, then $\overline{F}$ is irrotational, i.e. if $\oint_C \overline{F} \cdot d\overline{r} = 0, \overline{F}$ is irrotational.

1.9.3 Path Independence of Line Integral

If vector field $\overline{F}$ is conservative, i.e. $\overline{F} = \nabla\phi$ where ϕ is a scalar potential, then the line integral along the curve C from the points A to B is

$$\begin{aligned}\int_C \overline{F} \cdot d\overline{r} &= \int_A^B \nabla\phi \cdot d\overline{r} \\ &= \int_A^B \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right) \\ &= \int_A^B d\phi \\ &= \phi(B) - \phi(A)\end{aligned}$$

Thus, line integral depends only on the start and end values and therefore is independent of the path.
Hence, for a conservative vector field, line integral is independent of the path.

Note:

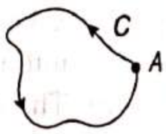

Fig. 1.3

(i) If vector field $\overline{F}$ is conservative and curve C is closed then

$$\oint_C \overline{F} \cdot d\overline{r} = \phi(A) - \phi(A) = 0$$

(ii) Work done in moving a particle from points A to B under a conservative force field is

$$\text{work done} = \phi(B) - \phi(A)$$

(iii) The integral $\int \overline{F} \cdot d\overline{r}$ is independent of the path C if $\nabla \times \overline{F} = 0$

1.9.4 Fundamental Theorem of Line Integrals

Let $\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector field whose component are continuous throughout an open connected region D in space. Then there exists a differentiable function ϕ such that

$$\overline{F} = \nabla\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

if and only if for all points A and B in D, the value of $\int_A^B \overline{F} \cdot d\overline{r}$ is independent of the path joining A to B in D.

If the integral is independent of the path from A to B, its value is

$$\int_A^B \bar{F}\cdot d\bar{r} = \phi(B) - \phi(A)$$

1.9.5 Flux

If the vector field $\bar{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ is some kind of fluid flow and C is a curve, then the line integral $\int_C \bar{F}\cdot\hat{n}\,dS$ represents the flux of $\bar{F}$ through the curve C, where $\hat{n}$ is unit normal vector to C.

Let $\bar{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ and equation of the curve C is $\bar{r} = x\hat{i} + y\hat{j}$.

$$d\bar{r} = dx\hat{i} + dy\hat{j} \qquad ...(1.2)$$

If $\hat{T}$ is the unit tangent vector to C then

$$d\bar{r} = \hat{T}\,ds \qquad ...(1.3)$$

From Eqs (1.2) and (1.3),

$$\hat{T}\,ds = dx\hat{i} + dy\hat{j}$$

The unit normal vector $\hat{n}$ is perpendicular to $\hat{T}$. Thus $\hat{n}$ is obtained by rotating $\hat{T}$ through $\frac{\pi}{2}$ in clockwise direction.

$$\hat{n}\,ds = dy\hat{i} + dx\hat{j}$$

$$\text{Flux} = \int_C \bar{F}\cdot\hat{n}\,dS$$

$$= \int (M\hat{i} + N\hat{j})\cdot(dy\hat{i} - dx\hat{j})$$

$$= \int (M\,dy - N\,dx)$$

Example 1

If $\bar{F} = (4xy - 3x^2z^2)\hat{i} + 2x^2\hat{j} - 2x^3z\hat{k}$ *then prove that the integral* $\int_C \bar{F}\cdot d\bar{r}$ *is independent of the path C.*

Solution

$$\bar{F} = (4xy - 3x^2z^2)\hat{i} + 2x^2\hat{j} - 2x^3z\hat{k}$$

The integral $\bar{F}\cdot d\bar{r}$ is independent of the path C if $\nabla\times\bar{F} = 0$.

$$\nabla\times\bar{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 4xy-3x^2z^2 & 2x^2 & -2x^3z\end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}(-2x^3z)-\frac{\partial}{\partial z}(2x^2)\right]-\hat{j}\left[\frac{\partial}{\partial x}(-2x^3z)-\frac{\partial}{\partial z}(4xy-3x^2z^2)\right]+\hat{k}\left[\frac{\partial}{\partial x}(2x^2)-\frac{\partial}{\partial y}(4xy-3x^2z^2)\right]$$

$$=\hat{i}[0-0]-\hat{j}[-6x^2z-(-6x^2z)]+\hat{k}(4x-4x)$$
$$=0$$

Hence, $\int_C \bar{F}\cdot d\bar{r}$ is independent of the path C.

Example 2

If $\bar{F}=x^2\hat{i}+xy^2\hat{j}$, *evaluate* $\int_C \bar{F}\cdot d\bar{r}$ *from* (0, 0) *to* (1, 1) *along the path* $y=x$.

Solution

(i) Let $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$$d\bar{r}=\hat{i}dx+\hat{j}dy+\hat{k}dz$$

(ii) $\bar{F}\cdot d\bar{r}=(x^2\hat{i}+xy^2\hat{j})\cdot(\hat{i}dx+\hat{j}dy+\hat{k}dz)$

$$=x^2dx+xy^2dy$$

(iii) Path of integration is along the line $y=x$.

$$dy=dx$$

Substituting in $\bar{F}\cdot d\bar{r}$ and integrating between the limits $x=0$ to $x=1$,

$$\int_C \bar{F}\cdot d\bar{r}=\int_0^1\left(x^2dx+x\cdot x^2dx\right)$$
$$=\int_0^1(x^2+x^3)\,dx$$
$$=\left|\frac{x^3}{3}+\frac{x^4}{4}\right|_0^1$$
$$=\frac{1}{3}+\frac{1}{4}$$
$$=\frac{7}{12}$$

Example 3

If $\bar{F} = 3xy\hat{i} - y^2\hat{j}$, *evaluate* $\int_C \bar{F} \cdot d\bar{r}$ *where C is the arc of parabola* $y = 2x^2$ *from* (0, 0) *to* (1, 2). **[Summer 2013]**

Solution

(i) Let $\bar{r} = x\hat{i} + y\hat{j}$

$d\bar{r} = \hat{i}dx + \hat{j}dy$

(ii) $\bar{F} \cdot d\bar{r} = (3xyi - y^2 j) \cdot (idx + jdy)$

$= 3xydx - y^2dy$

(iii) Path of integration is the parabola $y = 2x^2$

$$dy = 4xdx$$

Substituting in $\bar{F} \cdot d\bar{r}$ and integrating between the limits $y = 0$ to $y = 1$,

$$\int_C \bar{F} \cdot d\bar{r} = \int_0^1 (3x \cdot 2x^2 dx - 4x^4 \cdot 4xdx)$$

$$= \int_0^1 (6x^3 - 16x^5)\, dx$$

$$= \left| \frac{6x^4}{4} - \frac{16x^6}{6} \right|_0^1$$

$$= \frac{3}{2} - \frac{8}{3}$$

$$= -\frac{7}{6}$$

Example 4

Evaluate $\int_C \bar{F} \cdot d\bar{r}$ *along the parabola* $y^2 = x$ *between the points* (0, 0) *and* (1, 1) *where* $\bar{F} = x^2\hat{i} + xy\hat{j}$.

Solution

(i) Let $\bar{r} = x\hat{i} + y\hat{j}$

$d\bar{r} = \hat{i}dx + \hat{j}dy$

(ii) $\bar{F}\cdot d\bar{r} = (x^2\hat{i} + xy\hat{j})\cdot(\hat{i}\,dx + \hat{j}\,dy)$

$= x^2dx + xydy$

(iii) Path of integration is the parabola $x = y^2$.

$dx = 2ydy$

Substituting in $\bar{F}\cdot d\bar{r}$ and integrating between the limits $y = 0$ to $y = 1$,

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 (y^4 \cdot 2y\,dy + y^2 \cdot y\,dy)$$

$$= \int_0^1 (2y^5 + y^3)\,dy$$

$$= \left|2\frac{y^6}{6} + \frac{y^4}{4}\right|_0^1$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}$$

Example 5

If $\bar{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, *evaluate* $\int_C \bar{F}\cdot d\bar{r}$ *from* (0, 0, 0) *to* (1, 1, 1) *along the curve C given by* $x = t$, $y = t^2$, $z = t^3$.

Solution

(i) Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$

(ii) $x = t, \quad y = t^2, \quad z = t^3$

$dx = dt, \; dy = 2tdt, \; dz = 3t^2dt$

(iii) $\bar{F}\cdot d\bar{r} = [(3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}]\cdot[\hat{i}dx + \hat{j}dy + \hat{k}dz]$

$= (3x^2 + 6y)dx - 14yzdy + 20xz^2dz$

$= (3t^2 + 6t^2)dt - 14t^2t^3\,2tdt + 20tt^6\,3t^2dt$

$= 9t^2dt - 28t^6dt + 60t^9dt$

$= (9t^2 - 28t^6 + 60t^9)dt$

When $x = 0$, $t = 0$ and when $x = 1$, $t = 1$.

Substituting in $\bar{F}\cdot d\bar{r}$ and integrating between the limits $t = 0$ to $t = 1$,

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$$

$$= \left|\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right|_0^1$$

$$= 3 - 4 + 6$$

$$= 5$$

Example 6

If $\bar{F} = 5xy\hat{i} + 2y\hat{j}$, *evaluate* $\int_C \bar{F}\cdot d\bar{r}$ *where C is the part of the curve* $y = x^3$ *between* $x = 1$ *and* $x = 2$.

Solution

(i) Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

(ii) $\bar{F}\cdot d\bar{r} = (5xy\hat{i} + 2y\hat{j})\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$

$$= 5xy\,dx + 2y\,dy$$

(iii) Path of integration is the curve $y = x^3$.

$$dy = 3x^2 dx$$

Substituting in $\bar{F}\cdot d\bar{r}$ and integrating between the limits $x = 1$ to $x = 2$,

$$\int_C \bar{F}\cdot d\bar{r} = \int_1^2 (5x\cdot x^3 dx + 2\cdot x^3\cdot 3x^2 dx)$$

$$= \int_1^2 \left(5x^4 dx + 6x^5 dx\right)$$

$$= \int_1^2 (5x^4 + 6x^5)\,dx$$

$$= \left|\frac{5x^5}{5} - \frac{6x^6}{6}\right|_1^2$$

$$= (32 + 64) - (1 + 1)$$

$$= 94$$

Example 7

Prove that $\int_C \bar{F}\cdot d\bar{r} = 3\pi$, *where* $\bar{F} = z\hat{i} + x\hat{j} + y\hat{k}$ *and C is the arc of the curve* $\bar{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$ *from* $t = 0$ *to* $t = 2\pi$.

Solution

(i) Let $\bar{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$

$$d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

(ii) $x = \cos t,\ y = \sin t,\ z = t$

$$dx = -\sin t\,dt,\ dy = \cos t\,dt,\ dz = dt$$

(iii)
$$\begin{aligned}\bar{F}\cdot d\bar{r} &= (z\hat{i} + x\hat{j} + y\hat{k})\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)\\ &= zdx + xdy + ydz\\ &= t(-\sin t)dt + \cos t\cdot\cos t\,dt + \sin t\,dt\\ &= (-t\sin t + \cos^2 t + \sin t)dt\end{aligned}$$

(iv) Path of integration is the arc of the curve $\bar{r} = \cos t\hat{i} + \sin t\,\hat{j} + t\hat{k}$ from $t = 0$ to $t = 2\pi$.

$$\begin{aligned}\int_C \bar{F}\cdot d\bar{r} &= \int_0^{2\pi}(-t\sin t + \cos^2 t + \sin t)dt\\ &= \int_0^{2\pi}\left(-t\sin t + \frac{1+\cos 2t}{2} + \sin t\right)dt\\ &= -\left|t(-\cos t) - (-\sin t)\right|_0^{2\pi} + \left|\frac{t}{2} + \frac{\sin 2t}{4}\right|_0^{2\pi} + \left|-\cos t\right|_0^{2\pi}\\ &= -(-2\pi\cos 2\pi + \sin 2\pi - \sin 0) + \left(\frac{2\pi}{2} + \frac{\sin 4\pi}{4} - \frac{\sin 0}{4}\right) - (\cos 2\pi - \cos 0)\\ &= 2\pi + \pi\\ &= 3\pi\end{aligned}$$

Example 8

If $\bar{F} = (2x - y + 2z)\hat{i} + (x + y - z)\hat{j} + (3x - 2y - 5z)\hat{k}$, *calculate the circulation of* $\bar{F}$ *along the circle in the xy-plane of 2 unit radius and centre at the origin.*

Solution

$$\text{Circulation} = \oint_C \bar{F}\cdot d\bar{r}$$

(i) Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

(ii) $\bar{F}\cdot d\bar{r} = \left[(2x - y + 2z)\hat{i} + (x + y - z)\hat{j} + (3x - 2y - 5z)\hat{k}\right]\cdot(\hat{i}\,dx + \hat{j}dy + \hat{k}\,dz)$

$$= (2x - y + 2z)dx + (x + y - z)dy + (3x - 2y - 5z)dz$$

(iii) Path of integration is the circle in xy-plane of radius of 2 units and centre at the origin, i.e. $x^2 + y^2 = 4$ and in xy-plane $z = 0$

Parametric equation of the circle is

$$x = 2\cos\theta, \qquad y = 2\sin\theta$$

$$dx = -2\sin\theta\, d\theta, \qquad dy = 2\cos\theta\, d\theta$$

For the complete circle, θ varies from 0 to 2π.

Substituting in $\bar{F}\cdot d\bar{r}$ and integrating between the limits $\theta = 0$ to $\theta = 2\pi$,

$$\text{Circulation} = \int_0^{2\pi}[(2\cdot 2\cos\theta - 2\sin\theta)(-2\sin\theta\, d\theta) + (2\cos\theta + 2\sin\theta)(2\cos\theta\, d\theta)]$$

$$= 4\int_0^{2\pi}(-2\cos\theta\sin\theta + \sin^2\theta + \cos^2\theta + \cos\theta\sin\theta)\,d\theta$$

$$= 4\int_0^{2\pi}\left(1 - \frac{\sin 2\theta}{2}\right)d\theta$$

$$= 4\left|\theta + \frac{\cos 2\theta}{4}\right|_0^{2\pi}$$

$$= 8\pi$$

Example 9

Evaluate $\int_C \bar{F}\cdot d\bar{r}$ *over the circular path* $x^2 + y^2 = a^2$ *where* $\bar{F} = \sin y\hat{i} + x(1 + \cos y)\hat{j}$.

Solution

(i) Let $\bar{r} = x\hat{i} + y\hat{j}$

$$d\bar{r} = \hat{i}dx + \hat{j}dy$$

(ii) $\bar{F}\cdot d\bar{r} = [\sin y\hat{i} + x(1 + \cos y)\hat{j}]\cdot(\hat{i}\,dx + \hat{j}dy)$

$$= \sin y\, dx + x(1 + \cos y)dy$$

$$= \sin y\, dx + x\cos y\, dy + x\, dy$$

$$= d(x\sin y) + x dy$$

(iii) Path of integration is the circle $x^2 + y^2 = a^2$.

Parametric equation of the circle is

$$x = a\cos\theta, \qquad y = a\sin\theta$$
$$dx = -a\sin\theta\, d\theta, \qquad dy = a\cos\theta\, d\theta$$

For complete circle, θ varies from 0 to 2π.

Substituting in $\bar{F}\cdot d\bar{r}$ and integrating between the limits $\theta = 0$ to $\theta = 2\pi$,

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi}\left[d\{a\cos\theta\sin(a\sin\theta)\} + a\cos\theta\cdot a\cos\theta d\theta\right]$$
$$= \left|a\cos\theta\sin(a\sin\theta)\right|_0^{2\pi} + \frac{a^2}{2}\int_0^{2\pi}(1+\cos 2\theta)d\theta$$
$$= 0 + \frac{a^2}{2}\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{2\pi}$$
$$= \frac{a^2}{2}\left[2\pi + \frac{\sin 4\pi}{2} - \frac{\sin 0}{2}\right]$$
$$= \pi a^2$$

Example 10

Find the work done in moving a particle in the force field $\bar{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ *along the curve* $x^2 = 4y$ *and* $3x^3 = 8z$ *from* $x = 0$ *to* $x = 2$.

Solution

Work done $= \int_C \bar{F}\cdot d\bar{r}$

(i) Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

(ii) $\bar{F}\cdot d\bar{r} = \left[3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}\right]\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$

$$= 3x^2dx + (2xz - y)dy + zdz$$

(iii) Path of integration is the curve $x^2 = 4y$ and $3x^3 = 8z$.

$$y = \frac{x^2}{4}, \qquad z = \frac{3}{8}x^3$$
$$dy = \frac{x}{2}dx, \quad dz = \frac{9x^2}{8}dx$$

Substituting in $\bar{F}\cdot d\bar{r}$ and integrating between the limits $x = 0$ to $x = 2$,

$$\text{Work done} = \int_0^2 \left[3x^2 dx + \left(2x \cdot \frac{3x^3}{8} - \frac{x^2}{4} \right) \frac{x}{2} dx + \frac{3x^3}{8} \cdot \frac{9x^2}{8} dx \right]$$

$$= \int_0^2 \left(3x^2 + \frac{51x^5}{64} - \frac{x^3}{8} \right) dx$$

$$= \left| \frac{3x^3}{3} + \frac{51}{64} \cdot \frac{x^6}{6} - \frac{1}{8} \cdot \frac{x^4}{4} \right|_0^2$$

$$= 8 + \frac{51}{6} - \frac{1}{2}$$

$$= 16$$

Example 11

Find the work done, when a force $\overline{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ *moves a particle from the origin to the point* (1, 1) *along* $y^2 = x$.

[Summer 2014]

Solution

$$\text{Work done} = \int_C \overline{F} \cdot d\overline{r}$$

(i) Let $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\overline{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

(ii) $\overline{F} \cdot d\overline{r} = [(x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}] \cdot \left[\hat{i}dx + \hat{j}dy + \hat{k}dz \right]$

$$= (x^2 - y^2 + x)dx - (2xy + y)dy$$

(iii) Path of integration is the curve $y^2 = x$.

$$2y\,dy = dx$$

Substituting in $\overline{F} \cdot d\overline{r}$ and integrating between the limits $y = 0$ to $y = 1$,

$$\text{Work done} = \int_0^1 [(x^2 - y^2 + x)dx - (2xy + y)dy]$$

$$= \int_0^1 (2y^5 - 2y^3 - y)dy$$

$$= \left| \frac{2y^6}{6} - \frac{2y^4}{4} - \frac{y^2}{2} \right|_0^1$$

$$= \frac{1}{3} - \frac{1}{2} - \frac{1}{2}$$

$$= -\frac{2}{3}$$

Example 12

Find the work done when a force $\bar{F}=(y+3)\hat{i}+xz\hat{j}+(yz-x)\hat{k}$ *moves a particle from* (0, 0, 0) *to the point* (2, 1, 1) *along the curve* $x=2t^2,\ y=t,\ z=t^3$.

Solution

$$\text{Work done} = \int_C \bar{F}\cdot d\bar{r}$$

(i) Let $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$$d\bar{r}=\hat{i}dx+\hat{j}dy+\hat{k}dz$$

(ii) $x=2t^2,\quad y=t,\quad z=t^3$

$$dx=4t\,dt,\ dy=dt,\ dz=3t^2dt$$

(iii) $\bar{F}\cdot d\bar{r}=[(y+3)\hat{i}+xz\hat{j}+(yz-x)\hat{k}]\cdot\left[\hat{i}dx+\hat{j}dy+\hat{k}dz\right]$

$$=(y+3)dx+xzdy+(yz-x)dz$$
$$=(t+3)(4tdt)+(2t^2)(t^3)dt+[(t)(t^3)-2t^2](3t^2dt)$$
$$=(4t^2+12t)dt+2t^5dt+(t^4-2t^2)(3t^2)dt$$
$$=(4t^2+12t+2t^5+3t^6-6t^4)dt$$

Substituting in $\bar{F}\cdot d\bar{r}$ and integrating between the limits $t=0$ to $t=1$,

$$\text{Work done}=\int_0^1(4t^2+12t+2t^5+3t^6-6t^4)dt$$
$$=\left|\frac{4t^3}{3}+\frac{12t^2}{2}+\frac{2t^6}{6}+\frac{3t^7}{7}-\frac{6t^5}{5}\right|_0^1$$
$$=\frac{4}{3}+6+\frac{1}{3}+\frac{3}{7}-\frac{6}{5}$$
$$=\frac{724}{105}$$

Example 13

If $\bar{F}=2xyz\,\hat{i}+(x^2z+2y)\hat{j}+x^2y\hat{k}$ *then*

(i) *if* $\bar{F}$ *is conservative, find its scalar potential* ϕ

(ii) *find the work done in moving a particle under this force field from* (0, 1, 1) *to* (1, 2, 0)

Solution

(i) Since $\overline{F}$ is conservative,

$$\overline{F} = \nabla\phi$$

$$(2xyz)\hat{i} + (x^2z + 2y)\hat{j} + (x^2y)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Comparing coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ on both the sides,

$$\frac{\partial\phi}{\partial x} = 2xyz \quad \text{...(1)}$$

$$\frac{\partial\phi}{\partial y} = x^2z + 2y \quad \text{...(2)}$$

$$\frac{\partial\phi}{\partial z} = x^2y \quad \text{...(3)}$$

Integrating Eq. (1) partially w. r. t. x,

$$\phi = x^2yz + f_1(y, z)$$

Integrating Eq. (2) partially w. r. t. y,

$$\phi = x^2yz + y^2 + f_2(x, z)$$

Integrating Eq. (3) partially w. r. t. z,

$$\phi = x^2yz + f_3(x, y)$$

$$\therefore\ \phi = x^2yz + y^2 + c$$

where c is the constant of integration.

(ii) $\overline{F}$ is conservative and hence the work-done is independent of the path.

$$\begin{aligned}
\text{Work done } &= \int_C \overline{F}\cdot d\overline{r} \\
&= \int_{(0,1,1)}^{(1,2,0)} d\phi \\
&= \left|\phi\right|_{(0,1,1)}^{(1,2,0)} \\
&= \left|x^2yz + y^2 + c\right|_{(0,1,1)}^{(1,2,0)} \\
&= 4 - 1 \\
&= 3
\end{aligned}$$

Example 14

If $\overline{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ *then*

(i) *if* $\overline{F}$ *is conservative, find its scalar potential* ϕ.

(ii) *find the work done in moving a particle under this force field from* (1, 1, 0) *to* (2, 0, 1)

Solution

(i) Since $\overline{F}$ is conservative,

$$\overline{F} = \nabla\phi$$

$$(x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Comparing coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ on both the sides,

$$\frac{\partial\phi}{\partial x} = x^2 - yz \quad \text{...(1)}$$

$$\frac{\partial\phi}{\partial y} = y^2 - zx \quad \text{...(2)}$$

$$\frac{\partial\phi}{\partial z} = z^2 - xy \quad \text{...(3)}$$

Integrating Eq. (1) w. r. t. x,

$$\phi = \frac{x^3}{3} - xyz + f_1(y, z)$$

Integrating Eq. (2) w. r. t. y,

$$\phi = \frac{y^3}{3} - xyz + f_2(x, z)$$

Integrating Eq. (3) w. r. t. z,

$$\phi = \frac{z^3}{3} - xyz + f_3(x, y)$$

$$\therefore \phi = \frac{x^3}{3} - xyz + \frac{y^3}{3} + \frac{z^3}{3} + c$$

where C is the constant of integration.

(ii) $\overline{F}$ is conservative and hence the work done is independent of the path.

$$\text{Work done} = \int_C \overline{F} \cdot d\overline{r}$$

$$= \int_{(1,1,0)}^{(2,0,1)} d\phi$$

$$= \left|\phi\right|_{(1,1,0)}^{(2,0,1)}$$

$$= \left|\frac{x^3}{3} - xyz + \frac{y^3}{3} + \frac{z^3}{3} + c\right|_{(1,1,0)}^{(2,0,1)}$$

$$= \frac{7}{3}$$

Example 15

If $\overline{F} = 2\hat{i} + 0\hat{j}$, *calculate the flux of* $\overline{F}$ *through the line segment from* (3, 0) *to* (0, 3).

Solution

$$\text{Flux} = \int_C \overline{F} \cdot \hat{n}\, dS = \int_a^b (M\,dy - N\,dx)$$

(i) $M = 2, N = 0$

(ii) $M\,dy - N\,dx = 2\,dy - 0 \cdot dx = 2\,dy$

(iii) Path of integration is the line segment AB from (3, 0) to (0, 3).

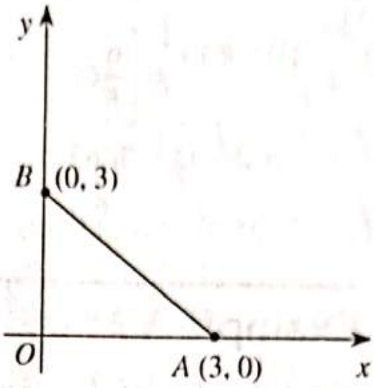

Fig. 1.4

Along AB, y varies from $y = 0$ to $y = 3$

Integrating $(M\,dy - N\,dx)$ between the limits $y = 0$ to $y = 3$,

$$\begin{aligned}\text{Flux} &= \int_0^3 2\,dy \\ &= 2\left|y\right|_0^3 \\ &= 6\end{aligned}$$

Example 16

Find the flux of $\overline{F} = 3xy\hat{i} + (x - y)\hat{j}$ *through the parabolic arc* $y = x^2$ *between* (–1, 1) *and* (4, 16).

Solution

$$\text{Flux} = \int_C \overline{F} \cdot \hat{n}\, dS = \int_a^b (M\,dy - N\,dx)$$

(i) $M = 3xy, N = (x - y)$

(ii) $M\,dy - N\,dx = 3xy\,dy - (x - y)\,dx$

(iii) Path of integration is the parabola

$$y = x^2$$
$$dy = 2x\,dx$$

Substituting in $M\,dy - N\,dx$ and integrating between the limits $x = -1$ to $x = 4$,

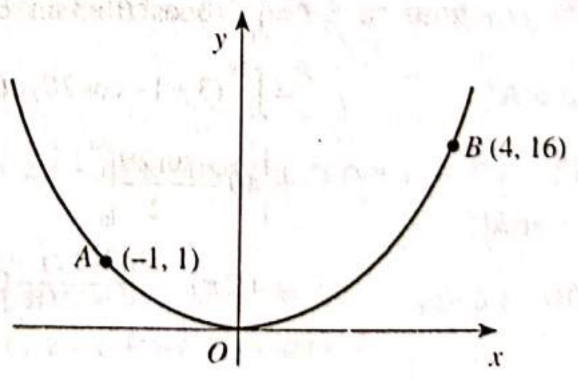

Fig. 1.5

$$\text{Flux} = \int_{-1}^{4}\left[3x\cdot x^2\cdot 2x\,dx-(x-x^2)\,dx\right]$$

$$= \int_{-1}^{4}(6x^4-x+x^2)\,dx$$

$$= \left|6\frac{x^5}{5}-\frac{x^2}{2}+\frac{x^3}{3}\right|_{-1}^{4}$$

$$= \left[\frac{6}{5}(4)^5-\frac{(4)^2}{2}+\frac{(4)^3}{3}\right]-\left[\frac{6}{5}(-1)^5-\frac{(-1)^2}{2}+\frac{(-1)^3}{3}\right]$$

$$= \frac{7465}{6}$$

Example 17

Find the flux of $\overline{F} = 3x\hat{i} + 5y\hat{j}$ through the circle to $x^2 + y^2 = 1$.

Solution

$$\text{Flux} = \int_C \overline{F}\cdot\hat{n}\,dS = \int_a^b (M\,dy - N\,dx)$$

(i) $M = 3x, N = 5y$

(ii) $M\,dy - N\,dx = 3x\,dy - 5y\,dx$

(iii) Path of integration is the circle $x^2 + y^2 = 1$.

Parametric equation of the circle is

$$x = \cos\theta, \qquad y = \sin\theta$$

$$dx = -\sin\theta\,d\theta, \qquad dy = -\cos\theta\,d\theta$$

For complete circle, θ varies from 0 to π.

Substituting in $M\,dy - N\,dx$ and integrating between the limits $\theta = 0$ to $\theta = 2\pi$,

$$\text{Flux} = \int_0^{2\pi}\left[3\cos\theta\cdot\cos\theta\,d\theta - 5\sin\theta(-\sin\theta)\,d\theta\right]$$

$$= \int_0^{2\pi}(3\cos^2\theta + 3\sin^2\theta + 2\sin^2\theta)\,d\theta$$

$$= \int_0^{2\pi}(3 + 1 - \cos 2\theta)\,d\theta$$

$$= \left|4\theta - \frac{\sin 2\theta}{2}\right|_0^{2\pi}$$

$$= 4(2\pi) - \frac{\sin 2(2\pi)}{2} - 0 + \frac{\sin 0}{2}$$

$$= 8\pi$$

EXERCISE 1.4

1. Evaluate $\int_C \bar{F}\cdot d\bar{r}$ $\bar{F}=(x+y)\hat{i}+(y-x)\hat{j}$ and C is

   (i) the parabola $y^2 = x$ between the points (1, 1) and (4, 2)

   (ii) the straight line joining the points (1, 1) and (4, 2)

   $\left[\text{Ans.: (i) } \frac{34}{3} \text{ (ii) } 11\right]$

2. Evaluate $\int_C \bar{F}\cdot d\bar{r}$, Where $\bar{F} = (3x-2y)\hat{i}+(y+2z)\hat{j}-x^2\hat{k}$ and C is

   (i) the curve $x = t$, $y = t^2$, $z = t^3$ between the points (0, 0, 0) to (1, 1, 1)

   (ii) the straight line joining the points (0, 0, 0) to (1, 1, 1).

   (iii) the straight lines from (0, 0, 0) to (0, 1, 0) then to (0, 1, 1) and then to (1, 1, 1).

   $\left[\text{Ans.: (i) } \frac{23}{15} \text{ (ii) } \frac{5}{3} \text{ (iii) } 0\right]$

3. Evaluate $\int_C \bar{F}\cdot d\bar{r}$, where $\bar{F} = (2x+y^2)\hat{i}+(3y-4x)\hat{j}$ and C is the triangle in the xy–plane with vertices (0, 0), (2, 0) and (2, 1).

   $\left[\text{Ans.: } -\frac{14}{3}\right]$

4. Evaluate $\int_C \bar{F}\cdot d\bar{r}$, where $\bar{F}=yz\hat{i}+zx\hat{j}+xy\hat{k}$ and C is the curve $y^2=x$, $z=0$ from (0, 0, 0) to (1, 1, 0) followed by the straight line from (1, 1, 0) to (1, 1, 1).

   $\left[\text{Ans.: } \frac{3}{4}\right]$

5. Evaluate $\int_C \bar{F}\cdot d\bar{r}$, where $\bar{F} = 2x\hat{i}+4y\hat{j}-3z\hat{k}$ and C is the curve $\bar{r}=\cos t\,\hat{i}+\sin t\,\hat{j}+t\,\hat{k}$ from $t = 0$ to $t = \pi$.

   $\left[\text{Ans.: } -\frac{3\pi^2}{2}\right]$

6. Find the circulation of $\bar{F}=(x-3y)\hat{i}+(y-2x)\hat{j}$ around the ellipse in the xy–plane with the origin as centre and 2 and 3 as semi–major and semi–minor axes respectively.

   [Ans.: 6π]

7. Find the circulation of $\bar{F} = y\hat{i}+z\hat{j}+x\hat{k}$ around the curve $x^2+y^2=1$, $z = 0$.

   [Ans.: $-\pi$]

8. Find the work done in moving a particle in a force field $\bar{F} = 3xy\hat{i}-5z\hat{j}+10x\hat{k}$ along the curve $x=1+t^2$, $y=2t^2$, $z=t^3$ from $t=1$ to $t=2$.

   [Ans.: 303]

9. Find the work done in moving a particle in a force field $\bar{F} = 3x^2\,\hat{i} + (2xz - y)\,\hat{j} + z\,\hat{k}$ along the

(i) straight line joining the points (0, 0, 0) and (2, 1, 3)

(ii) curve $x = 2t^2$, $y = t$, $z = 4t^2 - t$ from $t = 0$ to $t = 1$

$\left[\text{Ans.: (i) } 16 \text{ (ii) } \frac{71}{5}\right]$

10. Find the work done in moving a particle in a force field $\bar{F} = (2x - y + z)\,\hat{i} + (x + y - z^2)\,\hat{j} + (3x - 2y + 4z)\,\hat{k}$ once around the circle in xy–plane with centre at the origin and radius of 3 units.

$[\text{Ans.: } 18\pi]$

11. If $\bar{F} = (2xy + z^3)\,\hat{i} + x^2\,\hat{j} + 3xz^2\,\hat{k}$ is conservative then

(i) find its scalar potential ϕ

(ii) find the work done in moving a particle under this force field from (1, –2, 1) to (3, 1, 4)

$\left[\text{Ans.: (i) } \phi = x^2y + xz^3 + c \text{ (ii) } 202\right]$

12. If $\bar{F} = 3x^2y\hat{i} + (x^3 - 2yz^2)\,\hat{j} + (3z^2 - 2y^2z)\,\hat{k}$, is conservative

(i) find its scalar potential ϕ

(ii) find the work done in moving a particle under this force field from (2, 1, 1) to (2, 0, 1)

$\left[\text{Ans.: (i) } \phi = x^3y + z^3 - y^2z^2 + c \text{ (ii) } -7\right]$

13. If $\bar{F} = 2xye^z\,\hat{i} + x^2e^z\,\hat{j} + x^2ye^z\,\hat{k}$ is conservative then find

(i) the scalar potential ϕ

(ii) the work done in moving a particle under this force field from (0, 0, 0) to (1, 1, 1)

$\left[\text{Ans.: (i) } \phi = x^2ye^z + c \text{ (ii) } e\right]$

14. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = \cos y\,\hat{i} - x\sin y$ and C is the curve $y = \sqrt{1 - x^2}$ in the xy-plane from (1, 0) to (0, 1).

$[\text{Ans.: } -1]$

15. If $\bar{F} = x\hat{i} + y\hat{j}$, calculate the flux of $\bar{F}$ (i) through the line connecting (0, 0) to (a, b); (ii) through the circle $x^2 + y^2 = 1$.

[Ans.: (i) 0, (ii) 0]

16. Find the flux of $\bar{F} = x^2\hat{i} + y\hat{j}$ through the circle of radius 3 and centre at the origin.

$[\text{Ans.: } 9\pi]$

1.10 GREEN'S THEOREM IN THE PLANE

If $M(x, y)$, $N(x, y)$ and their partial derivatives $\frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are continuous in some region R of xy-plane bounded by a closed curve C then

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Proof Let the region R be bounded by the curve C.

Let the curve C be divided into two parts, the curves EAB and BDE.

Let the equations of the curves EAB and BDE are $x = f_1(y)$, $x = f_2(y)$ respectively and are bounded between the lines $y = c$ and $y = d$.

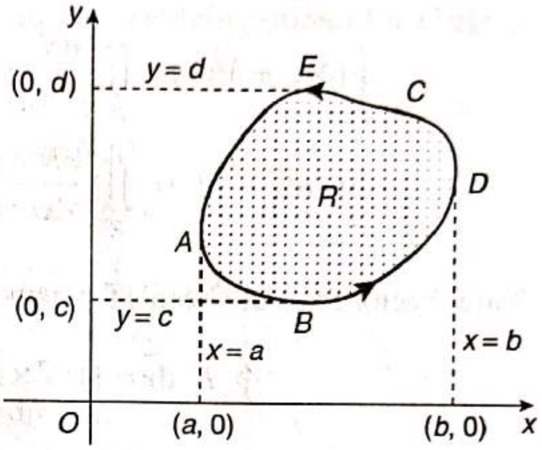

Fig. 1.6

$$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_c^d \left[\int_{f_1(y)}^{f_2(y)} \frac{\partial N}{\partial x}\,dx\right] dy$$

$$= \int_c^d \left|N(x, y)\right|_{f_1(y)}^{f_2(y)} dy$$

$$= \int_c^d [N(f_2, y) - N(f_1, y)]\,dy$$

$$= \int_c^d N(f_2, y)\,dy + \int_d^c N(f_1, y)\,dy$$

$$= \int_{BDE} N(x, y)\,dy + \int_{EAB} N(x, y)\,dy$$

$$= \oint_C N(x, y)\,dy$$

$$\oint_C N(x, y)\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy \qquad \text{... (1.4)}$$

Similarly, let the curve C be divided into two parts, the curves ABD and DEA.

Let the equations of the curves ABD and DEA are $y = g_1(x)$, $y = g_2(x)$ respectively and are bounded between the lines $x = a$ and $x = b$.

$$\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} \frac{\partial M}{\partial y}\,dy\right] dx$$

$$= \int_a^b \left|M(x, y)\right|_{g_1(x)}^{g_2(x)} dx$$

$$= \int_a^b [M(x, g_2) - M(x, g_1)]\,dx$$

$$= -\int_b^a M(x, g_2)\,dx - \int_a^b M(x, g_1)\,dx$$

$$= -\left[\int_{DEA} M(x,y)\,dx + \int_{ABD} M(x,y)\,dx\right]$$

$$= -\oint_C M(x,y)\,dx$$

$$\oint_C M(x,y)\,dx = -\iint_R \frac{\partial M}{\partial y}\,dx\,dy \qquad \dots(1.5)$$

Adding Eqs. (1.4) and (1.5),

$$\oint_C (N\,dy + M\,dx) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Note: Vector form of Green's theorem is given as

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_R (\nabla \times \bar{F})\cdot \hat{k}\,dx\,dy$$

where $\bar{F} = M\hat{i} + N\hat{j}$, $\bar{r} = x\hat{i} + y\hat{j}$, $\hat{k}$ is the unit vector along z-axis.

Area of the Plane Region Let A be the area of the plane region R bounded by a closed curve C.

Let
$$M = -y, \quad N = x$$

$$\frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1$$

Using Green's theorem,

$$\oint_C (-y\,dx + x\,dy) = \iint_R (1+1)\,dx\,dy = 2\iint_R dx\,dy = 2A$$

Hence,
$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

Note: In polar coordinates,

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$dx = \cos\theta\,dr - r\sin\theta\,d\theta, \quad dy = \sin\theta\,dr + r\cos\theta\,d\theta$$

$$A = \frac{1}{2}\oint_C [r\cos\theta(\sin\theta\,dr + r\cos\theta\,d\theta) - r\sin\theta(\cos\theta\,dr - r\sin\theta\,d\theta)]$$

$$= \frac{1}{2}\oint_C r^2\,d\theta$$

Example 1

Find the area of a circle of radius 'a' using Green's theorem.

Solution

(i) By Green's theorem, the area of a circle is

$$A = \frac{1}{2}\oint_C (x\mathrm{d}y - y\mathrm{d}x) \qquad \ldots(1)$$

(ii) Path of integration is the circle of radius a. Parametric equation of a circle of radius a is

$$x = a\cos\theta, \qquad y = a\sin\theta$$

$$\mathrm{d}x = -a\sin\theta\,\mathrm{d}\theta, \qquad \mathrm{d}y = a\cos\theta\,\mathrm{d}\theta$$

Substituting in Eq. (1) and integrating between the limits $\theta = 0$ to $\theta = 2\pi$,

$$A = \frac{1}{2}\int_0^{2\pi} [(a\cos\theta)(a\cos\theta\,\mathrm{d}\theta) - (a\sin\theta)(-a\sin\theta\,\mathrm{d}\theta)]$$

$$= \frac{1}{2}\int_0^{2\pi} (a^2\cos^2\theta + a^2\sin^2\theta)\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{2\pi} a^2\,\mathrm{d}\theta$$

$$= \frac{1}{2}a^2 |\theta|_0^{2\pi}$$

$$= \frac{1}{2}a^2(2\pi)$$

$$= \pi a^2$$

Example 2

Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution

(i) By Green's theorem, the area of the region bounded by a closed curve C is

$$A = \frac{1}{2}\int_C (x\,\mathrm{d}y - y\,\mathrm{d}x) \qquad \ldots(1)$$

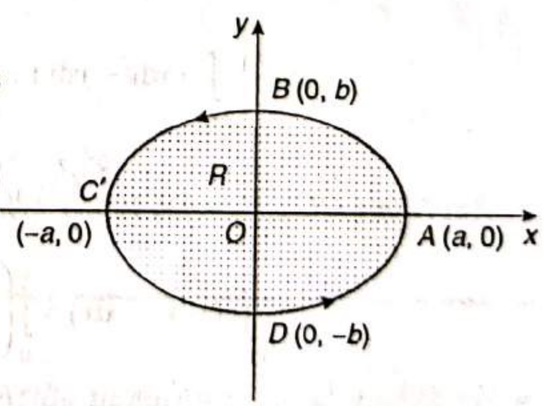

Fig. 1.7

(ii) Parametric equation of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$x = a\cos\theta, \qquad y = b\sin\theta$$

$$\mathrm{d}x = -a\sin\theta\,\mathrm{d}\theta, \qquad \mathrm{d}y = b\cos\theta\,\mathrm{d}\theta$$

For the given ellipse, θ varies from 0 to 2π.
Substituting in Eq. (1),

$$A = \frac{1}{2}\int_0^{2\pi} [a\cos\theta(b\cos\theta\, d\theta) - b\sin\theta(-a\sin\theta\, d\theta)]$$

$$= \frac{1}{2}\int_0^{2\pi} ab\, d\theta$$

$$= \frac{1}{2}ab|\theta|_0^{2\pi}$$

$$= \pi ab$$

Example 3

Find the area between the curves $y^2 = 4x$ and $x^2 = 4y$ by using Green's theorem.

Solution

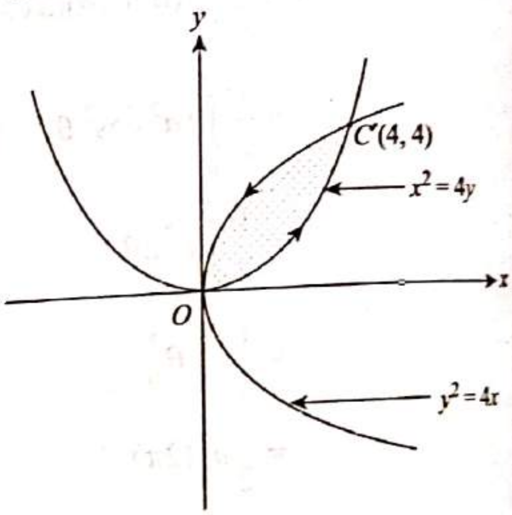

Fig. 1.8

(i) The points of intersection of the parabolas $y^2 = 4x$ and $x^2 = 4y$ are obtained as

$$\left(\frac{y^2}{4}\right)^2 = 4y,$$

$$y(y^3 - 64) = 0$$

$$y = 0, 4$$

$$x = 0, 4$$

Hence, $O(0, 0)$ and $C'(4, 4)$ are the points of intersection.

(ii) By Green's theorem, the area of the region bounded by a closed curve C is

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

$$= \frac{1}{2}\left[\int_{OC'} (x\,dy - y\,dx) + \int_{C'O} (x\,dy - y\,dx)\right] \quad \text{...(1)}$$

(a) Along OC': $x^2 = 4y$, $y = \frac{x^2}{4}$, $dy = \frac{x}{2}dx$ and x varies from 0 to 4.

$$\int_{OC'} (x\,dy - y\,dx) = \int_0^4 \left(x \cdot \frac{x}{2}dx - \frac{x^2}{4}dx\right)$$

$$= \int_0^4 \left(\frac{x^2}{2} - \frac{x^2}{4}\right)dx$$

$$= \int_0^4 \frac{1}{4} x^2 \mathrm{d}x$$

$$= \frac{1}{4}\left|\frac{x^3}{3}\right|_0^4$$

$$= \frac{1}{4}\left(\frac{64}{3}\right)$$

$$= \frac{16}{3}$$

(b) Along $C'O$: $y^2 = 4x, x = \frac{y^2}{4}, \mathrm{d}x = \frac{y}{2}\mathrm{d}y$ and y varies from 4 to 0.

$$\int_{C'O} (x\mathrm{d}y - y\mathrm{d}x) = \int_4^0 \left(\frac{y^2}{4}\mathrm{d}y - y \cdot \frac{y}{2}\mathrm{d}y\right)$$

$$= \int_4^0 \left(\frac{y^2}{4} - \frac{y^2}{2}\right)\mathrm{d}y$$

$$= \int_4^0 -\frac{y^2}{4}\mathrm{d}y$$

$$= -\frac{1}{4}\int_4^0 y^2 \mathrm{d}y$$

$$= -\frac{1}{4}\left|\frac{y^3}{3}\right|_4^0$$

$$= -\frac{1}{4}\left(0 - \frac{64}{3}\right)$$

$$= \frac{16}{3}$$

Substituting in Eq. (1),

$$A = \frac{1}{2}\left(\frac{16}{3} + \frac{16}{3}\right) = \frac{16}{3}$$

Example 4

Find the area of the region bounded by the parabola $y = x^2$ and the line $y = x + 2$.

Solution

(i) The points of intersection of the parabola $y = x^2$ and the line $y = x + 2$ are obtained as

$x + 2 = x^2$, $x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$,

$x = 2, -1$ and $y = 4, 1$

Hence, $A(-1, 1)$ and $B(2, 4)$ are the points of intersections.

(ii) By Green's theorem, the area of the region bounded by a closed curve C is

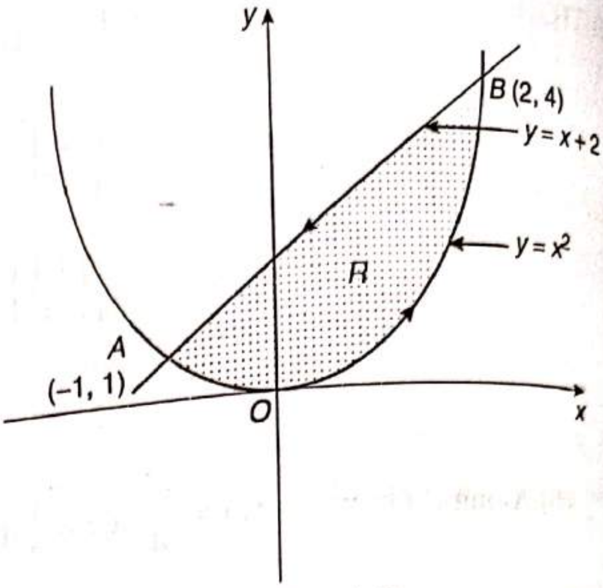

Fig. 1.9

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

$$= \frac{1}{2}\left[\int_{AOB} (x\,dy - y\,dx) + \int_{BA} (x\,dy - y\,dx)\right] \quad \text{...(1)}$$

(a) Along AOB: $y = x^2$, $dy = 2x\,dx$ and x varies from -1 to 2.

$$\int_{AOB} (x\,dy - y\,dx) = \int_{-1}^{2} (x \cdot 2x\,dx - x^2 dx)$$

$$= \left|\frac{x^3}{3}\right|_{-1}^{2}$$

$$= \frac{8}{3} + \frac{1}{3}$$

$$= 3$$

(b) Along BA: $y = x + 2$, $dy = dx$ and x varies from 2 to -1.

$$\int_{BA} (x\,dy - y\,dx) = \int_{2}^{-1} \left[x\,dx - (x+2)\,dx\right]$$

$$= -2|x|_2^{-1}$$

$$= -2(-1-2)$$

$$= 6$$

Substituting in Eq. (1),

$$A = \frac{1}{2}(3+6) = \frac{9}{2}$$

Example 5

Evaluate $\int_C (x\,dy - y\,dx)$ *around the circle* $x^2 + y^2 = 1$.

Solution

By Green's theorem,

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \qquad \cdots(1)$$

where R is region bounded by the circle $x^2 + y^2 = 1$

$$M = -y, \qquad N = x$$

$$\frac{\partial M}{\partial y} = -1, \qquad \frac{\partial N}{\partial x} = 1$$

Substituting in Eq. (1),

$$\oint_C (x\,dy - y\,dx) = \iint_R [1-(-1)]\,dx\,dy$$

$$= 2\iint dx\,dy$$

$$= 2(\text{Area of circle})$$

$$= 2(\pi r^2)$$

$$= 2\pi \qquad (\because r = 1)$$

Example 6

Evaluate $\int_C \left[(x^2 + xy)\,dx + (x^2 + y^2)\,dy\right]$ *where C is the square bounded by* $x = \pm 1, y = \pm 1$.

Solution

By Green's theorem,

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \qquad \cdots(1)$$

where R is the region bounded by the square $ABC'D$.

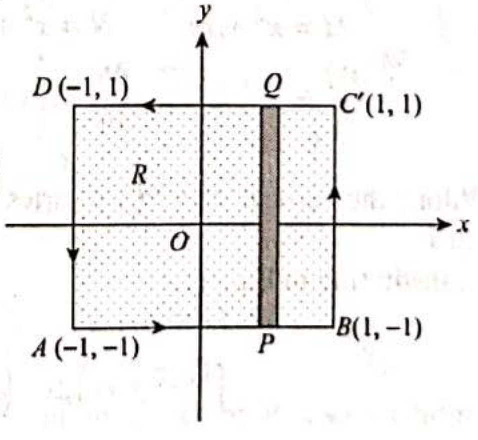

Fig. 1.10

$$M = x^2 + xy, \quad N = x^2 + y^2$$

$$\frac{\partial M}{\partial y} = x, \qquad \frac{\partial N}{\partial x} = 2x$$

Along the vertical strip PQ, y varies from -1 to 1 and in the region R, x varies from -1 to 1.

$$\oint_C \left[\left(x^2+xy\right)dx+\left(x^2+y^2\right)dy\right]=\int_{-1}^{1}\int_{-1}^{1}(2x-x)\,dx\,dy$$

$$=\int_{-1}^{1}\int_{-1}^{1}x\,dx\,dy$$

$$=\int_{-1}^{1}\left|\frac{x^2}{2}\right|_{-1}^{1}dy$$

$$=\int_{-1}^{1}\left(\frac{1}{2}-\frac{1}{2}\right)dy$$

$$=0$$

Example 7

Evaluate $\int_C\left[\left(x^2+xy\right)dx+\left(x^2+y^2\right)dy\right]$, *where C is the square bounded by the lines* $x=0$, $x=1$, $y=0$ *and* $y=1$.

Solution

By Green's theorem,

$$\oint_C (M\,dx+N\,dy)=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy \quad ...(1)$$

where R is the region bounded by the square $OABC'$.

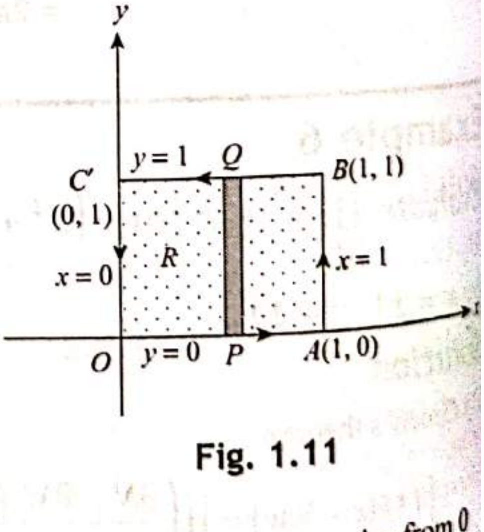

Fig. 1.11

$$M=x^2+xy, \qquad N=x^2+y^2$$

$$\frac{\partial M}{\partial y}=x, \qquad \frac{\partial N}{\partial x}=2x$$

Along the vertical strip PQ, y varies from 0 to 1 and in the region R, x varies from 0 to 1.

Substituting in Eq. (1),

$$\oint_C\left[\left(x^2+xy\right)dx+\left(x^2+y^2\right)dy\right]=\int_0^1\int_0^1(2x-x)\,dx\,dy$$

$$= \int_0^1 \int_0^1 x \, dx dy$$

$$= \int_0^1 \left| \frac{x^2}{2} \right| dy$$

$$= \int_0^1 \frac{1}{2} dy$$

$$= \frac{1}{2} \int_0^1 dy$$

$$= \frac{1}{2} |y|_0^1$$

$$= \frac{1}{2}(1-0)$$

$$= \frac{1}{2}$$

Example 8

Evaluate by Green's theorem $\int_C e^{-x}(\sin y \, dx + \cos y \, dy)$, *C being the rectangle with vertices* $(0,0), (\pi,0), \left(\pi, \frac{\pi}{2}\right)$ *and* $\left(0, \frac{\pi}{2}\right)$.

Solution

By Green's theorem,

$$\oint_C (M dx + N dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \quad \cdots (1)$$

where R is the region bounded by the rectangle $OABC'$,

$$M = e^{-x} \sin y, \qquad N = e^{-x} \cos y$$

$$\frac{\partial M}{\partial y} = e^{-x} \cos y, \qquad \frac{\partial N}{\partial x} = -e^{-x} \cos y$$

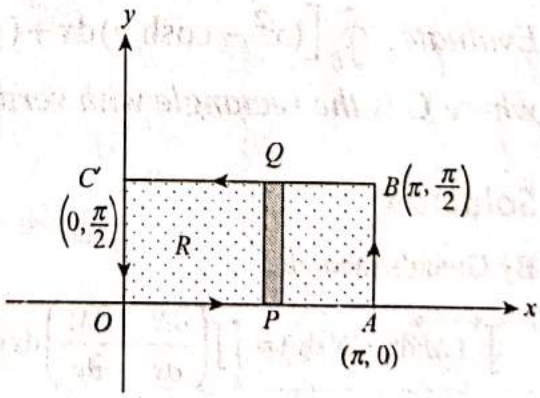

Fig. 1.12

Along the vertical strip PQ, y varies from 0 to $\frac{\pi}{2}$ and in the region R, x varies from 0 to π.

Substituting in Eq. (1),

$$\oint_C e^{-x}(\sin y\,dx + \cos y\,dy) = \int_{y=0}^{\frac{\pi}{2}}\int_{x=0}^{\pi}\left(-e^{-x}\cos y - e^{-x}\cos y\right)dx\,dy$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\pi} -2e^{-x}\cos y\,dx\,dy$$

$$= -2\int_0^{\frac{\pi}{2}}\int_0^{\pi} e^{-x}\cos y\,dx\,dy$$

$$= -2\left[\int_0^{\frac{\pi}{2}}\cos y\,dy\int_0^{\pi}e^{-x}dx\right]$$

$$= -2\left[\left|\sin y\right|_0^{\frac{\pi}{2}}\cdot\left|\frac{e^{-x}}{-1}\right|_0^{\pi}\right]$$

$$= 2\sin\frac{\pi}{2}\left(e^{-\pi} - e^0\right)$$

$$= 2\left(e^{-\pi} - 1\right)$$

Example 9

Evaluate $\oint_C\left[(x^2 - \cosh y)dx + (y + \sin x)dy\right]$ *by Green's theorem where C is the rectangle with vertices* (0, 0), (π, 0), (π, 1), (0, 1).

Solution

By Green's theorem,

$$\oint_C (M\,dx + N\,dy) = \iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy \qquad \dots (1)$$

where R is the region bounded by the rectangle $OABC'$.

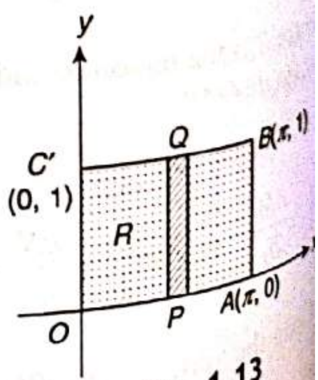

Fig. 1.13

$$M = x^2 - \cosh y,\ N = y + \sin x$$

$$\frac{\partial M}{\partial y} = -\sinh y, \quad \frac{\partial N}{\partial x} = \cos x$$

Along the vertical strip PQ, y varies from 0 to 1 and in the region R, x varies from 0 to π.

Substituting in Eq. (1),

$$\oint_C \left[(x^2 - \cosh y)\mathrm{d}x + (y + \sin x)\mathrm{d}y\right] = \int_{x=0}^{\pi}\int_{y=0}^{1}(\cos x + \sinh y)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^{\pi}\left|y\cos x + \cosh y\right|_0^1 \mathrm{d}x$$

$$= \int_0^{\pi}(\cos x + \cosh 1 - 0 - \cosh 0)\,\mathrm{d}x$$

$$= \int_0^{\pi}(\cos x + \cosh 1 - 1)\,\mathrm{d}x$$

$$= \left|\sin x + x\cosh 1 - x\right|_0^{\pi}$$

$$= \sin\pi + \pi\cosh 1 - \pi - \sin 0$$

$$= \pi(\cosh 1 - 1)$$

Example 10

Evaluate $\oint_C \left[(x^2 + 2y)\mathrm{d}x + (4x + y^2)\mathrm{d}y\right]$ *by Green's theorem where C is the boundary of the region bounded by* $y = 0$, $y = 2x$ *and* $x + y = 3$.

Solution

By Green's theorem,

$$\oint_C (M\,\mathrm{d}x + N\,\mathrm{d}y) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y \quad \ldots (1)$$

where R is the region bounded by the triangle OAB.

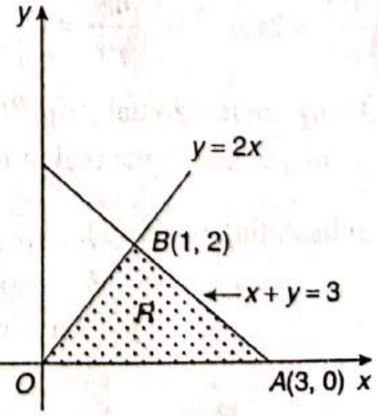

Fig. 1.14

$$M = x^2 + 2y, \qquad N = 4x + y^2$$

$$\frac{\partial M}{\partial y} = 2, \qquad \frac{\partial N}{\partial x} = 4$$

Substituting in Eq. (1),

$$\oint_C \left[(x^2+2y)dx+(4x+y^2)dy\right]=\iint_R (4-2)dx\,dy$$
$$=2\iint_R dx\,dy$$
$$=2(\text{Area of }\Delta OAB)$$
$$=2\left(\frac{1}{2}\times 3\times 2\right)$$
$$=6$$

Example 11

Evaluate $\int_C \left[\left(2xy-x^2\right)dx+\left(x+y^2\right)dy\right]$ *using Green's theorem where C is closed curve formed by* $y=x^2$ *and* $y^2=x$.

Solution

By Green's theorem,

$$\oint_C (M dx+N dy)=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy \quad \cdots(1)$$

where R is the region bounded by the curves $y=x^2$ and $y^2=x$

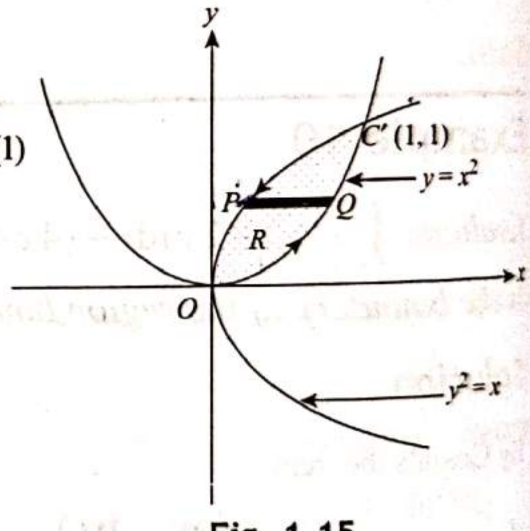

Fig. 1.15

$$M=2xy-x^2,\quad N=x+y^2$$
$$\frac{\partial M}{\partial y}=2x,\quad \frac{\partial N}{\partial x}=1$$

Along the horizontal strip PQ, x varies from y^2 to $\sqrt{y}$ and in the region R, y varies from 0 to 1.

Substituting in Eq. (1),

$$\oint_C\left[\left(2xy-x^2\right)dx+\left(x+y^2\right)dy\right]=\int_0^1\int_{y^2}^{\sqrt{y}}(1-2x)\,dxdy$$
$$=\int_0^1\int_{y^2}^{\sqrt{y}}(1-2x)\,dxdy$$

$$= \int_0^1 \left| x - \frac{2x^2}{2} \right|_{y^2}^{\sqrt{y}} dy$$

$$= \int_0^1 \left[\left(\sqrt{y} - y\right) - \left(y^2 - y^4\right) \right] dy$$

$$= \left| \frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^2}{2} - \frac{y^3}{3} + \frac{y^5}{5} \right|_0^1$$

$$= \frac{2}{3} - \frac{1}{2} - \frac{1}{3} + \frac{1}{5}$$

$$= \frac{1}{30}$$

Example 12

Verify Green's theorem in a plane with respect to $\int_C \left(x^2 dx - xy dy\right)$, *where C is the boundary of the square formed by* $x = 0, x = a, y = 0, y = a$.

Solution

(i) $M = x^2$ $\quad\quad N = -xy$

$\frac{\partial M}{\partial y} = 0 \quad\quad \frac{\partial N}{\partial x} = -y$

(ii) $$\oint_C (M dx + N dy) = \int_{OA} (M dx + N dy) + \int_{AB} (M dx + N dy) + \int_{BC'} (M dx + N dy) + \int_{C'O} (M dx + N dy) \quad \cdots (1)$$

(a) Along OA: $y = 0$, $dy = 0$ and x varies from 0 to a.

$$\int_{OA} (M dx + N dy) = \int_{OA} \left(x^2 dx - xy dy\right)$$

$$= \int_0^a x^2 dx$$

$$= \left| \frac{x^3}{3} \right|_0^a$$

$$= \frac{a^3}{3}$$

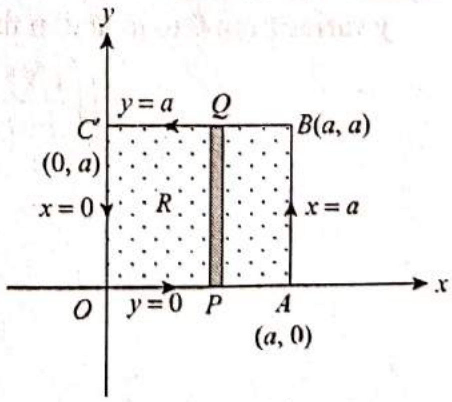

Fig. 1.16

(b) Along AB: $x = a$, $dx = 0$ and y varies from 0 to a.

$$\int_{AB} (M dx + N dy) = \int_{AB} \left(x^2 dx - xy dy\right)$$
$$= \int_0^a (0 - ay) dy$$
$$= -a \left| \frac{y^2}{2} \right|_0^a$$
$$= -\frac{a^3}{2}$$

(c) Along BC': $y = a$, $dy = 0$ and x varies from a to 0.

$$\int_{BC'} (M dx + N dy) = \int_{BC'} \left(x^2 dx - xy\, dy\right)$$
$$= \int_a^0 x^2 dx$$
$$= \left| \frac{x^3}{3} \right|_a^0$$
$$= -\frac{a^3}{3}$$

(d) Along $C'O$: $x = 0$, $dx = 0$, y varies from a to 0.

$$\int_{C'O} (M dx + N dy) = \int_{C'O} \left(x^2 dx - xy dy\right)$$
$$= 0$$

Substituting in Eq. (1),

$$\oint_C (M\, dx + N\, dy) = \frac{a^3}{3} - \frac{a^3}{2} - \frac{a^3}{3} + 0 = -\frac{a^3}{2} \qquad \cdots(2)$$

(iii) Let R be the region bounded by the square $OABC'$. Along the vertical strip PQ, y varies from 0 to a and in the region R, x varies from 0 to a.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \int_0^a \int_0^a -y\, dx\, dy$$
$$= -\int_0^a y |x|_0^a\, dy$$
$$= -a \int_0^a y\, dy$$
$$= -a \left| \frac{y^2}{2} \right|_0^a$$

$$= -a\left(\frac{a^2}{2}\right)$$

$$= -\frac{a^3}{2} \qquad \cdots(3)$$

From Eqs (2) and (3),

$$\oint (M\,dx - N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy = -\frac{a^3}{2}$$

Hence, Green's theorem is verified.

Example 13

Verify Green's theorem in a plane for $\overline{F} = (x^2 + y^2)i - 2xyj$ taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$ and $y = b$.

Solution

(i) Let $\overline{r} = x\hat{i} + y\hat{j}$

$$d\overline{r} = dx\hat{i} + dy\hat{j}$$

$$\overline{F}\cdot d\overline{r} = \left[\left(x^2 + y^2\right)\hat{i} - 2xy\hat{j}\right]\cdot\left[dx\hat{i} + dy\hat{j}\right]$$

$$= \left(x^2 + y^2\right)dx - 2xydy$$

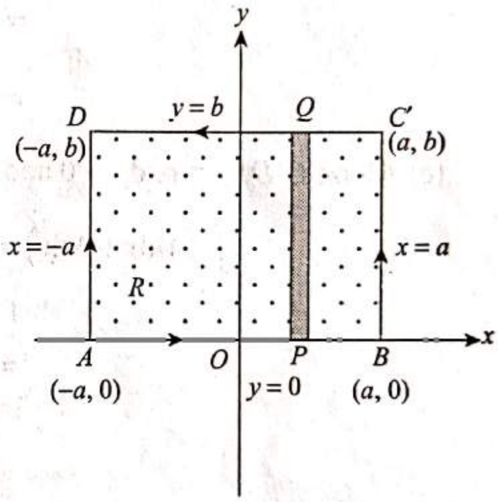

Fig. 1.17

(ii) $M = x^2 + y^2, N = -2xy$

$$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = -2y$$

(iii) $$\oint_C (Mdx + Ndy) = \int_{AB} (Mdx + Ndy) + \int_{BC'} (Mdx + Ndy) + \int_{C'D} (Mdx + Ndy) + \int_{DA} (Mdx + Ndy) \qquad \cdots(1)$$

(a) Along AB: $y = 0$, $dy = 0$ and x varies from $-a$ to a.

$$\int_{AB} (Mdx + Ndy) = \int_{AB} \left[\left(x^2 + y^2\right)dx - 2xydy\right]$$

$$= \int_{-a}^{a} x^2 dx$$

$$= \left|\frac{x^3}{3}\right|_{-a}^{a}$$

$$= \frac{a^3}{3} - \left(-\frac{a^3}{3}\right)$$

$$= \frac{2a^3}{3}$$

(b) Along BC': $x = a$, $dx = 0$ and y varies from 0 to b.

$$\int_{BC'} (M dx + N dy) = \int_{BC'} \left[\left(x^2 + y^2\right) dx - 2xy\, dy\right]$$

$$= \int_0^b -2ay\, dy$$

$$= -2a \left|\frac{y^2}{2}\right|_0^b$$

$$= -2a \frac{b^2}{2}$$

$$= -ab^2$$

(c) Along $C'D$: $y = b$, $dy = 0$ and x varies from a to $-a$.

$$\int_{C'D} (M dx + N dy) = \int_{C'D} \left[\left(x^2 + y^2\right) dx - 2xy\, dy\right]$$

$$= \int_a^{-a} \left(x^2 + b^2\right) dx$$

$$= \left|\frac{x^3}{3} + b^2 x\right|_a^{-a}$$

$$= -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$$

$$= -\frac{2a^3}{3} - 2ab^2$$

(d) Along DA: $x = -a$, $dx = 0$ and y varies from b to 0.

$$\int_{DA} (M\, dx + N\, dy) = \int_{DA} \left[\left(x^2 + y^2\right) dx - 2xy\, dy\right]$$

$$= \int_b^0 -2(-a)\, y\, dy$$

$$= 2a \left|\frac{y^2}{2}\right|_b^0$$

$$= 2a\left(-\frac{b^2}{2}\right)$$

$$= -ab^2$$

Substituting in Eq. (1),

$$\oint_C (M\mathrm{d}x + N\mathrm{d}y) = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 = -4ab^2$$

(iv) Let R be region bounded by the rectangle $ABC'D$. Along the vertical strip PQ, y varies from 0 to b and in the region R, x varies from $-a$ to a.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = \int_{x=-a}^{a}\int_{y=0}^{b} (-2y - 2y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= -4\int_{-a}^{a}\int_{0}^{b} y\,\mathrm{d}x\,\mathrm{d}y$$

$$= -4\int_{-a}^{a}\left|\frac{y^2}{2}\right|_0^b \mathrm{d}x$$

$$= -4\frac{b^2}{2}\int_{-a}^{a}\mathrm{d}x$$

$$= -2b^2 \left|x\right|_{-a}^{a}$$

$$= -2b^2(a + a)$$

$$= -4ab^2 \qquad \cdots(3)$$

From Eqs (2) and (3),

$$\oint_C (M\mathrm{d}x + N\mathrm{d}y) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

Hence, Green's theorem is verified.

Example 14

Verify Green's theorem in the plane for $\int_C \left[\left(3x^2 - 8y^2\right)\mathrm{d}x + \left(4y - 6xy\right)\mathrm{d}y\right]$, *where C is the boundary of the region bounded by* $x = 0$, $y = 0$, $x + y = 1$. **[Summer 2014]**

Solution

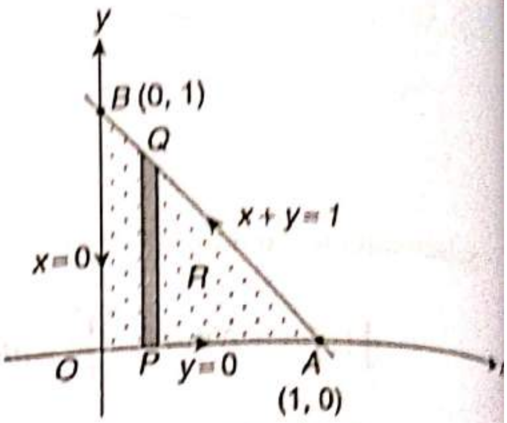

Fig. 1.18

(i) (a) The point of intersection of the lines $y = 0$ and $x + y = 1$ is obtained as $x = 1$. Hence, $A(1, 0)$ is the point of intersection.

(b) The point of intersection of the lines $x = 0$ and $x + y = 1$ is obtained as $y = 1$. Hence, $B(0, 1)$ is the point of intersection.

(ii) $M = 3x^2 - 8y^2 \qquad N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$$

(iii) $$\oint_C (M\mathrm{d}x + N\mathrm{d}y) = \int_{OA} (M\mathrm{d}x + N\mathrm{d}y) + \int_{AB} (M\mathrm{d}x + N\mathrm{d}y) + \int_{BO} (M\mathrm{d}x + N\mathrm{d}y)$$

(a) Along $OA : y = 0, \mathrm{d}y = 0$ and x varies from 0 to 1.

$$\int_{OA} (M\mathrm{d}x + N\mathrm{d}y) = \int_{OA} \left[\left(3x^2 - 8y^2\right)\mathrm{d}x + (4y - 6xy)\,\mathrm{d}y\right]$$

$$= \int_0^1 3x^2\mathrm{d}x$$

$$= 3\left|\frac{x^3}{3}\right|_0^1$$

$$= 1$$

(b) Along $AB : x + y = 1 \Rightarrow y = 1 - x, \mathrm{d}y = -\mathrm{d}x$ and x varies from 1 to 0.

$$\int_{AB} (M\,\mathrm{d}x + N\mathrm{d}y) = \int_{AB} \left[\left(3x^2 - 8y^2\right)\mathrm{d}x + (4y - 6xy)\,\mathrm{d}y\right]$$

$$= \int_1^0 \left[\left\{3x^2 - 8(1-x)^2\right\}\mathrm{d}x + \left\{4(1-x) - 6x(1-x)\right\}(-\mathrm{d}x)\right]$$

$$= \int_1^0 \left[3x^2 - 8(1-x)^2 - 4(1-x) + 6\left(x - x^2\right)\right]\mathrm{d}x$$

$$= \left|x^3 - \frac{8(1-x)^3}{-3} - 4\frac{(1-x)^2}{-2} + 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\right|_1^0$$

$$= \left[\left(0 + \frac{8}{3} + 2 + 0\right) - \left\{1 + 6\left(\frac{1}{2} - \frac{1}{3}\right)\right\}\right]$$

$$= \frac{8}{3}$$

(c) Along BO: $x = 0, \mathrm{d}x = 0$ and y varies from 1 to 0.

$$\int_{BO} (M\mathrm{d}x + N\mathrm{d}y) = \int_{BO} \left(3x^2 - 8y^2\right)\mathrm{d}x + (4y - 6xy)\,\mathrm{d}y$$

$$= \int_1^0 4y\,\mathrm{d}y$$

$$= 2\left|y^2\right|_1^0$$

$$= -2$$

Substituting in Eq. (1),

$$\oint_C (M\mathrm{d}x + N\mathrm{d}y) = 1 + \frac{8}{3} - 2 = \frac{5}{3} \qquad \cdots(2)$$

(iv) Let R be the region bounded by the triangle OAB. Along the vertical strip PQ, y varies from 0 to $(1-x)$ and in the region R, x varies from 0 to 1.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{1-x} \left[-6y - (-16y)\right]\mathrm{d}y\mathrm{d}x$$

$$= \int_0^1 \int_0^{1-x} 10y\,\mathrm{d}y\mathrm{d}x$$

$$= 10\int_0^1 \left|\frac{y^2}{2}\right|_0^{1-x} \mathrm{d}x$$

$$= 5\int_0^1 (1-x)^2\,\mathrm{d}x$$

$$= 5\left|\frac{(1-x)^3}{-3}\right|_0^1$$

$$= -\frac{5}{3}(0-1)$$

$$= \frac{5}{3} \qquad \cdots(3)$$

From Eqs (2) and (3),

$$\oint_C (M\mathrm{d}x + N\mathrm{d}y) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \frac{5}{3}$$

Hence, Green's theorem is verified.

Example 15

Verify Green's theorem for $\oint_C [(y-\sin x)dx + \cos x\, dy]$ *where C is the plane triangle enclosed by the lines* $y=0,\ x=\frac{\pi}{2},\ y=\frac{2x}{\pi}$.

Solution

(i) The point of intersection of the lines $y=\frac{2x}{\pi}$ and $x=\frac{\pi}{2}$ is obtained as $y=\frac{2}{\pi}\cdot\frac{\pi}{2}=1$.

Hence $B\left(\frac{\pi}{2},1\right)$ is the point of intersection.

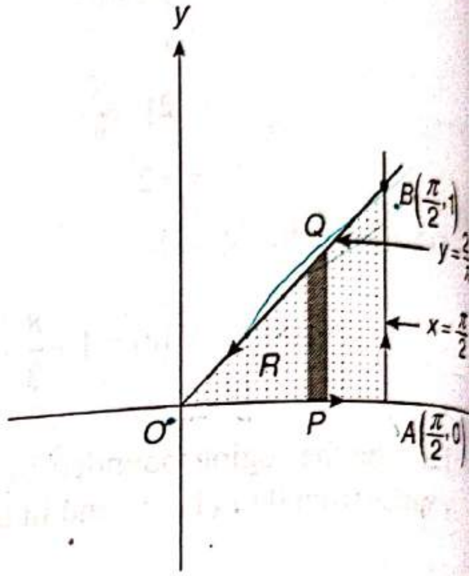

Fig. 1.19

(ii) $M=y-\sin x,\ N=\cos x$

$$\frac{\partial M}{\partial y}=1, \qquad \frac{\partial N}{\partial x}=-\sin x$$

(iii) $\oint_C (M\,dx+N\,dy)$

$$=\int_{OA}(M\,dx+N\,dy)+\int_{AB}(M\,dx+N\,dy)+\int_{BO}(M\,dx+N\,dy)$$

(a) Along OA : $y=0$, $dy=0$ and x varies from 0 to $\frac{\pi}{2}$.

$$\int_{OA}(M\,dx+N\,dy)=\int_{OA}[(y-\sin x)dx+\cos x\,dy]$$
$$=\int_0^{\frac{\pi}{2}}(-\sin x)\,dx$$
$$=|\cos x|_0^{\frac{\pi}{2}}$$
$$=-1$$

(b) Along AB : $x=\frac{\pi}{2}$, $dx=0$ and y varies from 0 to 1.

$$\int_{AB}(M\,dx+N\,dy)=\int_{AB}[(y-\sin x)dx+\cos x\,dy]$$
$$=\int_0^1\cos\frac{\pi}{2}\,dy$$
$$=0$$

(c) Along BO : $y=\frac{2x}{\pi}$, $dy=\frac{2}{\pi}dx$ and x varies from $\frac{\pi}{2}$ to 0.

$$\int_{BO}(M\,dx+N\,dy)=\int_{BO}[(y-\sin x)dx+\cos x dy]$$
$$=\int_{\frac{\pi}{2}}^{0}\left[\left(\frac{2x}{\pi}-\sin x\right)dx+\cos x\cdot\frac{2}{\pi}dx\right]$$

$$= \left| \frac{2}{\pi} \cdot \frac{x^2}{2} + \cos x + \frac{2}{\pi} \sin x \right|_{\frac{\pi}{2}}^{0}$$

$$= \cos 0 - \frac{1}{\pi} \cdot \frac{\pi^2}{4} - \cos\frac{\pi}{2} - \frac{2}{\pi}\sin\frac{\pi}{2}$$

$$= 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

Substituting in Eq. (1),

$$\oint_C (M\,dx + N\,dy) = -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\left(\frac{\pi^2 + 8}{4\pi}\right) \qquad \dots (2)$$

(iv) Let R be the region bounded by the triangle OAB.

Along the vertical strip PQ, y varies from 0 to $\frac{2x}{\pi}$ and in the region R, x varies from 0 to $\frac{\pi}{2}$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^{\frac{2x}{\pi}} (-\sin x - 1)\,dx\,dy$$

$$= \int_0^{\frac{\pi}{2}} \left| -y \sin x - y \right|_0^{\frac{2x}{\pi}} dx$$

$$= \int_0^{\frac{\pi}{2}} \left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right) dx$$

$$= -\frac{2}{\pi}\left| x(-\cos x) - (-\sin x) + \frac{x^2}{2} \right|_0^{\frac{\pi}{2}}$$

$$= -\frac{2}{\pi}\left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + \frac{\pi^2}{8} - 0\right)$$

$$= -\frac{2}{\pi}\left(1 + \frac{\pi^2}{8}\right)$$

$$= -\left(\frac{\pi^2 + 8}{4\pi}\right) \qquad \dots (3)$$

From Eqs (2) and (3),

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = -\left(\frac{\pi^2 + 8}{4\pi}\right)$$

Hence, Green's theorem is verified.

Example 16

Verify Green's theorem for $\oint_C \left[(x^2-2xy)dx+(x^2y+3)dy\right]$ *where C is the boundary of the region bounded by the parabola* $y=x^2$ *and the line* $y=x$.

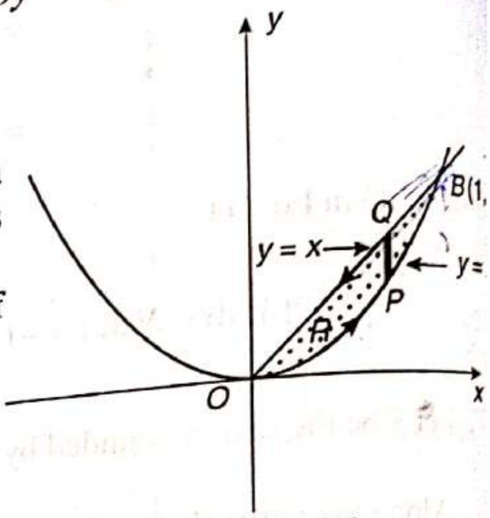

Fig. 1.20

Solution

(i) The points of intersection of the parabola $y = x^2$ and the line $y = x$ are obtained as $x = x^2$, $x = 0, 1$ and $y = 0, 1$.
Hence, O (0,0) and B (1, 1) are the points of intersection.

(ii) $M = x^2 - 2xy, \quad N = x^2y+3$

$$\frac{\partial M}{\partial y} = -2x, \qquad \frac{\partial N}{\partial x} = 2xy$$

(iii) $$\oint_C (M\,dx + N\,dy) = \int_{OAB} (M\,dx+N\,dy) + \int_{BO} (M\,dx + N\,dy) \quad ...(1)$$

(a) Along $OPB: y = x^2$, $dy = 2x\,dx$ and x varies from 0 to 1.

$$\int_{OPB} (M\,dx+N\,dy) = \int_{OPB}\left[(x^2-2xy)dx+(x^2y+3)dy\right]$$
$$= \int_0^1\left[(x^2-2x\cdot x^2)dx+(x^2\cdot x^2+3)2x\,dx\right]$$
$$= \int_0^1 (x^2-2x^3+2x^5+6x)\,dx$$
$$= \left|\frac{x^3}{3}-\frac{2x^4}{4}+\frac{2x^6}{6}+\frac{6x^2}{2}\right|_0^1$$
$$= \frac{1}{3}-\frac{1}{2}+\frac{1}{3}+3$$
$$= \frac{19}{6}$$

(b) Along $BO: y = x$, $dy = dx$ and x varies from 0 to 1.

$$\int_{BO} (M\,dx+N\,dy) = \int_{BO}\left[(x^2-2\,xy)dx+(x^2y+3)dy\right]$$
$$= \int_1^0\left[(x^2-2x^2)dx+(x^3+3)dx\right]$$
$$= \left|-\frac{x^3}{3}+\frac{x^4}{4}+3x\right|_1^0$$
$$= \frac{1}{3}-\frac{1}{4}-3$$
$$= -\frac{35}{12}$$

Substituting in Eq. (1),

$$\oint_C (M\,dx + N\,dy) = \frac{19}{6} - \frac{35}{12} = \frac{1}{4} \quad \text{... (2)}$$

(iv) Let R be the region bounded by the line $y = x$ and the parabola $y = x^2$.
Along the vertical strip PQ, y varies from x^2 to x and in the region R, x varies from 0 to 1.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_{x^2}^x (2xy + 2x)\,dy\,dx$$

$$= \int_0^1 \left| xy^2 + 2xy \right|_{x^2}^x dx$$

$$= \int_0^1 (x^3 + 2x^2 - x^5 - 2x^3)\,dx$$

$$= \left| \frac{-x^4}{4} + \frac{2x^3}{3} - \frac{x^6}{6} \right|_0^1$$

$$= -\frac{1}{4} + \frac{2}{3} - \frac{1}{6}$$

$$= \frac{1}{4} \quad \text{... (3)}$$

From Eqs. (2) and (3),

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{1}{4}$$

Hence, Green's theorem is verified.

Example 17

Verify Green's theorem for $\oint_C \left[(x - y)\,dx + 3xy\,dy\right]$, *where* C *is the boundary of the region bounded by the parabolas* $x^2 = 4y$ *and* $y^2 = 4x$.

Solution

(i) The points of intersection of the parabolas $x^2 = 4y$ and $y^2 = 4x$ are obtained as

$$\left(\frac{y^2}{4}\right)^2 = 4y, \; y(y^3 - 64) = 0$$

$$y = 0, 4$$

$$x = 0, 4$$

Hence, O (0,0) and C' (4, 4) are the points of intersection.

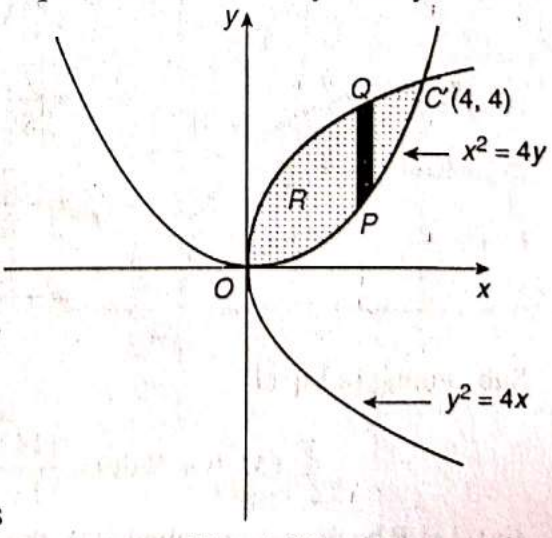

Fig. 1.21

(ii) $M = x - y, \qquad N = 3xy$

$\frac{\partial M}{\partial y} = -1, \qquad \frac{\partial N}{\partial x} = 3y$

(iii) $\oint_C (M\,dx + N dy) = \int_{OAC'} (M\,dx + N\,dy) + \int_{C'BO} (M\,dx + N\,dy)$ ...(1)

(a) Along OPC' : $x^2 = 4y, y = \frac{x^2}{4}$, $dy = \frac{x}{2}dx$ and x varies from 0 to 4.

$$\int_{OPC'} (M\,dx + N\,dy) = \int_{OPC'} [(x - y)dx + (3xy)dy]$$

$$= \int_0^4 \left(x - \frac{x^2}{4}\right)dx + \left(3x \cdot \frac{x^2}{4}\right)\frac{x}{2}dx$$

$$= \int_0^4 \left(x - \frac{x^2}{4} + \frac{3}{8}x^4\right)dx$$

$$= \left|\frac{x^2}{2} - \frac{x^3}{12} + \frac{3}{8} \cdot \frac{x^5}{5}\right|_0^4$$

$$= 8 - \frac{16}{3} + \frac{384}{5}$$

$$= \frac{1192}{15}$$

(b) Along $C'QO$: $y^2 = 4x$, $x = \frac{y^2}{4}$, $dx = \frac{y}{2}dy$ and y varies from 4 to 0.

$$\int_{C'QO} (M\,dx + N\,dy) = \int_{C'QO} [(x - y)dx + 3xy\,dy]$$

$$= \int_4^0 \left(\frac{y^2}{4} - y\right)\frac{y}{2}dy + \left(3 \cdot \frac{y^2}{4} \cdot y\right)dy$$

$$= \int_4^0 \left(\frac{7y^3}{8} - \frac{y^2}{2}\right)dy$$

$$= \left|\frac{7}{8} \cdot \frac{y^4}{4} - \frac{1}{2} \cdot \frac{y^3}{3}\right|_4^0$$

$$= -\frac{7}{8} \cdot 64 + \frac{1}{2} \cdot \frac{64}{3}$$

$$= -\frac{136}{3}$$

Substituting in Eq. (1),

$$\oint_C (M\,dx + N\,dy) = \frac{1192}{15} - \frac{136}{3} = \frac{512}{15} \qquad \text{...(2)}$$

(iv) Let R be the region bounded by the parabolas $x^2 = 4y$ and $y^2 = 4x$.

Along the vertical strip PQ, y varies from $\frac{x^2}{4}$ to $2\sqrt{x}$ and in the region R, x varies from 0 to 4.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} (3y+1)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^4 \left| \frac{3y^2}{2} + y \right|_{\frac{x^2}{4}}^{2\sqrt{x}} \mathrm{d}x$$

$$= \int_0^4 \left(6x + 2\sqrt{x} - \frac{3}{32}x^4 - \frac{x^2}{4}\right) \mathrm{d}x$$

$$= \left| 3x^2 + \frac{4}{3}x^{\frac{3}{2}} - \frac{3}{32}\cdot\frac{x^5}{5} - \frac{1}{4}\cdot\frac{x^3}{3} \right|_0^4$$

$$= 48 + \frac{32}{3} - \frac{96}{5} - \frac{16}{3}$$

$$= \frac{512}{15} \qquad \text{... (3)}$$

From Eqs. (2) and (3),

$$\oint_C (M\,\mathrm{d}x + N\,\mathrm{d}y) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \frac{512}{15}$$

Hence, Green's theorem is verified.

Example 18

Verify Green's theorem for $\int_C \left(\frac{1}{y}\mathrm{d}x + \frac{1}{x}\mathrm{d}y\right)$ *where C is the boundary of the region bounded by the parabola* $y = \sqrt{x}$ *and the lines* $x = 1, x = 4, y = 1$.

Solution

(i) The point of intersection of the

(a) parabola $y = \sqrt{x}$ and the line $x = 1$ is obtained as

$$y = \sqrt{1} = 1$$

Hence, A (1, 1) is the point of intersection.

(b) parabola $y = \sqrt{x}$ and the line $x = 4$ is obtained as

$$y = \sqrt{4} = 2$$

Hence, D (4, 2) is the point of intersection.

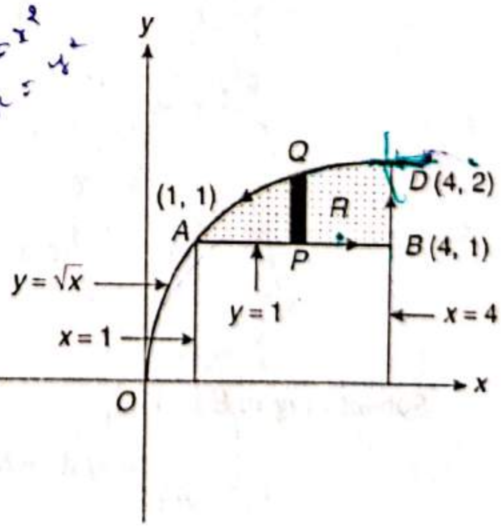

Fig. 1.22

(ii) $M = \frac{1}{y}$, $\qquad N = \frac{1}{x}$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2}, \qquad \frac{\partial N}{\partial x} = -\frac{1}{x^2}$$

(iii) $$\oint_C (M\,dx + N\,dy) = \int_{AB} (M\,dx + N\,dy) + \int_{BD} (M\,dx + N\,dy) + \int_{DQA} (M\,dx + N\,dy) \quad \dots(1)$$

(a) Along AB: $y = 1$, $dy = 0$ and x varies from 1 to 4.

$$\int_{AB} (M\,dx + N\,dy) = \int_{AB} \left(\frac{1}{y} dx + \frac{1}{x} dy \right)$$
$$= \int_1^4 dx$$
$$= |x|_1^4$$
$$= 3$$

(b) Along BD: $x = 4$, $dx = 0$ and y varies from 1 to 2.

$$\int_{BD} (M\,dx + N\,dy) = \int_{BD} \left(\frac{1}{y} dx + \frac{1}{x} dy \right)$$
$$= \int_1^2 \frac{1}{4} dy$$
$$= \frac{1}{4} |y|_1^2$$
$$= \frac{1}{4}$$

(c) Along DQA: $y = \sqrt{x}$, $dy = \frac{1}{2\sqrt{x}} dx$ and x varies from 4 to 1.

$$\int_{DQA} (M\,dx + N\,dy) = \int_{DQA} \left(\frac{1}{y} dx + \frac{1}{x} dy \right)$$
$$= \int_4^1 \left(\frac{1}{\sqrt{x}} dx + \frac{1}{x} \cdot \frac{1}{2\sqrt{x}} dx \right)$$
$$= \left| 2\sqrt{x} - \frac{1}{\sqrt{x}} \right|_4^1$$
$$= 2 - 1 - 4 + \frac{1}{2}$$
$$= -\frac{5}{2}$$

Substituting in Eq. (1),

$$\oint_C (M\,dx + N\,dy) = 3 + \frac{1}{4} - \frac{5}{2} = \frac{3}{4} \quad \dots(2)$$

(iv) Let R be the region bounded by the parabola $y = \sqrt{x}$ and the lines $x = 1, x = 4, y = 1$. Along the vertical strip PQ, y varies from 1 to $\sqrt{x}$ and in the region R, x varies from 1 to 4.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_1^4 \int_1^{\sqrt{x}} \left(-\frac{1}{x^2} + \frac{1}{y^2}\right) dx\, dy$$

$$= \int_1^4 \left| -\frac{1}{x^2} \cdot y - \frac{1}{y} \right|_1^{\sqrt{x}} dx$$

$$= \int_1^4 \left(-x^{-\frac{3}{2}} - x^{-\frac{1}{2}} + \frac{1}{x^2} + 1\right) dx$$

$$= \left| 2x^{-\frac{1}{2}} - 2x^{\frac{1}{2}} - \frac{1}{x} + x \right|_1^4$$

$$= 1 - 4 - \frac{1}{4} + 4 - 2 + 2 + 1 - 1$$

$$= \frac{3}{4} \qquad \text{... (3)}$$

From Eqs (2) and (3),

$$\oint (M\, dx + N\, dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \frac{3}{4}$$

Hence, Green's theorem is verified.

Example 19

Verify Green's theorem in a plane for the integral $\int_C \left[(x - 2y)\, dx + x\, dy\right]$ *taken around the circle* $x^2 + y^2 = 4$.

Solution

(i) $M = x - 2y, \quad N = x$

$$\frac{\partial M}{\partial y} = -2, \qquad \frac{\partial N}{\partial x} = 1$$

(ii) $$\oint_C (M\, dx + N\, dy) = \oint_C \left[(x - 2y)\, dx + x\, dy\right] \qquad \cdots (1)$$

where C is the circle $x^2 + y^2 = 4$.

Parametric equation of the circle is

$x = 2\cos\theta, \qquad y = 2\sin\theta$

$dx = -2\sin\theta\, d\theta, \quad dy = 2\cos\theta\, d\theta$

For the given circle, θ varies from 0 to 2π.

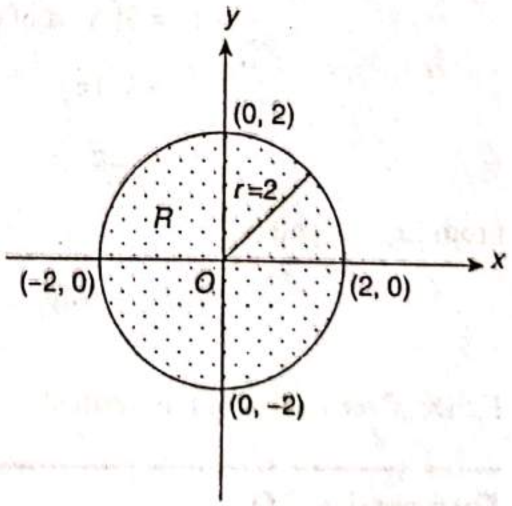

Fig. 1.23

Substituting in Eq. (1),

$$\oint_C (M\mathrm{d}x + N\mathrm{d}y) = \int_0^{2\pi} \left[(2\cos\theta - 4\sin\theta)(-2\sin\theta\mathrm{d}\theta) + 2\cos\theta \cdot 2\cos\theta\mathrm{d}\theta\right]$$

$$= \int_0^{2\pi} \left(-4\sin\theta\cos\theta + 8\sin^2\theta + 4\cos^2\theta\right)\mathrm{d}\theta$$

$$= \int_0^{2\pi} \left[-2\sin 2\theta + 4\sin^2\theta + 4\left(\sin^2\theta + \cos^2\theta\right)\right]\mathrm{d}\theta$$

$$= \int_0^{2\pi} \left[-2\sin 2\theta + 2(1 - \cos 2\theta) + 4\right]\mathrm{d}\theta$$

$$= \int_0^{2\pi} \left[-2\sin 2\theta - 2\cos 2\theta + 6\right]\mathrm{d}\theta$$

$$= \left|\frac{2\cos 2\theta}{2} - \frac{2\sin 2\theta}{2} + 6\theta\right|_0^{2\pi}$$

$$= \left[\cos 4\pi - \sin 4\pi + 6(2\pi)\right] - \left[\cos 0 - \sin 0 + 0\right]$$

$$= 1 + 12\pi - 1$$

$$= 12\pi \qquad \text{...(2)}$$

(iii) Let R be the region bounded by the circle $x^2 + y^2 = 4$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \iint_R 3\,\mathrm{d}x\mathrm{d}y$$

$$= 3\iint_R \mathrm{d}x\mathrm{d}y$$

$$= 3(\text{Area of circle})$$

$$= 3(4\pi) \qquad \left[\because \text{Area of circle} = \pi(2)^2 = 4\pi\right]$$

$$= 12\pi \qquad \text{...(3)}$$

From Eqs (2) and (3),

$$\oint_C (M\mathrm{d}x + N\mathrm{d}y) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x\mathrm{d}y = 12\pi$$

Hence, Green's theorem is verified.

Example 20

Verify Green's theorem for $\oint_C (2xy\,\mathrm{d}x - y^2\mathrm{d}y)$ *where C is the boundary of the region bounded by the ellipse* $3x^2 + 4y^2 = 12$.

Solution

(i) $M = 2xy, \qquad N = -y^2$

$$\frac{\partial M}{\partial y} = 2x, \qquad \frac{\partial N}{\partial x} = 0$$

(ii) $\oint_C (M\,dx + N\,dy) = \oint_C (2xy\,dx - y^2\,dy)$ ...(1)

where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$.

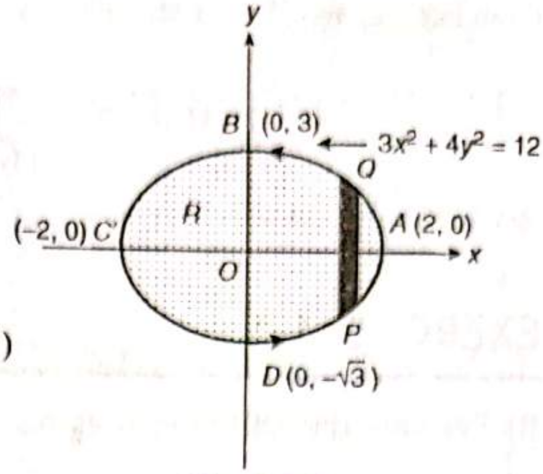

Fig. 1.24

Parametric equation of the ellipse is

$$x = 2\cos\theta, \qquad y = \sqrt{3}\sin\theta$$

$$dx = -2\sin\theta\, d\theta, \qquad dy = \sqrt{3}\cos\theta\, d\theta$$

For the given ellipse, θ varies from 0 to 2π.

Substituting in Eq. (1),

$$\oint_C (M\,dx + N\,dy) = \int_0^{2\pi} \left[(2\cdot 2\cos\theta\cdot\sqrt{3}\sin\theta)(-2\sin\theta\, d\theta) - 3\sin^2\theta\cdot\sqrt{3}\cos\theta\, d\theta\right]$$

$$= \int_0^{2\pi} \left(-11\sqrt{3}\cos\theta\sin^2\theta\right) d\theta$$

$$= -11\sqrt{3}\cdot 2\int_0^{\pi} \cos\theta\sin^2\theta\, d\theta$$

$$= 0 \quad ...(2) \qquad \left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x) \atop = 0, \text{ if } f(2a-x) = -f(x)\right]$$

(iii) Let R be the region bounded by the ellipse, $\frac{x^2}{4} + \frac{y^2}{3} = 1$.

Along the vertical strip PQ, y varies from $-\sqrt{3 - \frac{3x^2}{4}}$ to $\sqrt{3 - \frac{3x^2}{4}}$ and in the region R, x varies from -2 to 2.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{-2}^{2}\int_{-\sqrt{3-\frac{3x^2}{4}}}^{\sqrt{3-\frac{3x^2}{4}}} (0 - 2x)\,dy\,dx$$

$$= \int_{-2}^{2} -2x\,|y|_{-\sqrt{3-\frac{3x^2}{4}}}^{\sqrt{3-\frac{3x^2}{4}}} dx$$

$$= -4\int_{-2}^{2} x\sqrt{3 - \frac{3x^2}{4}}\, dx$$

$$= 0 \quad ...(3) \qquad \left[\because \int_{-a}^{a} f(x)dx = 0, \text{ if } f(-x) = -f(x)\right]$$

From Eqs. (2) and (3),

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 0$$

Hence, Green's theorem is verified.

EXERCISE 1.5

(I) Evaluate the following integrals using Green's theorem:

1. $\oint_C e^{-x}(\cos y\,dx - \sin y\,dy)$, where C is the boundary of the region bounded by the rectangle with vertices $(0,0), (\pi,0), \left(\pi, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$.

   $\left[\text{Ans.: } 2(1-e^{-\pi})\right]$

2. $\oint_C \left[(x^2+y^2)\,dx + (5x^2-3y)\,dy\right]$, where C is the boundary of the region bounded by the parabola $x^2 = 4y$ and the line $y = 4$.

   $\left[\text{Ans.: } -\frac{512}{5}\right]$

3. $\oint_C \left[(y^3 - xy)\,dx + (xy + 3xy^2)\,dy\right]$, where C is the boundary of the region in the first quadrant bounded by the y–axis and the parabolas $y = 1 - x^2, y = x^2$.

   $\left[\text{Ans.: } \left(\frac{1}{8} + \frac{\sqrt{2}}{6}\right)\right]$

4. $\oint_C (xy\,dx + x^3\,dy)$, where C is the boundary of the region bounded by the x–axis and the circle $y = \sqrt{4 - x^2}$.

   [Ans.: 6p]

5. $\oint_C e^x(\sin y\,dx + \cos y\,dy)$, where C is the boundary of the region bounded by the ellipse $4(x+1)^2 + 9(y-3)^2 = 36$.

   [Ans.: 0]

(II) Verify Green's theorem in plane for the following:

1. $\oint_C \left[(x^2 - 2xy)\,dx + (x^2y + 3)\,dy\right]$, where C is the boundary of the region bounded by the parabola $y^2 = 8x$ and the line $x = 2$.

$$\left[\textbf{Ans.: } \frac{128}{5}\right]$$

2. $\oint_C \left[(xy - x^2)\,dx + x^2y\,dy\right]$, where C is the boundary of the triangle formed by the lines $y = 0$, $x = 1$ and $y = x$.

$$\left[\textbf{Ans.: } -\frac{1}{12}\right]$$

3. $\oint_C \left[(3x^2 - 8y^2)\,dy + (4y - 6xy)\,dy\right]$, where C is the boundary of the region bounded by $y = x^2$ and $y = \sqrt{x}$.

$$\left[\textbf{Ans.: } \frac{3}{2}\right]$$

4. $\oint_C (e^{-x}\sin y\,dx + e^{-x}\cos y\,dy)$, where C is the boundary of the region bounded by the square with vertices $(0,0), \left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(0, \frac{\pi}{2}\right)$.

$$\left[\textbf{Ans.: } 2\left(e^{-\frac{\pi}{2}} - 1\right)\right]$$

5. $\oint_C (xy^2 - 2xy)\,dx + (x^2y + 3)\,dy$, where C is the boundary of the region bounded by the rectangle with vertices $(-1, 0)$, $(1, 0)$, $(1, 1)$ and $(-1, 1)$.

[**Ans.:** 0]

6. $\oint_C (x^3dy - y^3dx)$, where C is the circle $x^2 + y^2 = 4$.

[**Ans.:** 48π]

7. $\oint_C \left[(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy\right.$, where C is the boundary of the region bounded by the x–axis and the circle $y = \sqrt{1 - x^2}$.

$$\left[\textbf{Ans.: } \frac{4}{3}\right]$$

1.11 SURFACE INTEGRALS

The surface integral over a curved surface S is the generalisation of a double integral over a plane region R. Let $\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a continuous vector field defined over a two-sided surface S. Divide S into a finite number of subsurfaces $S_1, S_2, ..., S_m$ with surface areas $\delta S_1, \delta S_2, ..., \delta S_m$. Let δS_r be the surface area of S_r and $\hat{n}_r$ be the unit vector at some point P_r (in S_r) in the direction of the outward normal to S_r.

If we increase the number of subsurfaces then the surface area δS_r of each subsurface will decrease. Thus, as $m \to \infty, \delta S_r \to 0$

Then

$$\lim_{m\to\infty} \sum_{r=1}^{m} \overline{F}(P_r)\cdot\hat{n}_r\delta S_r = \iint_S \overline{F}\cdot\hat{n}\, ds$$

This is called surface integral of $\overline{F}$ over the surface S.

The surface integral can also be written as

$$\iint_S \overline{F}\cdot d\overline{S}, \text{ where } d\overline{S} = \hat{n}\, dS$$

If equation of the surface S is $\phi(x, y, z) = 0$ then $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$.

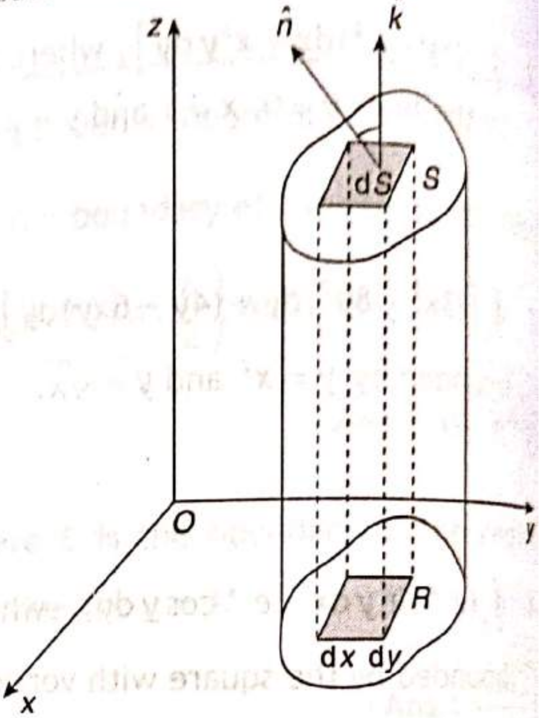

Fig. 1.25

1.11.1 Flux

If $\overline{F}$ represents velocity of the fluid at any point P on a closed surface S then surface integral $\iint_S \overline{F}\cdot\hat{n}\, dS$ represents the flux of $\overline{F}$ over S, i.e., volume of the fluid flowing out from S per unit time.

Note: If $\iint_S \overline{F}\cdot\hat{n}\, dS = 0$ then $\overline{F}$ is called a solenoidal vector field.

1.11.2 Evaluation of Surface Integral

A surface integral is evaluated by expressing it as a double integral over the region R. The region R is the orthogonal projection of S on one of the coordinate planes (xy, yz or zx). Let R be the orthogonal projection of S on the xy-plane and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of $\hat{n}$.

$$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

$$dx\,dy = \text{Projection of } dS \text{ on } xy\text{-plane}$$

$$= \cos\gamma\, dS$$

$$dS = \frac{dx\,dy}{\cos\gamma} = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

Hence, $$\iint_S \bar{F}\cdot\hat{n}\,\mathrm{d}S = \iint_R \bar{F}\cdot\hat{n}\frac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|}$$

Similarly, taking projection on yz and zx-plane,

$$\iint_S \bar{F}\cdot\hat{n}\,\mathrm{d}S = \iint_R \bar{F}\cdot\hat{n}\frac{\mathrm{d}y\,\mathrm{d}z}{|\hat{n}\cdot\hat{i}|} \quad \text{and} \quad \iint_S \bar{F}\cdot\hat{n}\,\mathrm{d}S = \iint_R \bar{F}\cdot\hat{n}\frac{\mathrm{d}z\,\mathrm{d}x}{|\hat{n}\cdot\hat{j}|}$$

Component Form of surface integral

$$\iint_S \bar{F}\cdot\hat{n}\,\mathrm{d}S = \iint_S (F_1\hat{i}+F_2\hat{j}+F_3\hat{k})\cdot(\cos\alpha\,\hat{i}+\cos\beta\,\hat{j}+\cos\gamma\,\hat{k})\,\mathrm{d}S$$
$$= \iint_S (F_1\cos\alpha\,\mathrm{d}S + F_2\cos\beta\,\mathrm{d}S + F_3\cos\gamma\,\mathrm{d}S)$$
$$= \iint_S F_1\,\mathrm{d}y\,\mathrm{d}z + F_2\,\mathrm{d}z\,\mathrm{d}x + F_3\,\mathrm{d}x\,\mathrm{d}y)$$

Example 1

Evaluate $\iint_S \bar{F}\cdot\hat{n}\,\mathrm{d}S$, *where* $\bar{F} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ *and* S *is the part of the plane* $2x + 3y + 6z = 12$ *in the first octant.*

Solution

(i) The given surface is the plane $2x + 3y + 6z = 12$ in the first octant.

Let $\phi = 2x + 3y + 6z$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{4+9+36}} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}$$

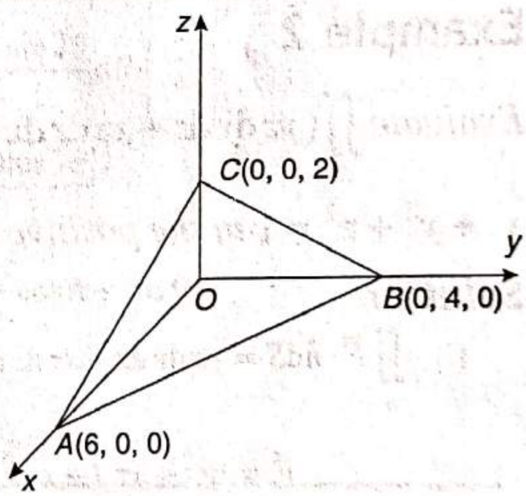

Fig. 1.26

(ii) Let R be the projection of the plane $2x + 3y + 6z = 12$ (in the first octant) on the xy-plane, which is a triangle OAB bounded by the lines $y = 0$, $x = 0$ and $2x + 3y = 12$.

(iii) $$\mathrm{d}S = \frac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|} = \frac{7}{6}\mathrm{d}x\,\mathrm{d}y$$

(iv) Along the vertical strip PQ, y varies from 0 to $\frac{12-2x}{3}$ and in the region R, x varies from 0 to 6.

$$\iint_S \bar{F}\cdot\hat{n}\mathrm{d}S = \iint_R (18z\hat{i} - 12\hat{j} + 3y\hat{k})\cdot\left(\frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}\right)\frac{7}{6}\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{6}\iint_R (36z - 36 + 18y)\,dx\,dy$$

$$= 3\iint_R \left[2\left(\frac{12-2x-3y}{6}\right) - 2 + y\right] dx\,dy \qquad \text{[Substituting } z\text{]}$$

$$= \int_0^6 \int_0^{\frac{12-2x}{3}} (6-2x)\,dy\,dx$$

$$= 2\int_0^6 (3-x)\left|y\right|_0^{\frac{12-2x}{3}} dx$$

$$= 2\int_0^6 (3-x)\frac{(12-2x)}{3}\,dx$$

$$= \frac{4}{3}\int_0^6 (x^2 - 9x + 18)\,dx$$

$$= \frac{4}{3}\left|\frac{x^3}{3} - \frac{9x^2}{2} + 18x\right|_0^6$$

$$= \frac{4}{3}(72 - 162 + 108)$$

$$= 24$$

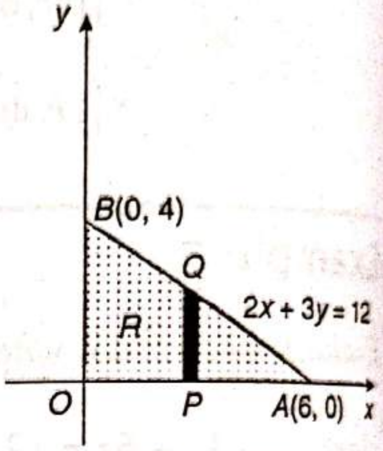

Fig. 1.27

Example 2

Evaluate $\iint_S (yz\,dy\,dz + xz\,dz\,dx + xy\,dx\,dy)$ *over the surface of the sphere* $x^2 + y^2 + z^2 = 1$ *in the positive octant.* **[Winter 2014]**

Solution

(i) $\iint_S \bar{F}\cdot\hat{n}\,dS = yz\,dy\,dz + xz\,dz\,dx + xy\,dx\,dy$

$$\bar{F} = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$$

(ii) The given surface is the sphere $x^2 + y^2 + z^2 = 1$.

Let $\phi = x^2 + y^2 + z^2$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= x\hat{i} + y\hat{j} + z\hat{k} \qquad [\because x^2 + y^2 + z^2 = 1]$$

(iii) Let R be the projection of the sphere $x^2 + y^2 + z^2 = 1$ (in the positive octant) on the xy-plane ($z = 0$), which is the part of the circle $x^2 + y^2 = 1$ in the first quadrant.

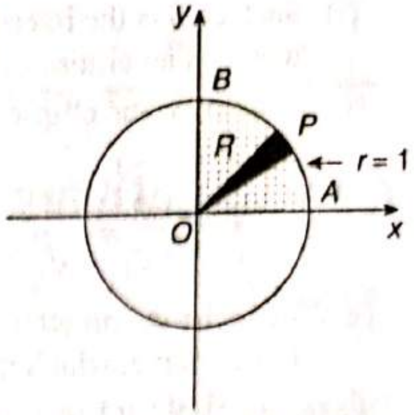

Fig. 1.28

(iv) $dS = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \dfrac{dx\,dy}{z}$

(v) $\iint\limits_S (yz\,dy\,dz + xz\,dz\,dx + xy\,dx\,dy) = \iint\limits_S \bar{F}\cdot\hat{n}\,dS$

$$= \iint\limits_R (yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k})\cdot(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})\frac{dx\,dy}{z}$$

$$= \iint\limits_R (3xyz)\frac{dx\,dy}{z}$$

$$= 3\iint\limits_R xy\,dx\,dy$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, the equation of the circle $x^2 + y^2 = 1$ reduces to $r = 1$ and $dx\,dy = r\,dr\,d\theta$.

Along the radius vector OP, r varies from 0 to 1 and in the first quadrant of the circle, θ varies from 0 to $\dfrac{\pi}{2}$.

$$\iint\limits_S (yz\,dy\,dz + xz\,dz\,dx + xy\,dx\,dy) = 3\int_0^{\frac{\pi}{2}}\int_0^1 r\cos\theta\cdot r\sin\theta\cdot r\,dr\,d\theta$$

$$= 3\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{2}\,d\theta\cdot\left|\frac{r^4}{4}\right|_0^1$$

$$= \frac{3}{8}\left|\frac{-\cos 2\theta}{2}\right|_0^{\frac{\pi}{2}}$$

$$= \frac{3}{16}(-\cos\pi + \cos 0)$$

$$= \frac{3}{8}$$

Example 3

Find the flux of $\bar{F} = \hat{i} - \hat{j} + xyz\,\hat{k}$ through the circular region S obtained by cutting the sphere $x^2 + y^2 + z^2 = a^2$ with a plane $y = x$.

Solution

$$\text{Flux} = \iint\limits_S \bar{F}\cdot\hat{n}\,dS$$

(i) Surface S is the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ with a plane $y = x$, which is an ellipse $2x^2 + z^2 = a^2$.

(ii) Normal to the ellipse $2x^2 + z^2 = a^2$ is also normal to the plane $y = x$.

Let $\phi = x - y$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\hat{i} - \hat{j}}{\sqrt{2}}$$

(iii) Let R be the projection of the surface S on the xz-plane, which is an ellipse $2x^2 + z^2 = a^2$.

(iv) $$dS = \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|} = \sqrt{2}\,dx\,dz$$

(v) $$\iint_S \bar{F}\cdot\hat{n}\,dS = \iint_R (\hat{i} - \hat{j} + xyz\,\hat{k})\cdot\left(\frac{\hat{i} - \hat{j}}{\sqrt{2}}\right)\sqrt{2}\,dx\,dz$$

$$= \iint_R 2\,dx\,dz$$

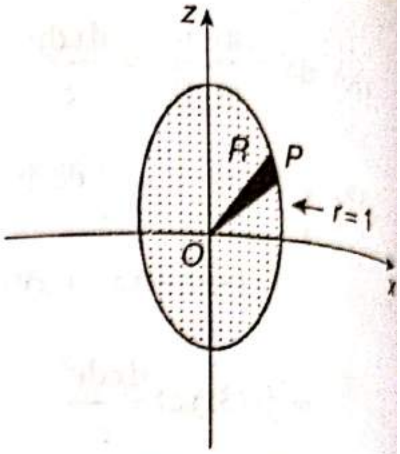

Fig. 1.29

Putting $x = \frac{a}{\sqrt{2}}r\cos\theta$, $z = ar\sin\theta$, the equation of the ellipse $2x^2 + z^2 = a^2$ reduces to $r = 1$ and $dx\,dz = \frac{a^2}{\sqrt{2}}r\,dr\,d\theta$.

Along the radius vector OP, r varies from 0 to 1 and for a complete ellipse, θ varies from 0 to 2π.

$$\iint_S \bar{F}\cdot\hat{n}\,dS = 2\int_0^{2\pi}\int_0^1 \frac{a^2}{\sqrt{2}}r\,dr\,d\theta$$

$$= \frac{2a^2}{\sqrt{2}}\int_0^{2\pi}\left|\frac{r^2}{2}\right|_0^1 d\theta$$

$$= \sqrt{2}a^2\cdot\frac{1}{2}|\theta|_0^{2\pi}$$

$$= \sqrt{2}\pi a^2$$

Aliter

$$\iint_S \bar{F}\cdot\hat{n}\,dS = 2\iint_R dx\,dz$$

$$= 2\left[\text{Area of the ellipse } \frac{x^2}{\left(\frac{a}{\sqrt{2}}\right)^2} + \frac{y^2}{a^2} = 1\right]$$

$$= 2\cdot\pi\frac{a}{\sqrt{2}}\cdot a$$

$$= \sqrt{2}\,\pi a^2$$

Hence, flux $= \sqrt{2}\,\pi a^2$

EXERCISE 1.6

Evaluate the following integrals:

1. $\iint \bar{F} \cdot \hat{n} ds$, where $\bar{F} = (x + y^2)\hat{i} - 2x - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.

[Ans.: 81]

2. $\iint_S \bar{F} \cdot \hat{n} dS$, where $\bar{F} = 2xy\,\hat{i} + yz^2\,\hat{j} + xz\,\hat{k}$ and S is the surface of the parallelepiped $0 \le x \le 1$, $0 \le y \le 2$ and $0 \le z \le 3$.

[Ans.: 33]

3. $\iint_S \bar{F} \cdot \hat{n} dS$, *where* $\bar{F} = x\,\hat{i} + (z^2 - zx)\,\hat{j} - xy\,\hat{k}$ and *S* is the triangular surface with vertices (2, 0, 0), (0, 2, 0) and (0, 0, 4).

$\left[\text{Ans.: } -\frac{22}{3}\right]$

4. $\iint_S \nabla \times \bar{F} \cdot \hat{n} dS$, *where* $\bar{F} = y^2\,\hat{i} + y\,\hat{j} - xz\,\hat{k}$ the sphere $x^2 + y^2 + z^2 = a^2$.

[Ans.: 0]

5. Find the flux of the vector field $\bar{F}$ through the portion of the sphere $x^2 + y^2 + z^2 = 36$ lying between the planes $z = \sqrt{11}$ *and* $z = \sqrt{20}$ *where* $\bar{F} = x\hat{i} + y\hat{j} + z\hat{k}$.

$\left[\text{Ans.: } 72\pi\sqrt{20} - \sqrt{11}\right]$

6. Find the flux of the vector field $\bar{F} = x\hat{i} + y\hat{j} + \sqrt{x^2 + y^2 - 1}\,\hat{k}$ through the outer side of the hyperboloid $z = \sqrt{x^2 + y^2 - 1}$ bounded by the planes $z = 0$ and $z = \sqrt{3}$.

$\left[\text{Ans.: } 2\sqrt{3}\pi\right]$

7. Find the flux of the vector field $\bar{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$ across the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$.

[Ans.: 132]

1.12 STOKES' THEOREM

If S be an open surface bounded by a closed curve C and $\bar{F}$ be a continuous and differentiable vector function then

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS$$

where $\hat{n}$ is the unit outward normal at any point of the surface S.

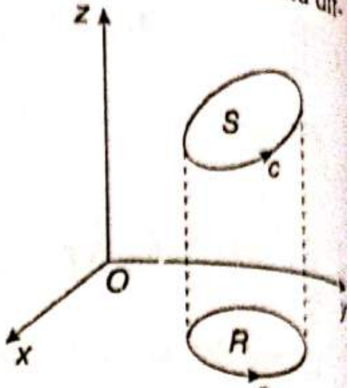

Fig. 1.30

Proof Let $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS = \iint_S \nabla\times(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\cdot\hat{n}\,dS$$

$$= \iint_S (\nabla\times F_1\hat{i})\cdot\hat{n}\,dS + \iint_S (\nabla\times F_2\hat{j})\cdot\hat{n}\,dS + \iint_S (\nabla\times F_3\hat{k})\cdot\hat{n}\,dS \quad \text{...(1.6}$$

Consider, $\iint_S (\nabla\times F_1\hat{i})\cdot\hat{n}\,dS = \iint_S \left[\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times F_1\hat{i}\right]\cdot\hat{n}\,dS$

$$= \iint_S \left(-\hat{k}\frac{\partial F_1}{\partial y} + \hat{j}\frac{\partial F_1}{\partial z}\right)\cdot\hat{n}\,dS$$

$$= \iint \left(\frac{\partial F_1}{\partial z}\hat{j}\cdot\hat{n} - \frac{\partial F_1}{\partial y}\hat{k}\cdot\hat{n}\right)dS \quad \text{...(1.7}$$

Let equation of the surface S be $z = f(x, y)$.

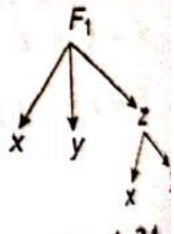

Fig. 1.31

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= x\hat{i} + y\hat{j} + f(x, y)\hat{k}$$

Differentiating partially w.r.t. y,

$$\frac{\partial\bar{r}}{\partial y} = \hat{j} + \frac{\partial f}{\partial y}\hat{k}$$

Taking dot product with $\hat{n}$,

$$\frac{\partial\bar{r}}{\partial y}\cdot\hat{n} = \hat{j}\cdot\hat{n} + \frac{\partial f}{\partial y}\hat{k}\cdot\hat{n} \quad \text{...(1.8}$$

$\frac{\partial\bar{r}}{\partial y}$ is tangential and $\hat{n}$ is normal to the surface S.

$$\frac{\partial\bar{r}}{\partial y}\cdot\hat{n} = 0$$

Substituting in Eq. (1.8),

$$0 = \hat{j}\cdot\hat{n} + \frac{\partial f}{\partial y}\hat{k}\cdot\hat{n}$$

$$\hat{j}\cdot\hat{n} = -\frac{\partial f}{\partial y}\hat{k}\cdot\hat{n} = -\frac{\partial z}{\partial y}\hat{k}\cdot\hat{n} \qquad [\because z = f(x, y)]$$

Substituting in Eq. (1.7),

$$\iint_S (\nabla\times F_1\hat{i})\cdot\hat{n}\,\mathrm{d}S = \iint_S \left[\frac{\partial F_1}{\partial z}\left(-\frac{\partial z}{\partial y}\hat{k}\cdot\hat{n}\right) - \frac{\partial F_1}{\partial y}\hat{k}\cdot\hat{n}\right]\mathrm{d}S$$

$$= -\iint_S \left(\frac{\partial F_1}{\partial z}\cdot\frac{\partial z}{\partial y} + \frac{\partial F_1}{\partial y}\right)\hat{k}\cdot\hat{n}\,\mathrm{d}S \qquad \text{... (1.9)}$$

Equation of the surface is $z = f(x, y)$.

$$F_1(x, y, z) = F_1[x, y, f(x, y)] = G(x, y) \text{ say}$$

Differentiating partially w.r.t. y,

$$\frac{\partial G}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\cdot\frac{\partial z}{\partial y}$$

Substituting in Eq. (1.9),

$$\iint_S (\nabla\times F_1\hat{i})\cdot\hat{n}\,\mathrm{d}S = -\iint_S \frac{\partial G}{\partial y}\hat{k}\cdot\hat{n}\,\mathrm{d}S$$

Let R is the projection of S on the xy-plane and dxdy is the projection of dS on the xy-plane, then $\hat{k}\cdot\hat{n}\,\mathrm{d}S = \mathrm{d}x\,\mathrm{d}y$.

Thus, $\iint_S (\nabla\times F_1\hat{i})\cdot\hat{n}\,\mathrm{d}S = -\iint_R \frac{\partial G}{\partial y}\mathrm{d}x\,\mathrm{d}y$

$$= \oint_{C_1} G\,\mathrm{d}x \qquad \text{[Using Green's theorem]}$$

Since the value of G at each point (x, y) of C_1 is same as the value of F_1 at each point (x, y, z) of C and dx is same for both the curves C_1 and C,

$$\iint_S (\nabla\times F_1\hat{i})\cdot\hat{n}\,\mathrm{d}S = \oint_C F_1\,\mathrm{d}x \qquad \text{... (1.10)}$$

Similarly, by projecting the surface S on to yz and zx planes,

$$\iint_S (\nabla\times F_2\hat{j})\cdot\hat{n}\,\mathrm{d}S = \oint_C F_2\,\mathrm{d}y \qquad \text{... (1.11)}$$

and

$$\iint_S (\nabla\times F_3\hat{k})\cdot\hat{n}\,\mathrm{d}S = \oint_C F_3\,\mathrm{d}z \qquad \text{... (1.12)}$$

Substituting Eqs. (1.10), (1.11) and (1.12) in Eq. (1.6),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \oint_C (F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z) = \oint_C \left(\overline{F}\cdot\mathrm{d}\overline{r}\right)$$

Note: If surfaces S_1 and S_2 have the same bounding curve C then

$$\iint_{S_1} \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_2} \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \oint_C \overline{F}\cdot \mathrm{d}\overline{r}$$

Example 1

If $\overline{F}$ is irrotational and C is a closed curve then find the value of $\int_C \overline{F}\cdot \mathrm{d}\overline{r}$.

Solution

If $\overline{F}$ is irrotational, $\nabla\times\overline{F} = 0$.

By Stokes' theorem,

$$\oint_C \overline{F}\cdot \mathrm{d}\overline{r} = \iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S$$

$$= 0$$

Example 2

Prove that $\int_C \overline{r}\cdot \mathrm{d}\overline{r} = 0$, *where C is the simple closed curve.*

Solution

Let $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

By Stokes' theorem,

$$\oint_C \overline{r}\cdot \mathrm{d}\overline{r} = \iint_S \nabla\times\overline{r}\cdot\hat{n}\,\mathrm{d}S$$

$$\nabla\times\overline{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$= 0$$

$$\therefore \oint_C \overline{r}\cdot \mathrm{d}\overline{r} = 0$$

Example 3

Evaluate $\int_C (yz\,dx + zx\,dy + xy\,dz)$ *where C is the curve* $x^2 + y^2 = 1, z = y^2$.

Solution

By Stokes' theorem,

$$\oint_C F \cdot d\bar{r} = \iint_S \nabla \times \bar{F} \cdot \hat{n}\, dS$$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \hat{i}(x - x) - \hat{j}(y - y) + \hat{k}(z - z)$$

$$= 0$$

$$\therefore \oint_C F \cdot d\bar{r} = \int_C (yz\,dx + zx\,dy + xy\,dz)$$

$$= 0$$

Example 4

Using Stokes' theorem, prove that curl $(grad\,\phi) = 0$.

Solution

By Stokes' theorem,

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S \nabla \times \bar{F} \cdot \hat{n}\, dS$$

Putting $\bar{F} = \nabla\phi$,

$$\iint_S (\nabla \times \nabla\phi) \cdot \hat{n}\, dS = \oint_C \nabla\phi \cdot d\bar{r}$$

$$= \oint_C \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right)$$

$$= \oint_C d\phi$$

$$= 0 \qquad [\because C \text{ is a closed curve}]$$

$$\therefore \nabla \times \nabla\phi = 0$$

$$\text{curl}(\text{grad}\,\phi) = 0$$

Example 5

Evaluate $\oint_C \left(xy\,dx + xy^2 dy\right)$ *by Stokes' theorem where C is the square in the xy-plane with vertices* $(1,0), (-1,0), (0,1), (0,-1)$.

Solution

By Stokes' theorem,

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS$$

$$\text{(i) } \nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}\left(y^2 - x\right)$$

$$= \left(y^2 - x\right)\hat{k}$$

(ii) S is the surface of the square $ABC'D$ which lies in the xy-plane.

$$\hat{n} = \hat{k} \text{ and } dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{|\hat{k}\cdot\hat{k}|} = dx\,dy$$

(iii) Let R be the region bounded by the square $ABC'D$ in xy-plane.

$$\iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS = \iint_R \left(y^2 - x\right)\hat{k}\cdot\hat{k}\,dx\,dy$$

$$= \iint_R \left(y^2 - x\right)dx\,dy$$

where R is the region inside the square $ABC'D$.

(iv) Equation of $AB: \frac{x}{1} + \frac{y}{1} = 1$

$$x + y = 1$$

$$y = -x + 1$$

$$y = -(x-1)$$

Equation of $BC': \frac{x}{-1} + \frac{y}{1} = 1$

$$-x + y = 1$$

$$y = x + 1$$

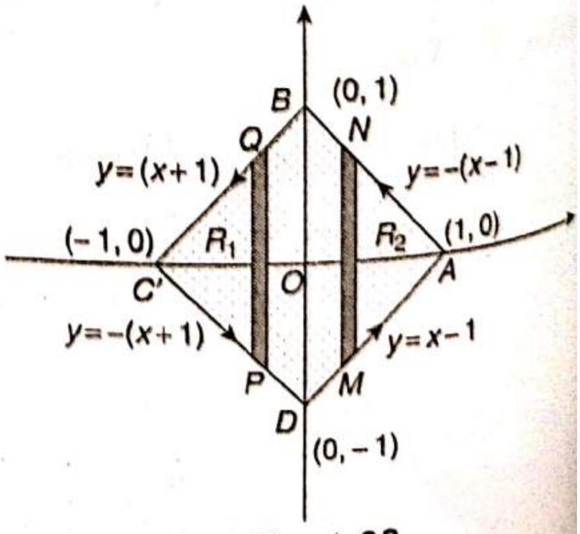

Fig. 1.32

Equation of $C'D$: $\frac{x}{-1}+\frac{y}{-1}=1$

$$x+y=-1$$
$$y=-(x+1)$$

Equation of AD : $\frac{x}{1}+\frac{y}{-1}=1$

$$x-y=1$$
$$y=x-1$$

R is the region bounded by the square $ABC'D$ in xy-plane. Divide the region into two parts. In the left region R_1, draw vertical strip PQ. Along PQ, y varies from $-(x+1)$ to $(x+1)$ and in the region R_1, x varies from -1 to 0.

In the right region R_2, draw vertical strip MN. Along MN, y varies from $(x-1)$ to $-(x-1)$ and in the region R_2, x varies from 0 to 1.

$$\iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS = \int_{-1}^{0}\int_{-(x+1)}^{x+1}\left(y^2-x\right)dy\,dx+\int_{0}^{1}\int_{x-1}^{-(x-1)}\left(y^2-x\right)dy\,dx$$

$$=\int_{-1}^{0}\left|\frac{y^3}{3}-xy\right|_{-(x+1)}^{x+1}dx+\int_{0}^{1}\left|\frac{y^3}{3}-xy\right|_{x-1}^{-(x-1)}dx$$

$$=\int_{-1}^{0}\left[\frac{1}{3}\left\{(x+1)^3-\left(-(x+1)^3\right)\right\}-x\left\{(x+1)-(-(x+1))\right\}dx\right]$$

$$+\int_{0}^{1}\left[\frac{1}{3}\left\{\left(-(x-1)^3\right)-(x-1)^3\right\}-x\left\{-(x-1)-(x-1)\right\}dx\right]$$

$$=\int_{-1}^{0}\left[\frac{1}{3}\left\{(x+1)^3+(x+1)^3\right\}-x\left\{(x+1)+(x+1)\right\}\right]dx$$

$$+\int_{0}^{1}\left[-\frac{1}{3}\left\{(x-1)^3+(x-1)^3\right\}+x\left\{(x-1)+(x+1)\right\}\right]dx$$

$$=\int_{-1}^{0}\left[\frac{2}{3}(x+1)^3-2x(x+1)\right]dx+\int_{0}^{1}\left[-\frac{2}{3}(x-1)^3+2x(x-1)\right]dx$$

$$=\left|\frac{2}{3}\frac{(x+1)^4}{4}-2\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\right|_{-1}^{0}+\left|-\frac{2}{3}\frac{(x-1)^4}{4}+2\left(\frac{x^3}{3}-\frac{x^2}{2}\right)\right|_{0}^{1}$$

$$=\frac{2}{3}\left(\frac{1}{4}\right)-2\left[0-\left\{\frac{1}{3}(-1)^3+\frac{(-1)^2}{2}\right\}\right]-\frac{2}{3}\left[0-\left(\frac{1}{4}\right)\right]+2\left(\frac{1}{3}-\frac{1}{2}\right)$$

$$= \frac{1}{6} - \frac{2}{3} + 1 + \frac{1}{6} + \frac{2}{3} - 1$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

$$\therefore \oint_C \left(xy\,dx + xy^2\,dy\right) = \frac{1}{3}$$

Example 6

Evaluate $\oint_C (\sin z\,dx - \cos x\,dy + \sin y\,dz)$ *by using Stokes' theorem where C is the boundary of the rectangle defined by* $0 \le x \le \pi, 0 \le y \le 1, z = 3$.

Solution

$$\bar{F} \cdot d\bar{r} = \sin z\,dx - \cos x\,dy + \sin y\,dz$$

$$\bar{F} = \sin z\hat{i} - \cos x\hat{j} + \sin y\hat{k}$$

By Stokes' theorem,

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S \nabla \times \bar{F} \cdot \hat{n}\,dS$$

(i) $\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix}$

$$= \hat{i}(\cos y - 0) - \hat{j}(0 - \cos z) + \hat{k}(\sin x - 0)$$

$$= \cos y\,\hat{i} + \cos z\,\hat{j} + \sin x\,\hat{k}$$

(ii) Surface S is the rectangular parallelepiped.

$$\hat{n} = \hat{k} \text{ and } dS = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|} = \frac{dx\,dy}{|\hat{k} \cdot \hat{k}|} = dx\,dy$$

(iii) Let R be the region bounded by the rectangle $OABC'$ in xy-plane. Along the vertical strip PQ, y varies from 0 to 1 and in the region R, x varies from 0 to π.

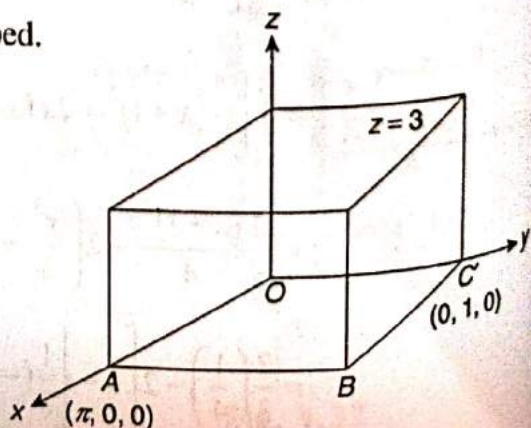

Fig. 1.33

$$\iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS = \iint_R \left(\cos y\,\hat{i} + \cos z\,\hat{j} + \sin x\,\hat{k}\right)\cdot\hat{k}\,dx dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{\pi} \sin x\,dx\,dy$$

$$= \int_0^1 \left|-\cos x\right|_0^{\pi} dy$$

$$= -\int_0^1 (\cos\pi - \cos 0)\,dy$$

$$= -\int_0^1 -2\,dy$$

$$= 2\left|y\right|_0^1$$

$$= 2(1-0)$$

$$= 2$$

$$\therefore \oint_C (\sin z\,dx - \cos x\,dy + \sin y\,dz) = 2$$

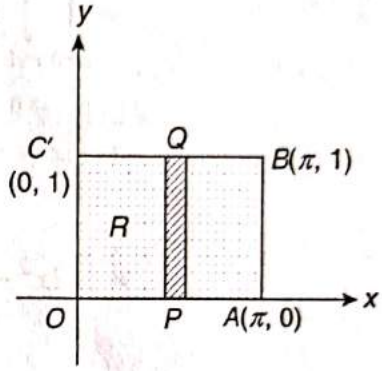

Fig. 1.34

Example 7

Evaluate $\oint_C \bar{F}\cdot d\bar{r}$ *where* $\bar{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ *and C is the boundary of the triangle with vertices at* (0,0,0), (1,0,0) *and* (1,1,0).

Solution

By Stokes' theorem,

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS$$

(i) $$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(-1-0) + \hat{k}(2x-2y)$$

$$= \hat{j} + (2x-2y)\hat{k}$$

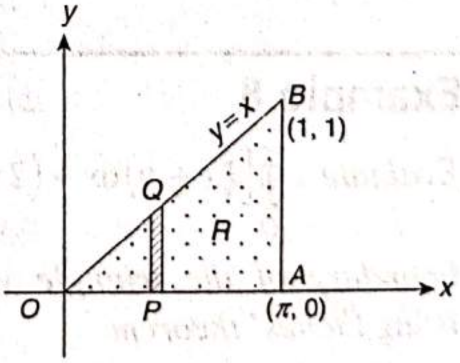

Fig. 1.35

(ii) S is the surface of the triangle OAB

$$\hat{n} = \hat{k} \text{ and } dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{|\hat{k}\cdot\hat{k}|} = dx\,dy$$

(iii) Let R be the region bounded by the triangle OAB in the xy-plane.
Along the vertical strip PQ, y varies from 0 to x and in the region R, x varies from 0 to 1.

$$\iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS = \iint_R\left[\hat{j}+(2x-2y)\hat{k}\right]\cdot\hat{k}\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{x}(2x-2y)\,dy\,dx$$

$$= \int_0^1 \left|2xy-y^2\right|_0^x dx$$

$$= \int_0^1 \left(2x^2-x^2\right)dx$$

$$= \int_0^1 x^2 dx$$

$$= \left|\frac{x^3}{3}\right|_0^1$$

$$= \frac{1}{3}$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \frac{1}{3}$$

Example 8

Evaluate $\oint_C\left[(x+y)dx+(2x-z)dy+(y+z)dz\right]$, *where* C *is the boundary of the triangle with vertices* $(2,0,0),(0,3,0)$ *and* $(0,0,6)$ *using Stokes' theorem.*

Solution

By Stokes' theorem,

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS$$

(i) $$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \hat{i}\left[1-(-1)\right]-\hat{j}\left[0-0\right]+\hat{k}\left[2-1\right]$$

$$= 2\hat{i}+\hat{k}$$

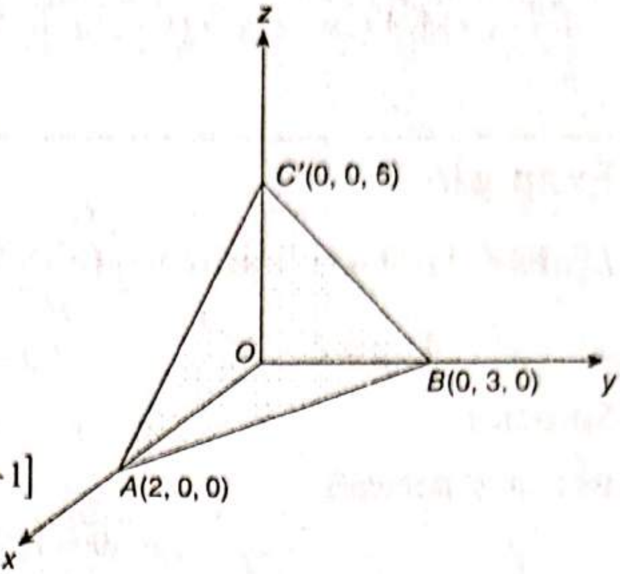

Fig. 1.36

(ii) Equation of the plane ABC' is

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$$

$$3x+2y+z=6$$

Let $\phi = 3x+2y+z$

$$\nabla\phi = 3\hat{i}+2\hat{j}+\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{9+4+1}} = \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$$

$$\hat{n}\cdot\hat{k} = \left(\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}\right)\cdot\hat{k} = \frac{1}{\sqrt{14}}$$

$$dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{\frac{1}{\sqrt{14}}} = \sqrt{14}\,dx\,dy$$

(iii) Let R be the region bounded by the triangle ABC'.

$$\iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS = \iint_R \left(2\hat{i}+\hat{k}\right)\cdot\left(\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}\right)\cdot\sqrt{14}\,dx\,dy$$

$$= \iint_R \frac{6+1}{\sqrt{14}}\sqrt{14}\,dx\,dy$$

$$= 7\iint_R dx\,dy$$

$$= 7\left(\text{Area of }\Delta OAB\right)$$

$$= 7\times\frac{1}{2}\times 2\times 3$$

$$= 21$$

$$\therefore \oint_C \left[(x+y)\mathrm{d}x + (2x-z)\mathrm{d}y + (y+z)\mathrm{d}z\right] = 21$$

Example 9

Evaluate by Stokes' theorem $\oint_C (e^x \mathrm{d}x + 2y\,\mathrm{d}y - \mathrm{d}z)$, *where C is the curve* $x^2 + y^2 = 4,\ z = 2.$

Solution

By Stokes' theorem,

$$\oint_C \overline{F}\cdot \mathrm{d}\overline{r} = \iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S \quad \text{...(1)}$$

$$\overline{F} = e^x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$= 0$$

Substituting in Eq. (1),

$$\oint_C \overline{F}\cdot\mathrm{d}\overline{r} = 0$$

$$\oint_C (e^x\mathrm{d}x + 2y\,\mathrm{d}y - \mathrm{d}z) = 0$$

Example 10

Evaluate $\iint_S (\nabla\times\overline{F})\cdot\hat{n}\,\mathrm{d}S$ *for the vector field* $\overline{F} = (2y^2+3z^2-x^2)\hat{i} + (2z^2+3x^2-y^2)\hat{j} + (2x^2+3y^2-z^2)\hat{k}$ *over the part of the sphere* $x^2+y^2+z^2-2ax+az=0$ *cut off by the plane* $z=0$.

Solution

By Stokes' theorem,

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \oint_C \overline{F}\cdot\mathrm{d}\overline{r} \quad \text{...(1)}$$

(i) $\overline{F}\cdot\mathrm{d}\overline{r} = (2y^2+3z^2-x^2)\mathrm{d}x + (2z^2+3x^2-y^2)\mathrm{d}y + (2x^2+3y^2-z^2)\mathrm{d}z$

(ii) Let C be the boundary of the part of the sphere $x^2+y^2+z^2-2ax+az=0$ cut off by the plane $z = 0$, which is a circle, $x^2+y^2-2ax=0, (x-a)^2+y^2=a^2$.

Parametric equation of the circle

$$x-a=a\cos\theta, \qquad y=a\sin\theta$$

$$dx=-a\sin\theta\,d\theta, \quad dy=a\cos\theta\,d\theta$$

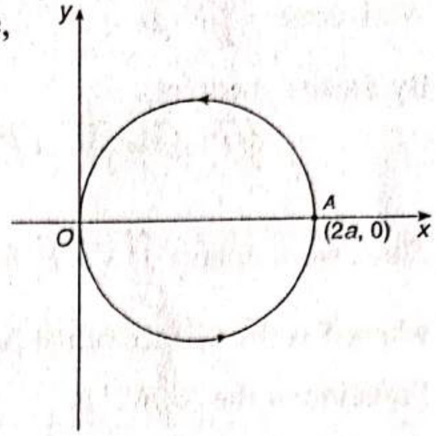

Fig. 1.37

For the complete circle, θ varies from 0 to 2π.

$$\oint_C \overline{F}\cdot d\bar{r} = \oint_C \left[(2y^2-x^2)dx+(3x^2-y^2)dy\right] \qquad [\because z=0, dz=0]$$

$$= \int_0^{2\pi}\left[\left\{2a^2\sin^2\theta-(a+a\cos\theta)^2\right\}(-a\sin\theta\,d\theta) + \left\{3(a+a\cos\theta)^2-a^2\sin^2\theta\right\}(a\cos\theta\,d\theta)\right]$$

$$= a^3\int_0^{2\pi}(-2\sin^3\theta+\sin\theta+\sin\theta\cos^2\theta+2\cos\theta\sin\theta + 3\cos\theta+3\cos^3\theta+6\cos^2\theta-\sin^2\theta\cos\theta)d\theta$$

$$= 2a^3\int_0^{\pi}(3\cos\theta+3\cos^3\theta + 6\cos^2\theta-\sin^2\theta\cos\theta)d\theta \quad \left[\because \int_0^{2a} f(\theta)d\theta=0, \text{if } f(2a-\theta)=-f(\theta) \; = 2\int_0^{a} f(\theta)d\theta, \text{ if } f(2a-\theta)=f(\theta)\right]$$

$$= 4a^3\int_0^{\frac{\pi}{2}} 6\cos^2\theta\,d\theta$$

$$= 24a^3\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 12a^3\left|\theta+\frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{2}}$$

$$= 12a^3\left(\frac{\pi}{2}+\frac{\sin\pi-0}{2}\right)$$

$$= 6\pi a^3$$

From Eq. (1),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,ds = 6\pi a^3$$

Example 11

Using Stokes' theorem, find the work done in moving a particle once around the perimeter of the triangle with vertices at (2, 0, 0), (0, 3, 0) *and* (0, 0, 6) *under the force field* $\overline{F}=(x+y)\hat{i}+(2x-z)\hat{j}+(y+z)\hat{k}$.

Solution

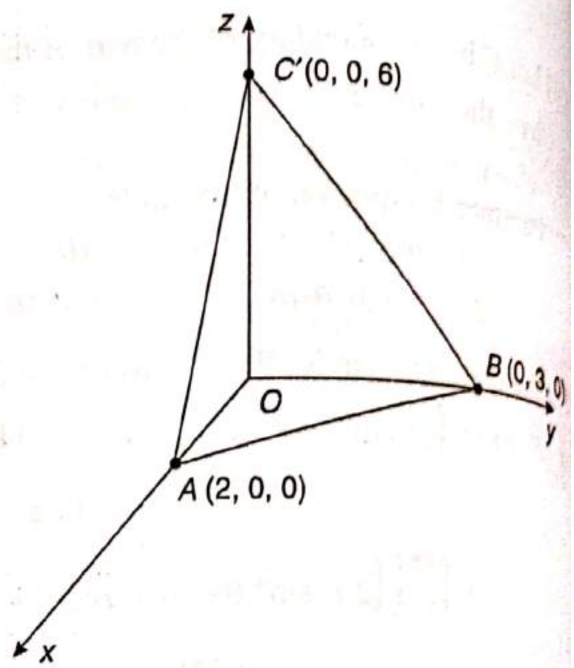

Fig. 1.38

Work done $= \oint_C \overline{F}\cdot d\overline{r}$

By Stokes' theorem,

$$\oint_C \overline{F}\cdot d\overline{r} = \iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS$$

Thus, work done $= \iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS$

where S is the surface of the $\Delta ABC'$.

Equation of the $\Delta ABC'$ is

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$$

$$3x+2y+z=6$$

(i) $$\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \hat{i}(1+1)-\hat{j}(0-0)+\hat{k}(2-1)$$

$$= 2\hat{i}+\hat{k}$$

(ii) Let $\phi = 3x+2y+z$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{9+4+1}}$$

$$= \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$$

(iii) Projection of $\Delta ABC'$ on the xy-plane is the ΔOAB bounded by the lines $y=0$, $3x+2y=6$, $x=0$.

$$dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \sqrt{14}\;dx\,dy$$

(iv) Let R be the region bounded by the ΔOAB. Along the vertical strip PQ, y varies from 0 to $\frac{6-3x}{2}$ and in the region R, x varies from 0 to 2.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = \iint_R (2\hat{i}+\hat{k})\cdot\left(\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}\right)\sqrt{14}\,dx\,dy$$

$$= \int_0^2\int_0^{\frac{6-3x}{2}} 7\,dy\,dx$$

$$= 7\int_0^2 |y|_0^{\frac{6-3x}{2}}\,dx$$

$$= 7\int_0^2\left(\frac{6-3x}{2}\right)dx$$

$$= 7\left|3x-\frac{3x^2}{4}\right|_0^2$$

$$= 21$$

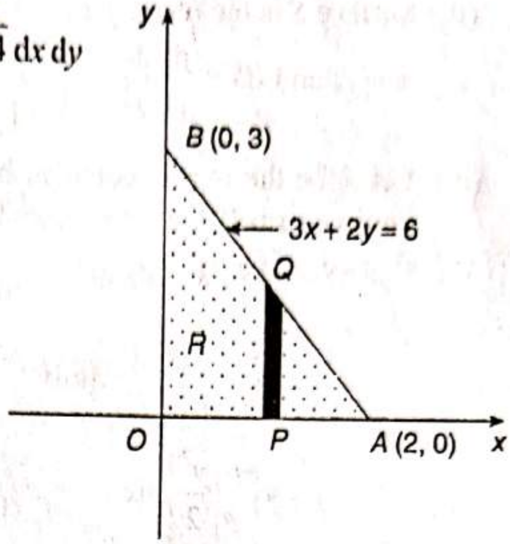

Fig. 1.39

Aliter

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = 7\iint_R dx\,dy$$

$$= 7(\text{Area of }\Delta OAB)$$

$$= 7\cdot\frac{1}{2}\cdot2\cdot3$$

$$= 21$$

Example 12

Verify Stokes' theorem for the vector field $\overline{F}=(x^2-y^2)\hat{i}+2xy\,\hat{j}$ *in the rectangular region in the xy-plane bounded by the lines* $x=-a$, $x=a$, $y=0$, $y=b$.

Solution

By Stokes' theorem,

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = \oint_C \overline{F}\cdot d\overline{r}$$

(i) $$\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \hat{i}(0)-\hat{j}(0)+\hat{k}(2y+2y)$$

$$= 4y\,\hat{k}$$

(ii) Surface S is the rectangle $ABC'D$ in xy-plane.

$$\hat{n} = \hat{k} \text{ and } dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{|\hat{k}\cdot\hat{k}|} = dx\,dy$$

(iii) Let R be the region bounded by the rectangle $ABC'D$ in xy-plane. Along the vertical strip PQ, y varies from 0 to b and in the region R, x varies from $-a$ to a.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = \iint_R 4y\hat{k}\cdot\hat{k}\,dx\,dy$$

$$= 4\int_{x=-a}^{a}\int_{y=0}^{b} y\,dy\,dx$$

$$= 4\int_{-a}^{a}\left|\frac{y^2}{2}\right|_0^b dx$$

$$= 2b^2\left|x\right|_{-a}^{a}$$

$$= 4ab^2 \qquad \dots (1)$$

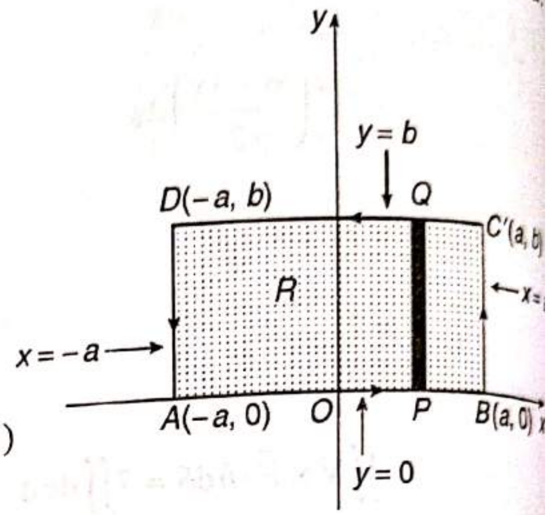

Fig. 1.40

(iv) $\overline{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$

$d\overline{r} = \hat{i}dx + \hat{j}dy$

$\overline{F}\cdot d\overline{r} = (x^2 - y^2)dx + 2xydy$

Let C be the boundary of the rectangle $ABC'D$.

$$\oint_C \overline{F}\cdot dr = \oint_{AB}\overline{F}\cdot d\overline{r} + \oint_{BC'}\overline{F}\cdot d\overline{r} + \oint_{C'D}\overline{F}\cdot d\overline{r} + \oint_{DA}\overline{F}\cdot d\overline{r} \qquad \dots (2)$$

(a) Along AB : $y = 0$, $dy = 0$ and x varies from $-a$ to a.

$$\oint_{AB}\overline{F}\cdot d\overline{r} = \int_{AB}\left[(x^2 - y^2)\,dx + 2xy\,dy\right]$$

$$= \int_{-a}^{a} x^2\,dx$$

$$= \left|\frac{x^3}{3}\right|_{-a}^{a}$$

$$= \frac{2a^3}{3}$$

(b) Along BC' : $x = a$, $dx = 0$ and y varies from 0 to b.

$$\int_{BC'}\overline{F}\cdot d\overline{r} = \int_{BC'}\left[(x^2 - y^2)\,dx + 2xy\,dy\right]$$

$$= \int_0^b 2ay\,dy$$

$$= 2a\left|\frac{y^2}{2}\right|_0^b$$

$$= ab^2$$

(c) Along $C'D$: $y = b$, $dy = 0$ and x varies from a to $-a$.

$$\int_{C'D} \overline{F} \cdot d\bar{r} = \int_{C'D} \left[(x^2 - y^2)\, dx + 2xy\, dy \right]$$

$$= \int_a^{-a} (x^2 - b^2)\, dx$$

$$= \left| \frac{x^3}{3} - b^2 x \right|_a^{-a}$$

$$= -\frac{2a^3}{3} + 2ab^2$$

(d) Along DA : $x = -a$, $dx = 0$ and y varies from b to 0.

$$\int_{DA} \overline{F} \cdot d\bar{r} = \int_{DA} \left[(x^2 - y^2)\, dx + 2xy\, dy \right]$$

$$= \int_b^0 (-2ay)\, dy$$

$$= -2a \left| \frac{y^2}{2} \right|_b^0$$

$$= ab^2$$

Substituting in Eq. (2),

$$\oint_C \overline{F} \cdot d\bar{r} = \frac{2a^3}{3} + ab^2 - \frac{2a^3}{3} + 2ab^2 + ab^2$$

$$= 4ab^2 \qquad \text{... (3)}$$

From Eqs (1) and (3),

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = \oint_C \overline{F} \cdot d\bar{r} = 4ab^2$$

Hence, Stokes' theorem is verified.

Example 13

Verify Stokes' theorem for $\overline{F} = \left(x^2 + y^2\right)\hat{i} - 2xy\hat{j}$ *taken around the rectangle bounded by the lines* $x = \pm a, y = 0, y = b.$

Solution

By Stokes' theorem,

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = \oint_C \overline{F} \cdot d\bar{r}$$

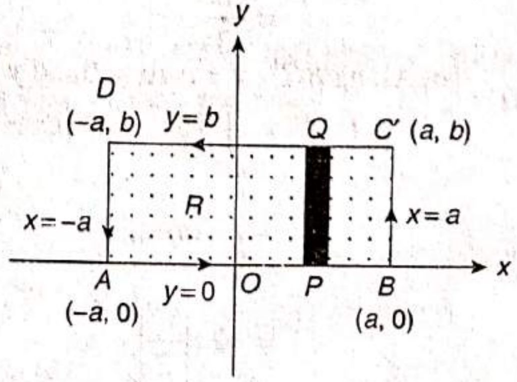

Fig. 1.41

$$\text{(i) } \nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(-2y - 2y)$$

$$= -4y\, \hat{k}$$

(ii) S is the surface of the rectangle $ABC'D$ which lies in xy-plane.

$$\hat{n} = \hat{k} \text{ and } dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{|\hat{k}\cdot\hat{k}|} = dx\,dy$$

(iii) Let R be the region bounded by the rectangle $ABC'D$ in the xy-plane. Along the vertical strip PQ, y varies 0 to b and in the region R, x varies from $-a$ to a.

$$\iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS = \iint_R \left(-4y\hat{k}\right)\cdot\hat{k}\,dx\,dy$$

$$= -4\int_{x=-a}^{a}\int_{y=0}^{b} y\,dy\,dx$$

$$= -4\int_{-a}^{a}\left|\frac{y^2}{2}\right|_0^b dx$$

$$= -2b^2\left|x\right|_{-a}^{a}$$

$$= -2b^2(a+a)$$

$$= -4ab^2$$

(iv) $\bar{F} = \left(x^2+y^2\right)\hat{i} - 2xy\hat{j}$

$$d\bar{r} = \hat{i}dx + \hat{j}dy \qquad \text{...(1)}$$

$$\bar{F}\cdot d\bar{r} = \left(x^2+y^2\right)dx - 2xy\,dy$$

Let C be the boundary of the rectangle $ABC'D$ in the xy plane.

$$\oint_C \bar{F}\cdot d\bar{r} = \int_{AB}\bar{F}\cdot d\bar{r} + \int_{BC'}\bar{F}\cdot d\bar{r} + \int_{C'D}\bar{F}\cdot d\bar{r} + \int_{DA}\bar{F}\cdot d\bar{r} \qquad \text{...(2)}$$

(*a*) Along AB : $y = 0$, $dy = 0$ and x varies from $-a$ to a.

$$\int_{AB}\bar{F}\cdot d\bar{r} = \int_{AB}\left[\left(x^2+y^2\right)dx - 2xy\,dy\right]$$

$$= \int_{-a}^{a} x^2\,dx$$

$$= \left|\frac{x^3}{3}\right|_{-a}^{a}$$

$$= \frac{2a^3}{3}$$

(*b*) Along BC' : $x = a$, $dx = 0$ and y varies from 0 to b.

$$\int_{BC'}\bar{F}\cdot d\bar{r} = \int_{BC'}\left[\left(x^2+y^2\right)dx - 2xy\,dy\right]$$

$$= \int_0^b -2ay\,dy$$

$$= -2a\left|\frac{y^2}{2}\right|_0^b$$

$$= -ab^2$$

(c) Along $C'D: y = b, \mathrm{d}y = 0$ and x varies from a to $-a$.

$$\int_{C'D} \bar{F} \cdot \mathrm{d}\bar{r} = \int_{C'D} \left[\left(x^2 + y^2\right)\mathrm{d}x - 2xy\,\mathrm{d}y\right]$$

$$= \int_a^{-a} \left(x^2 + b^2\right)\mathrm{d}x$$

$$= \left|\frac{x^3}{3} + b^2 x\right|_a^{-a}$$

$$= \left(-\frac{a^3}{3} - ab^2\right) - \left(\frac{a^3}{3} + ab^2\right)$$

$$= -\frac{2a^3}{3} - 2ab^2$$

(d) Along $DA: x = -a, \mathrm{d}x = 0$ and y varies from b to 0.

$$\int_{DA} \bar{F} \cdot \mathrm{d}\bar{r} = \int_{DA} \left[\left(x^2 + y^2\right)\mathrm{d}x - 2xy\,\mathrm{d}y\right]$$

$$= \int_b^0 2ay\,\mathrm{d}y$$

$$= 2a\left|\frac{y^2}{2}\right|_b^0$$

$$= -ab^2$$

Substituting in Eq. (2),

$$\oint_C \bar{F} \cdot \mathrm{d}\bar{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2$$

$$= -4ab^2 \qquad \cdots(3)$$

From Eqs (1) and (3),

$$\iint_S \nabla \times \bar{F} \cdot \hat{n}\,\mathrm{d}S = \oint_C \bar{F} \cdot \mathrm{d}\bar{r} = -4ab^2$$

Hence, Stokes' theorem is verified.

Example 14

Verify Stokes' theorem for the function $\bar{F} = x^2\hat{i} + xy\hat{j}$ *integrated round the square in the* $z = 0$ *plane whose sides are along the lines* $x = 0$, $y = 0$, $x = a$, $y = a$.

Solution

By Stokes' theorem,

$$\iint_S \nabla \times \bar{F} \cdot \hat{n}\,\mathrm{d}S = \oint_C \bar{F} \cdot \mathrm{d}\bar{r}$$

(i) $\nabla\times\bar{F}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$

$=i(0)-j(0)+k(y)$

$=yk$

(ii) S is the surface of the square $OABC'$ which lies in xy-plane.

$\hat{n}=\hat{k},\qquad dS=\dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}=\dfrac{dx\,dy}{|\hat{k}\cdot\hat{k}|}=dx\,dy$

(iii) Let R be the region bounded by the square $OABC'$ in the xy-plane. Along the vertical strip PQ, y varies from 0 to a and in the region R, x varies from 0 to a.

$$\iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS=\iint_R (y\hat{k})\cdot\hat{k}\,dx\,dy$$

$$=\int_0^a\int_0^a y\,dy\,dx$$

$$=\int_0^a\left|\frac{y^2}{2}\right|_0^a dx$$

$$=\frac{1}{2}a^2|x|_0^a$$

$$=\frac{a^3}{2}\qquad\text{...(1)}$$

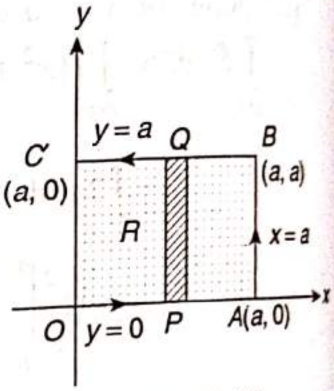

Fig. 1.42

(iv) $\bar{F}=x^2\hat{i}+xy\hat{j}$

$d\bar{r}=\hat{i}dx+\hat{j}dy$

$\bar{F}\cdot d\bar{r}=x^2dx+xy\,dy$

Let C be the boundary of the square $OABC'$ in $z=0$ plane.

$$\oint_C \bar{F}\cdot d\bar{r}=\oint_{OA}\bar{F}\cdot d\bar{r}+\oint_{AB}\bar{F}\cdot d\bar{r}+\oint_{BC'}\bar{F}\cdot d\bar{r}+\oint_{C'O}\bar{F}\cdot d\bar{r}\qquad\text{...(2)}$$

(a) Along $OA: y=0, dy=0$ and x varies from 0 to a.

$$\int_{OA}\bar{F}\cdot d\bar{r}=\int_{OA}\left(x^2dx+xy\,dy\right)$$

$$=\int_0^a x^2\,dx$$

$$=\left|\frac{x^3}{3}\right|_0^a$$

$$=\frac{a^3}{3}$$

(*b*) Along $AB: x = a, \mathrm{d}x = 0$ and y varies from 0 to a.

$$\int_{AB} \bar{F}\cdot \mathrm{d}\bar{r} = \int_{AB} \left(x^2\mathrm{d}x + xy\,\mathrm{d}y\right)$$

$$= \int_0^a ay\cdot \mathrm{d}y$$

$$= a\left|\frac{y^2}{2}\right|_0^a$$

$$= \frac{a^3}{2}$$

(*c*) Along $BC': y = a, \mathrm{d}y = 0$ and x varies from a to 0.

$$\int_{BC'} \bar{F}\cdot \mathrm{d}\bar{r} = \int_{BC'} \left(x^2\mathrm{d}x + xy\,\mathrm{d}y\right)$$

$$= \int_a^0 x^2\,\mathrm{d}x$$

$$= \left|\frac{x^3}{3}\right|_a^0$$

$$= -\frac{a^3}{3}$$

(*d*) Along $C'O: x = 0, \mathrm{d}x = 0$ and y varies from a to 0.

$$\int_{C'O} \bar{F}\cdot \mathrm{d}\bar{r} = \int_{C'O} \left(x^2\mathrm{d}x + xy\,\mathrm{d}y\right)$$

$$= 0$$

Substituting in Eq (2),

$$\oint_C \bar{F}\cdot \mathrm{d}\bar{r} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0$$

$$= \frac{a^3}{2} \qquad \cdots(3)$$

From Eqs (1) and (3),

$$\iint_S \nabla\times\bar{F}\cdot\hat{n}\,\mathrm{d}S = \oint_C \bar{F}\cdot\mathrm{d}\bar{r} = \frac{a^3}{2}$$

Hence, Stokes' theorem is verified.

Example 15

Verify Stokes' theorem for $\bar{F} = (y - z)\hat{i} + yz\hat{j} - xz\hat{k}$ *where S is the surface bounded by the planes* $x = 0, x = 1, y = 0, y = 1, z = 0$ *and* $z = 1$ *above the xy plane.*

Solution

By Stokes' theorem,

$$\iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS = \oint_C \bar{F}\cdot d\bar{r}$$

$$\text{(i)}\ \nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y-z & yz & -xz \end{vmatrix}$$

$$= \hat{i}[0-y] - \hat{j}[-z+1] + \hat{k}[0-1]$$

$$= -y\hat{i} + (z-1)\hat{j} - \hat{k}$$

(ii) Surface S is the cuboid with 5 faces except the face $OABC'$ which is not above the xy- plane.

$$\iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS = \iint_{S_1} \nabla\times\bar{F}\cdot\hat{n}\,dS + \iint_{S_2} \nabla\times\bar{F}\cdot\hat{n}\,dS + \iint_{S_3} \nabla\times\bar{F}\cdot\hat{n}\,dS + \iint_{S_4} \nabla\times\bar{F}\cdot\hat{n}\,dS + \iint_{S_5} \nabla\times\bar{F}\cdot\hat{n}\,dS \quad \dots(1)$$

Face	Equation	Normal $\hat{n}$	$\nabla\times\bar{F}\cdot\hat{n}$	dS
$S_1 : ABEF$	$x=1$	$\hat{i}$	$-y$	$dy\,dz$
$S_2 : OC'DG$	$x=0$	$-\hat{i}$	y	$dy\,dz$
$S_3 : BC'DE$	$y=1$	$\hat{j}$	$z-1$	$dx\,dz$
$S_4 : OAFG$	$y=0$	$-\hat{j}$	$-(z-1)$	$dx\,dz$
$S_5 : DEFG$	$z=1$	$\hat{k}$	-1	$dx\,dy$

$$(a)\iint_{S_1} \nabla\times\bar{F}\cdot\hat{n}\,dS = \int_0^1\int_0^1 -y\,dy\,dz$$

$$= \int_0^1 \left|-\frac{y^2}{2}\right|_0^1 dz$$

$$= \int_0^1 \left(-\frac{1}{2}\right) dz$$

$$= \left|-\frac{1}{2}z\right|_0^1$$

$$= -\frac{1}{2}$$

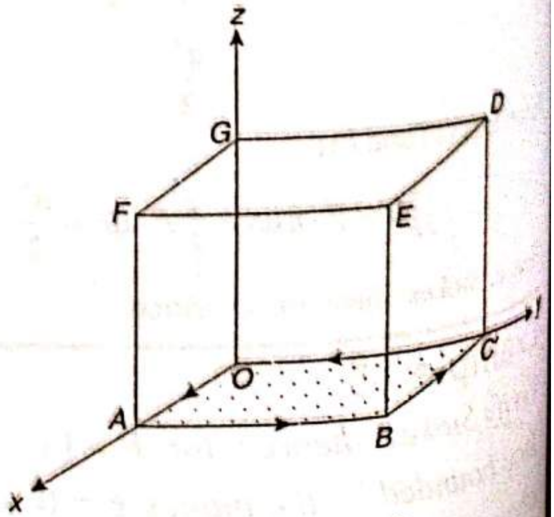

Fig. 1.43

$$(b)\iint_{S_2} \nabla\times\bar{F}\cdot\hat{n}\,dS = \int_0^1\int_0^1 y\,dy\,dz$$

$$= \int_0^1 \left|\frac{y^2}{2}\right|_0^1 dz$$

$$= \int_0^1 \left(\frac{1}{2}\right) dz$$

$$= \left|\frac{1}{2} z\right|_0^1$$

$$= \frac{1}{2}$$

$$(c) \iint_{S_3} \nabla \times \bar{F} \cdot \hat{n}\, dS = \int_0^1 \int_0^1 (z-1)\, dx\, dz$$

$$= \int_0^1 |xz - x|_0^1\, dz$$

$$= \int_0^1 |z-1|\, dz$$

$$= \left|\frac{z^2}{2} - z\right|_0^1$$

$$= \frac{1}{2} - 1$$

$$= -\frac{1}{2}$$

$$(d) \iint_{S_4} \nabla \times \bar{F} \cdot \hat{n}\, dS = \int_0^1 \int_0^1 (-z+1)\, dx\, dz$$

$$= \int_0^1 |-xz + x|_0^1\, dz$$

$$= \int_0^1 (-z+1)\, dz$$

$$= \left|-\frac{z^2}{2} + z\right|_0^1$$

$$= -\frac{1}{2} + 1$$

$$= \frac{1}{2}$$

$$(e) \iint_{S_5} \nabla \times \bar{F} \cdot \hat{n}\, dS = \int_0^1 \int_0^1 (-1)\, dx\, dy$$

$$= \int_0^1 |-x|_0^1\, dy$$

$$= \int_0^1 (-1)\, dy$$

$$= |-y|_0^1$$

$$= -1$$

Substituting in Eq (1),

$$\iint_S \nabla \times \bar{F} \cdot \hat{n}\, \mathrm{d}S = -\frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - 1 = -1 \qquad \ldots(2)$$

(iii)

$$\bar{F} = (y-z)\hat{i} + yz\hat{j} - xz\hat{k}$$

$$\mathrm{d}\bar{r} = \hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y + \hat{k}\mathrm{d}z$$

$$\bar{F} \cdot \mathrm{d}\bar{r} = (y-z)\,\mathrm{d}x + yz\,\mathrm{d}y - xz\,\mathrm{d}z$$

$$\bar{F} \cdot \mathrm{d}\bar{r} = y\,\mathrm{d}x \qquad (\because z = 0)$$

Let C be the boundary of the square $OABC'$ consisting of edges OA, AB, BC' and $C'O$ in $z = 0$ plane.

$$\oint_C \bar{F} \cdot \mathrm{d}\bar{r} = \oint_{OA} \bar{F} \cdot \mathrm{d}\bar{r} + \oint_{AB} \bar{F} \cdot \mathrm{d}\bar{r} + \oint_{BC'} \bar{F} \cdot \mathrm{d}\bar{r} + \oint_{C'O} \bar{F} \cdot \mathrm{d}\bar{r} \qquad \ldots(3)$$

(*a*) Along $OA : y = 0, \mathrm{d}y = 0$ and x varies from 0 to 1.

$$\int_{OA} \bar{F} \cdot \mathrm{d}\bar{r} = \int_{OA} y\,\mathrm{d}x$$
$$= 0$$

(*b*) Along $AB : x = 1, \mathrm{d}x = 0$ and y varies from 0 to 1.

$$\int_{AB} \bar{F} \cdot \mathrm{d}\bar{r} = \int_{AB} y\,\mathrm{d}x$$
$$= 0$$

(*c*) Along $BC' : y = 1, \mathrm{d}y = 0$ and x varies from 1 to 0.

$$\int_{BC'} \bar{F} \cdot \mathrm{d}\bar{r} = \int_{BC'} y\,\mathrm{d}x$$
$$= \int_1^0 1\,\mathrm{d}x$$
$$= |x|_1^0$$
$$= -1$$

(*d*) Along $C'O : x = 0, \mathrm{d}x = 0$ and y varies from 1 to 0.

$$\int_{C'O} \bar{F} \cdot \mathrm{d}\bar{r} = \int_{C'O} y\,\mathrm{d}x$$
$$= 0$$

Substituting in Eq (3),

$$\oint_C \bar{F} \cdot \mathrm{d}\bar{r} = 0 + 0 - 1 + 0 = -1 \qquad \ldots(4)$$

From Eqs (2) and (4),

$$\iint_S \nabla \times \bar{F} \cdot \hat{n}\, \mathrm{d}S = \oint_C \bar{F} \cdot \mathrm{d}\bar{r} = -1$$

Hence, Stokes' theorem is verified.

Example 16

Verify Stokes' theorem for $\overline{F} = xy\hat{i} - 2yz\hat{j} - zx\hat{k}$ where S is the open surface of the rectangular parallelepiped formed by the planes x = 0, x = 1, y = 0, y = 2 and z = 3 above the xy-plane.

Solution

By Stokes' theorem,

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = \oint_C \overline{F}\cdot d\overline{r}$$

$$\text{(i)}\ \nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -2yz & -zx \end{vmatrix}$$

$$= \hat{i}\left[0-(-2y)\right] - \hat{j}\left[-z-0\right] + \hat{k}\left[0-x\right]$$

$$= 2y\hat{i} + z\hat{j} - x\hat{k}$$

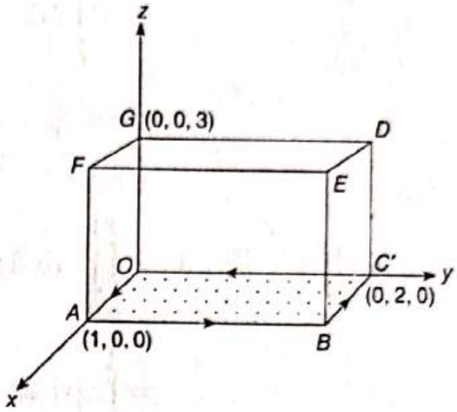

Fig. 1.44

(ii) Surface S is the open surface of the rectangular parallelopiped with 5 faces except the face $OABC'$ which is not above the xy-plane.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = \iint_{S_1} \nabla\times\overline{F}\cdot\hat{n}\,dS + \iint_{S_2} \nabla\times\overline{F}\cdot\hat{n}\,dS + \iint_{S_3} \nabla\times\overline{F}\cdot\hat{n}\,dS + \iint_{S_4} \nabla\times\overline{F}\cdot\hat{n}\,dS$$

$$+ \iint_{S_5} \nabla\times\overline{F}\cdot\hat{n}\,dS \quad ...(1)$$

Face	Equation	Outward Normal $\hat{n}$	$\nabla\times\overline{F}\cdot\hat{n}$	dS
$S_1 : ABEF$	$x=1$	$\hat{i}$	$2y$	dy dz
$S_2 : OC'DG$	$x=0$	$-\hat{i}$	$-2y$	dy dz
$S_3 : BC'DE$	$y=2$	$\hat{j}$	z	dx dz
$S_4 : OAFG$	$y=0$	$-\hat{j}$	$-z$	dx dz
$S_5 : DEFG$	$z=3$	$\hat{k}$	$-x$	dx dy

$$(a) \iint_{S_1} \nabla\times\overline{F}\cdot\hat{n}\,dS = \int_0^3\int_0^2 2y\,dy\,dz$$

$$= 2\int_0^3 \left|\frac{y^2}{2}\right|_0^2 dz$$

$$= 2\int_0^3 2\,dz$$

$$= 4\left|z\right|_0^3$$

$$= 12$$

$$(b)\iint_{S_2} \nabla\times\bar{F}\cdot\hat{n}\,dS = \int_0^3\int_0^2 -2y\,dy\,dz$$

$$= -2\int_0^3 \left|\frac{y^2}{2}\right|_0^2 dz$$

$$= -2\int_0^3 2\,dz$$

$$= -4|z|_0^3$$

$$= -12$$

$$(c)\iint_{S_3} \nabla\times\bar{F}\cdot\hat{n}\,dS = \int_0^3\int_0^1 z\,dx\,dz$$

$$= \int_0^3 z|x|_0^1 dz$$

$$= \int_0^3 z\,dz$$

$$= \left|\frac{z^2}{2}\right|_0^3$$

$$= \frac{9}{2}$$

$$(d)\iint_{S_4} \nabla\times\bar{F}\cdot\hat{n}\,dS = \int_0^3\int_0^1 (-z)\,dx\,dz$$

$$= -\int_0^3 z|x|_0^1 dz$$

$$= -\int_0^3 z\,dz$$

$$= -\left|\frac{z^2}{2}\right|_0^3$$

$$= -\frac{9}{2}$$

$$(e)\iint_{S_5} \nabla\times\bar{F}\cdot\hat{n}\,dS = \int_0^2\int_0^1 -x\,dx\,dy$$

$$= -\int_0^2 \left|\frac{x^2}{2}\right|_0^1 dy$$

$$= -\int_0^2 \frac{1}{2}\,dy$$

$$= -\frac{1}{2}\left|y\right|_0^2$$

$$= -\frac{1}{2}(2)$$

$$= -1$$

Substituting in Eq. (1),

$$\iint_S \nabla \times \bar{F} \cdot \hat{n}\,dS = 12 - 12 + \frac{9}{2} - \frac{9}{2} - 1 = -1 \qquad \cdots(2)$$

(iii) $\bar{F} = xy\hat{i} - 2yz\hat{j} - zx\hat{k}$

$$d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

$$\bar{F} \cdot d\bar{r} = xy\,dx - 2yz\,dy - zx\,dz$$

$$= xy\,dx \qquad (\because z = 0)$$

Let C be the boundary of the rectangle $OABC'$ consisting of edges OA, AB, BC' and $C'O$ in $z = 0$ plane.

$$\oint_C \bar{F} \cdot d\bar{r} = \int_{OA} \bar{F} \cdot d\bar{r} + \int_{AB} \bar{F} \cdot d\bar{r} + \int_{BC'} \bar{F} \cdot d\bar{r} + \int_{C'O} \bar{F} \cdot d\bar{r} \qquad \cdots(3)$$

(*a*) Along $OA : y = 0, dy = 0$ and x varies from 0 to 1.

$$\int_{OA} \bar{F} \cdot d\bar{r} = \int_{OA} xy\,dx$$

$$= 0$$

(*b*) Along $AB : x = 1, dx = 0$ and y varies from 0 to 2.

$$\int_{AB} \bar{F} \cdot d\bar{r} = \int_{AB} xy\,dx$$

$$= 0$$

(*c*) Along $BC' : y = 2, dy = 0$ and x varies from 1 to 0.

$$\int_{BC'} \bar{F} \cdot d\bar{r} = \int_{BC'} xy\,dx$$

$$= \int_1^0 2x\,dx$$

$$= 2\left|\frac{x^2}{2}\right|_1^0$$

$$= 2\left(-\frac{1}{2}\right)$$

$$= -1$$

(d) Along $C'O: x = 0, dx = 0$ and y varies from 2 to 0.

$$\int_{C'O} \bar{F}\cdot d\bar{r} = \int_{C'O} xy\,dx$$
$$= 0$$

Substituting in Eq. (3),

$$\oint_{C'} \bar{F}\cdot d\bar{r} = 0 + 0 - 1 + 0 = -1 \quad \cdots(4)$$

From Eqs (2) and (4),

$$\iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS = \oint_C \bar{F}\cdot d\bar{r} = -1$$

Hence, Stokes' theorem is verified.

Example 17

Verify Stokes' theorem for $\bar{F} = (x+y)\hat{i} + (y+z)\hat{j} - x\hat{k}$ *and S is the surface of the plane* $2x + y + z = 2$ *which is in the first octant.* **[Summer 2014]**

Solution

By Stokes' theorem,

$$\iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS = \oint_C \bar{F}\cdot d\bar{r}$$

(i) $$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & y+z & -x \end{vmatrix}$$
$$= \hat{i}(0-1) - \hat{j}(-1-0) + \hat{k}(0-1)$$
$$= -\hat{i} + \hat{j} - \hat{k}$$

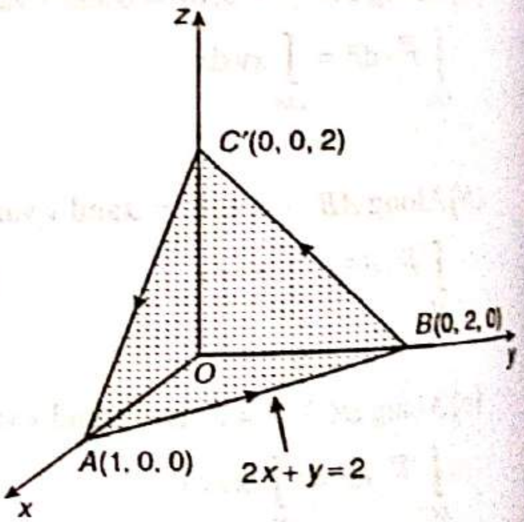

Fig. 1.45

(ii) Let $\phi = 2x + y + z$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$
$$= \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{4+1+1}}$$
$$= \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$$

(iii) Projection of the plane $2x + y + z = 2$ on xy-plane $(z = 0)$ is the triangle OAB bounded by the lines $x = 0, y = 0, 2x + y = 2$.

(iv) $$dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \sqrt{6}\,dx\,dy$$

(v) Let R be the region bounded by the triangle OAB in the xy-plane.
Along the vertical strip PQ, y varies from 0 to $(2-2x)$ and in the region R, x varies from 0 to 1.

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = \iint_R (-\hat{i}+\hat{j}-\hat{k}) \cdot \frac{(2\hat{i}+\hat{j}+\hat{k})}{\sqrt{6}} \sqrt{6}\, dx\, dy$$

$$= \int_0^1 \int_0^{2-2x} (-2+1-1)\, dx\, dy$$

$$= -2\int_0^1 |y|_0^{2-2x} dx$$

$$= -2\int_0^1 (2-2x)\, dx$$

$$= -4\left|x - \frac{x^2}{2}\right|_0^1$$

$$= -4\left(1 - \frac{1}{2}\right)$$

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = -2 \qquad \text{... (1)}$$

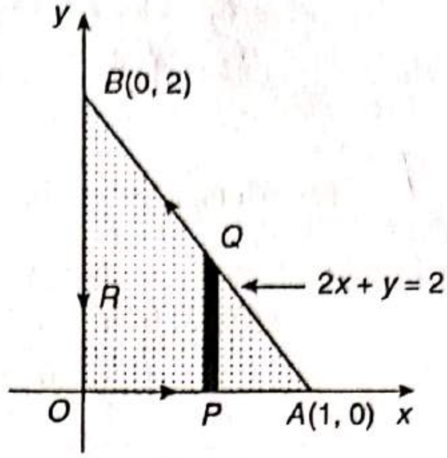

Fig. 1.46

(vi) Let C be the boundary of the triangle ABC'.

$$\overline{F} \cdot d\overline{r} = (x+y)\, dx + (y+z)\, dy - x\, dz$$

$$\oint \overline{F} \cdot d\overline{r} = \int_{AB} \overline{F} \cdot d\overline{r} + \int_{BC'} \overline{F} \cdot d\overline{r} + \int_{C'A} \overline{F} \cdot d\overline{r} \qquad \text{... (2)}$$

(a) Along AB: $z = 0$, $y = 2 - 2x$
$dz = 0$, $dy = -2dx$
x varies from 1 to 0.

$$\int_{AB} \overline{F} \cdot d\overline{r} = \int_{AB} [(x+y)dx + (y+z)dy - x\, dz]$$

$$= \int_1^0 [(x+2-2x)dx + (2-2x)(-2dx)]$$

$$= \int_1^0 (3x-2)\, dx$$

$$= \left|3 \cdot \frac{x^2}{2} - 2x\right|_1^0$$

$$= -\frac{3}{2} + 2$$

$$= \frac{1}{2}$$

(b) Along BC': $x = 0$, $y + z = 2$ (Equation of the line BC' is $y + z = 2$)
$dx = 0$, $dz = -dy$
y varies from 2 to 0.

$$\int_{BC'} \overline{F}\cdot d\overline{r} = \int_{BC'} [(x+y)dx + (y+z)dy - x\,dz]$$
$$= \int_2^0 2\,dy$$
$$= 2|y|_2^0$$
$$= -4$$

(c) Along $C'A: y = 0,\quad 2x + z = 2\left(\text{Equation of the line } C'A \text{ is } \frac{x}{1}+\frac{z}{2}=1\right)$

$dy = 0,\quad dz = -2dx$

x varies from 0 to 1.

$$\int_{C'A} \overline{F}\cdot d\overline{r} = \int_{C'A} [(x+y)dx + (y+z)dy - x\,dz]$$
$$= \int_0^1 [x\,dx - x(-2\,dx)]$$
$$= \int_0^1 3x\,dx$$
$$= 3\left|\frac{x^2}{2}\right|_0^1$$
$$= \frac{3}{2}$$

Substituting in Eq. (2),

$$\oint_C \overline{F}\cdot d\overline{r} = \frac{1}{2} - 4 + \frac{3}{2}$$
$$= -2 \quad \text{...(3)}$$

From Eqs (1) and (3),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = \oint_C \overline{F}\cdot d\overline{r} = -2$$

Hence, Stokes' theorem is verified.

Example 18

Verify Stokes' theorem for $\overline{F} = xz\hat{i} + y\hat{j} + xy^2\hat{k}$ where S is the surface of the region bounded by $y = 0$, $z = 0$ and $4x + y + 2z = 4$ which is not included in the yz-plane.

Solution

By Stokes' theorem,

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = \oint_C \overline{F}\cdot d\overline{r}$$

(i) $$\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & y & xy^2 \end{vmatrix}$$

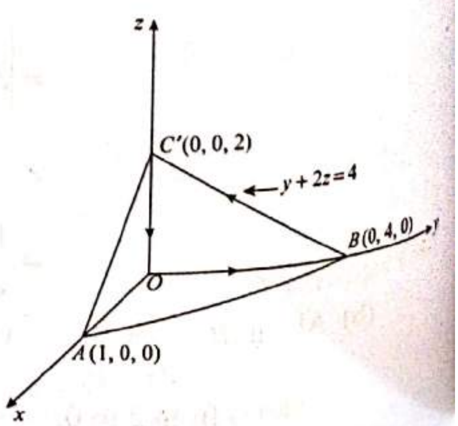

Fig. 1.47

$$= \hat{i}(2xy - 0) - \hat{j}(y^2 - x) + \hat{k}(0 - 0)$$
$$= 2xy\hat{i} + (x - y^2)\hat{j}$$

(ii) Surface S consists of three surfaces, $y = 0$, $z = 0$ and $4x + y + 2z = 4$.

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = \iint_{S_1} \nabla \times \overline{F} \cdot \hat{n}\, dS + \iint_{S_2} \nabla \times \overline{F} \cdot \hat{n}\, dS + \iint_{S_3} \nabla \times \overline{F} \cdot \hat{n}\, dS \quad \ldots (1)$$

(a) Surface S_1 ($\Delta OAC'$): $y = 0$, $\hat{n} = -\hat{j}$ and $dS = dx\, dz$.

Let R_1 be the region bounded by the $\Delta OAC'$. Along the vertical strip P_1Q_1, z varies from 0 to $2 - 2x$ and in the region R_1, x varies from 0 to 1.

$$\iint_{S_1} \nabla \times \overline{F} \cdot \hat{n}\, dS = \iint_{R_1} -(x - y^2)\, dx\, dz$$
$$= \int_0^1 \int_0^{2-2x} (-x)\, dx\, dz \quad [\because y = 0]$$
$$= -\int_0^1 x |z|_0^{2-2x}\, dx$$
$$= -\int_0^1 x(2 - 2x)\, dx$$
$$= -\left| x^2 - \frac{2x^3}{3} \right|_0^1$$
$$= -\left(1 - \frac{2}{3}\right)$$
$$= -\frac{1}{3}$$

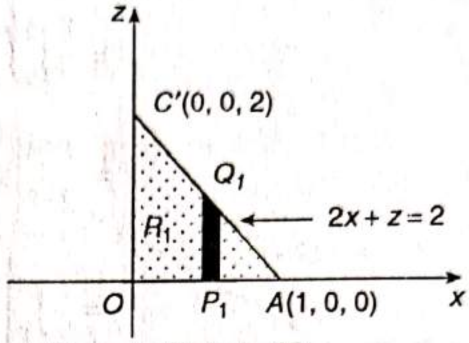

Fig. 1.48

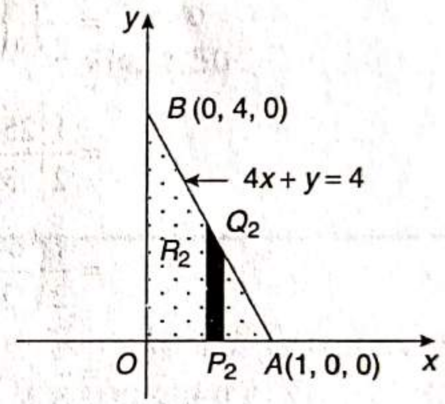

Fig. 1.49

(b) Surface S_2 (ΔOAB): $z = 0$, $\hat{n} = -\hat{k}$ and $dS = dx\, dy$.

Let R_2 be the region bounded by the ΔOAB. Along the vertical strip P_2Q_2, y varies from 0 to $4 - 4x$ and in the region R_2, x varies from 0 to 1.

$$\iint_{S_2} \nabla \times \overline{F} \cdot \hat{n}\, dS = \iint_{R_2} \left[2xy\hat{i} + (x - y^2)\hat{j}\right] \cdot (-\hat{k})\, dx\, dy$$
$$= 0$$

(c) Surface $S_3 (4x + y + 2z = 4)$:

Let $\phi = 4x + y + 2z$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$$
$$= \frac{4\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{16 + 1 + 4}}$$
$$= \frac{4\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{21}}$$

Projection of the plane $4x+y+2z=4$ on xy-plane is the ΔOAB.

$$dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{\sqrt{21}}{2}dx\,dy$$

Let R_2 be the region bounded by the ΔOAB. Along the vertical strip P_2Q_2, y varies from 0 to $4-4x$ and in the region R_2, x varies from 0 to 1.

$$\iint_{S_3} \nabla\times\overline{F}\cdot\hat{n}\,dS = \iint_{R_2}\left[2xy\hat{i}+(x-y^2)\hat{j}\right]\cdot\left(\frac{4\hat{i}+\hat{j}+2\hat{k}}{\sqrt{21}}\right)\frac{\sqrt{21}}{2}dx\,dy$$

$$= \frac{1}{2}\int_0^1\int_0^{4-4x}(8xy+x-y^2)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1\left|8x\frac{y^2}{2}+xy-\frac{y^3}{3}\right|_0^{4-4x}dx$$

$$= \frac{1}{2}\int_0^1\left[4x(4-4x)^2+x(4-4x)-\frac{(4-4x)^3}{3}\right]dx$$

$$= \frac{1}{2}\int_0^1\left(\frac{256}{3}x^3-196x^2+132x-\frac{64}{3}\right)dx$$

$$= \frac{1}{2}\left|\frac{256}{3}\cdot\frac{x^4}{4}-196\frac{x^3}{3}+132\frac{x^2}{2}-\frac{64}{3}x\right|_0^1$$

$$= \frac{1}{2}\left(\frac{64}{3}-\frac{196}{3}+66-\frac{64}{3}\right)$$

$$= \frac{1}{3}$$

Substituting in Eq. (1),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = -\frac{1}{3}+0+\frac{1}{3}=0 \qquad \text{...(2)}$$

(iii) Since the surface S does not include the yz-plane, it is open on the yz-plane. $\Delta OBC'$ is the boundary of the surface S.

Let C be the boundary of the $\Delta OBC'$ bounded by the lines $y=0$, $z=0$, $y+2z=4$

$$\oint_C \overline{F}\cdot d\overline{r} = \int_{C'O}\overline{F}\cdot d\overline{r}+\int_{OB}\overline{F}\cdot d\overline{r}+\int_{BC'}\overline{F}\cdot d\overline{r} \qquad \text{...(3)}$$

$$\overline{F}\cdot d\overline{r} = xz\,dx+y\,dy+xy^2dz = y\,dy \qquad [\because x=0, dx=0]$$

(a) Along $C'O: y=0 \quad dy=0$ and z varies from 2 to 0.

$$\int_{C'O}\overline{F}\cdot d\overline{r} = \int_2^0 y\,dy = 0$$

(b) Along $OB: z = 0$, $dz = 0$ and y varies from 0 to 4.

$$\int_{OB} \overline{F} \cdot d\overline{r} = \int_0^4 y\,dy$$

$$= \left|\frac{y^2}{2}\right|_0^4$$

$$= 8$$

(c) Along BC': $y = 4 - 2z$, $dy = -2\,dz$ and z varies from 0 to 2.

$$\int_{BC'} \overline{F} \cdot d\overline{r} = \int_0^2 y\,dy$$

$$= \int_0^2 (4 - 2z)(-2dz)$$

$$= -4\left|2z - \frac{z^2}{2}\right|_0^2$$

$$= -4(4 - 2)$$

$$= -8$$

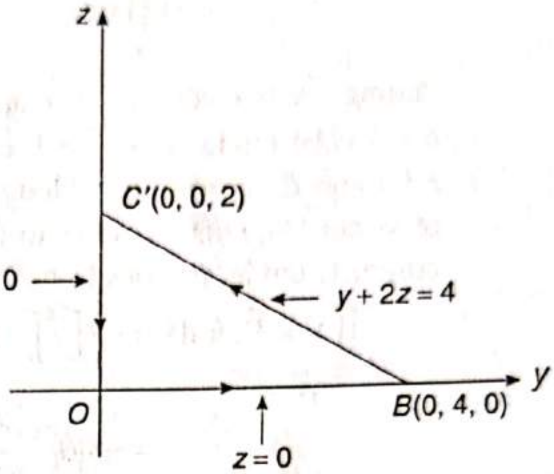

Fig. 1.50

Substituting in Eq. (3),

$$\oint_C \overline{F} \cdot d\overline{r} = 0 + 8 - 8 = 0 \qquad \text{... (4)}$$

From Eqs (2) and (4),

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = \oint_C \overline{F} \cdot d\overline{r} = 0$$

Hence, Stokes' theorem is verified.

Example 19

Verify Stokes' theorem for $\overline{F} = 4y\hat{i} - 4x\hat{j} + 3\hat{k}$, *where S is a disk of 1-unit radius lying on the plane z = 1 and C is its boundary.*

Solution

By Stokes' theorem,

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\, dS = \oint_C \overline{F} \cdot d\overline{r}$$

where S is the surface of the disk of 1-unit radius lying on the plane $z = 1$ and C is the circle $x^2 + y^2 = 1$.

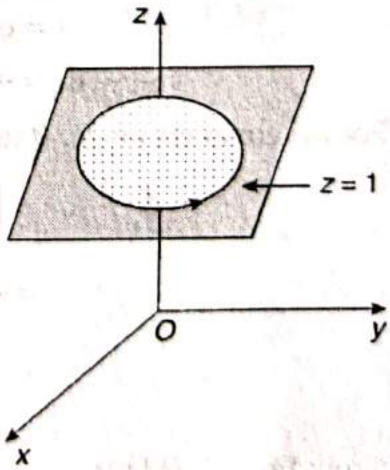

Fig. 1.51

(i) $$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & -4x & 3 \end{vmatrix}$$

$$= \hat{i}(0 - 0) - \hat{j}(0 - 0) + \hat{k}(-4 - 4)$$

$$= -8\hat{k}$$

(ii) Since disc lies on the plane $z = 1$, parallel to the xy-plane,

$$\hat{n} = \hat{k}$$

(iii) Projection of the disc in the xy-plane is the circle $x^2 + y^2 = 1$.

(iv) $dS = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = dx\,dy$

(v) Let R be the region bounded by the circle $x^2 + y^2 = 1$ in the xy-plane.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\, dS = \iint_R (-8\hat{k})\cdot\hat{k}\, dx\, dy$$
$$= -8\iint_R dx\,dy$$

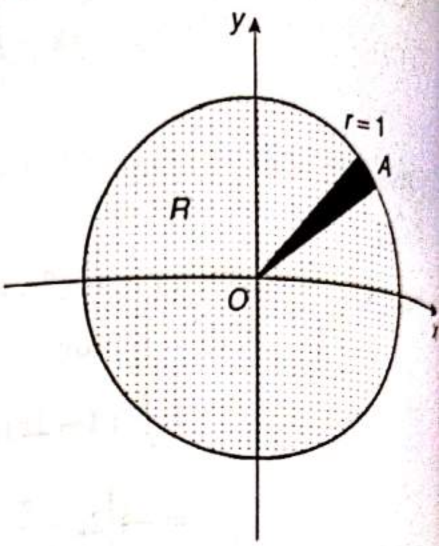

Fig. 1.52

Putting $x = r\cos\theta$, $y = r\sin\theta$, equation of the circle $x^2 + y^2 = 1$ reduces to $r = 1$ and $dx\,dy = r\,dr\,d\theta$. Along, the radius vector OA, r varies from 0 to 1 and for a complete circle, θ varies from 0 to 2π.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\, dS = -8\int_0^{2\pi}\int_0^1 r\, dr\, d\theta$$
$$= -8|\theta|_0^{2\pi}\left|\frac{r^2}{2}\right|_0^1$$
$$= -8(2\pi)\left(\frac{1}{2}\right)$$
$$= -8\pi \qquad \text{(1)}$$

(vi) C is the boundary of the disc, i.e., the circle $x^2 + y^2 = 1$ lying on the plane $z = 1$.

$$\overline{F}\cdot d\overline{r} = 4y\,dx - 4x\,dy + 3dz$$
$$= 4y\,dx - 4x\,dy \qquad [\because z = 1, dz = 0]$$
$$\oint_C \overline{F}\cdot d\overline{r} = \oint_C (4y\,dx - 4x\,dy)$$

Parametric equation of the circle is

$$x = \cos\theta, \qquad y = \sin\theta$$
$$dx = -\sin\theta\, d\theta, \quad dy = \cos\theta\, d\theta$$

For the complete circle, θ varies from 0 to 2π.

$$\oint_C \overline{F}\cdot d\overline{r} = \int_0^{2\pi}[4\sin\theta(-\sin\theta\, d\theta) - 4\cos\theta(\cos\theta\, d\theta)]$$
$$= -4\int_0^{2\pi} d\theta$$
$$= -4|\theta|_0^{2\pi} \qquad \text{...(2)}$$
$$= -8\pi$$

From Eqs (1) and (2),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\, dS = \oint_C \overline{F}\cdot d\overline{r} = -8\pi$$

Hence, Stokes' theorem is verified.

Example 20

Verify Stokes' theorem for $\overline{F} = (x^2 + y - 4)\hat{i} + 3xy\,\hat{j} + (2xz + z^2)\hat{k}$ *over the surface of the sphere* $x^2 + y^2 + z^2 = 16$ *above xy-plane.*

Solution

By Stokes' theorem,

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \oint_C \overline{F}\cdot\mathrm{d}\bar{r}$$

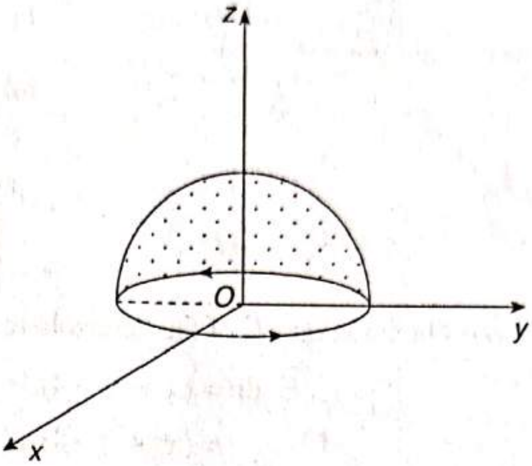

Fig. 1.53

(i) $\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$

$$= \hat{i}(0-0) - \hat{j}(2z-0) + \hat{k}(3y-1)$$

$$= -2z\,\hat{j} + (3y-1)\hat{k}$$

(ii) Let $\phi = x^2 + y^2 + z^2$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{4} \qquad [\because x^2 + y^2 + z^2 = 16]$$

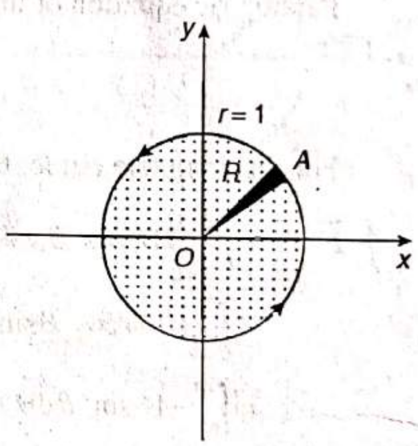

Fig. 1.54

(iii) Let R be the projection of the hemisphere $x^2 + y^2 + z^2 = 16$ on the xy-plane ($z = 0$) which is a circle, $x^2 + y^2 = 16$.

(iv) $\mathrm{d}S = \dfrac{\mathrm{d}x\,\mathrm{d}y}{|\hat{n}\cdot\hat{k}|}$

$$= \frac{4\,\mathrm{d}x\,\mathrm{d}y}{z}$$

(v) $\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S$

$$= \iint_R \left[-2z\,\hat{j} + (3y-1)\hat{k}\right]\cdot\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{4}\right)\frac{4\,\mathrm{d}x\,\mathrm{d}y}{z}$$

$$= \iint_R [-2zy + (3y-1)z]\frac{\mathrm{d}x\,\mathrm{d}y}{z}$$

$$= \iint_R (y-1)\,\mathrm{d}x\,\mathrm{d}y$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, equation of the circle $x^2 + y^2 = 16$ reduces to $r = 4$ and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$.

Along the radius vector OA, r varies from 0 to 4 and for the complete circle, θ varies from 0 to 2π.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \int_0^{2\pi}\int_0^4 (r\sin\theta - 1)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\left|\frac{r^3}{3}\right|_0^4 \sin\theta - \left|\frac{r^2}{2}\right|_0^4\right]d\theta$$

$$= \frac{64}{3}\left|-\cos\theta\right|_0^{2\pi} - \frac{16}{2}\left|\theta\right|_0^{2\pi}$$

$$= -\frac{64}{3}(\cos 2\pi - \cos 0) - \frac{16}{2}\cdot 2\pi$$

$$= -16\pi \qquad \text{...(1)}$$

(vi) The boundary C of the hemisphere S is the circle $x^2 + y^2 = 16$ in xy-plane $(z = 0)$.

$$\overline{F}\cdot \mathrm{d}\overline{r} = (x^2 + y - 4)\mathrm{d}x + 3xy\,\mathrm{d}y + (2xz + z^2)\mathrm{d}z$$

$$= (x^2 + y - 4)\,\mathrm{d}x + 3xy\,\mathrm{d}y \qquad [\because z = 0, \mathrm{d}z = 0]$$

$$\oint_C \overline{F}\cdot \mathrm{d}\overline{r} = \oint_C \left[(x^2 + y - 4)\,\mathrm{d}x + 3xy\,\mathrm{d}y\right]$$

Parametric equation of the circle $x^2 + y^2 = 16$ is

$$x = 4\cos\theta, \qquad y = 4\sin\theta$$

$$\mathrm{d}x = -4\sin\theta\,\mathrm{d}\theta, \qquad \mathrm{d}y = 4\cos\theta\,\mathrm{d}\theta$$

For the complete circle, θ varies from 0 to 2π.

$$\oint_C \overline{F}\cdot \mathrm{d}\overline{r} = \left[\int_0^{2\pi} (16\cos^2\theta + 4\sin\theta - 4)(-4\sin\theta\,\mathrm{d}\theta) + (3\cdot 4\cos\theta\cdot 4\sin\theta)(4\cos\theta\,\mathrm{d}\theta)\right]$$

$$= \int_0^{2\pi} (-64\cos^2\theta\sin\theta - 16\sin^2\theta + 16\sin\theta + 192\cos^2\theta\sin\theta)\,\mathrm{d}\theta$$

$$= \int_0^{2\pi} -16\sin^2\theta\,\mathrm{d}\theta \qquad \left[\because \int_0^{2a} f(2a-x) = 0, \text{ if } f(2a-x) = -f(x)\right]$$

$$= -16\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= -8\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{2\pi}$$

$$= -16\pi \qquad \text{...(2)}$$

From Eqs (1) and (2),

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,\mathrm{d}S = \oint_C \overline{F}\cdot \mathrm{d}\overline{r} = -16\pi$$

Hence, Stokes' theorem is verified.

Example 21

Verify Stokes' theorem for $\overline{F} = y\hat{i} + z\hat{j} + x\hat{k}$ *over the surface* $x^2 + y^2 = 1 - z,\ z > 0$.

Solution

By Stokes' theorem,

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = \oint_C \overline{F}\cdot d\overline{r}$$

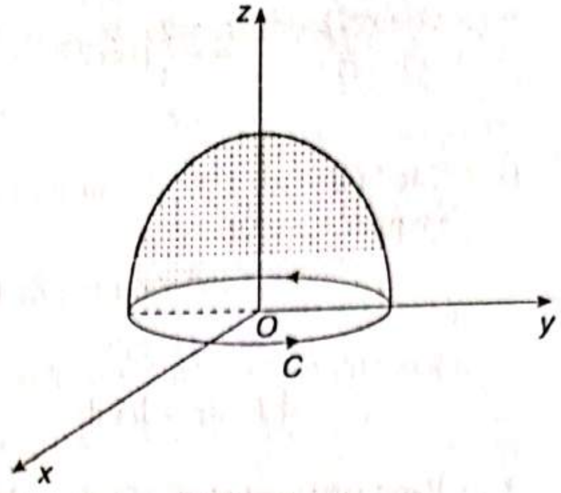

Fig. 1.55

(i) $$\nabla\times\overline{F} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & z & x\end{vmatrix}$$

$$= \hat{i}(0-1)-\hat{j}(1-0)+\hat{k}(0-1)$$

$$= -(\hat{i}+\hat{j}+\hat{k})$$

(ii) Let $\phi = x^2+y^2+z$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}+\hat{k}}{\sqrt{4x^2+4y^2+1}}$$

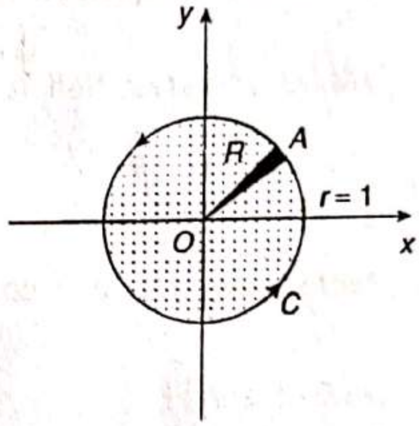

Fig. 1.56

(iii) Let R be the projection of the surface $x^2+y^2 = 1-z$ on the xy-plane ($z = 0$) which is a circle $x^2+y^2 = 1$.

(iv) $$dS = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \sqrt{4x^2+4y^2+1}\,dx\,dy$$

(v) $$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS$$

$$= \iint_R -(\hat{i}+\hat{j}+\hat{k})\cdot\frac{(2x\hat{i}+2y\hat{j}+\hat{k})}{\sqrt{4x^2+4y^2+1}}\sqrt{4x^2+4y^2+1}\,dx\,dy$$

$$= -\iint_R (2x+2y+1)\,dx\,dy$$

Putting $x = r\cos\theta,\ y = r\sin\theta$, circle $x^2+y^2 = 1$ reduces to $r = 1$ and $dx\,dy = r\,dr\,d\theta$

Along the radius vector OA, r varies from 0 to 1 and for the complete circle, θ varies from 0 to 2π.

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = -\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(2r\cos\theta+2r\sin\theta+1)r\,dr\,d\theta$$

$$= -\int_0^{2\pi}\left[2(\cos+\sin\theta)\left|\frac{r^3}{3}\right|_0^1+\left|\frac{r^2}{2}\right|_0^1\right]d\theta$$

$$= -\int_0^{2\pi}\left[\frac{2}{3}(\cos\theta+\sin\theta)+\frac{1}{2}\right]d\theta$$

$$= -\left|\frac{2}{3}(\sin\theta-\cos\theta)+\frac{1}{2}\theta\right|_0^{2\pi}$$

$$= -\frac{2}{3}(\sin 2\pi - \cos 2\pi - \sin 0 + \cos 0) - \frac{1}{2}(2\pi - 0)$$

$$= -\pi \qquad \ldots (1)$$

(vi) The boundary C of the surface $x^2 + y^2 = 1 - z$ is the circle $x^2 + y^2 = 1$ in the xy-plane ($z = 0$).

$$\overline{F} \cdot d\overline{r} = y\,dx + z\,dy + x\,dz$$

$$= y\,dx \qquad [\because z = 0, dz = 0]$$

$$\oint_C \overline{F} \cdot d\overline{r} = \oint_C y\,dx$$

Parametric equation of the circle $x^2 + y^2 = 1$ is

$$x = \cos\theta, \qquad y = \sin\theta$$

$$dx = -\sin\theta\,d\theta, \qquad dy = \cos\theta\,d\theta$$

For the complete circle, θ varies from 0 to 2π.

$$\oint_C \overline{F} \cdot d\overline{r} = \int_0^{2\pi} \sin\theta(-\sin\theta\,d\theta)$$

$$= -\int_0^{2\pi} \left(\frac{1 - \cos 2\theta}{2}\right) d\theta$$

$$= -\frac{1}{2}\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{2\pi}$$

$$= -\frac{1}{2}\left(2\pi - \frac{\sin 4\pi}{2} - 0\right)$$

$$= -\pi \qquad \ldots (2)$$

From Eqs (1) and (2),

$$\iint_S \nabla \times \overline{F} \cdot \hat{n}\,dS = \oint_C \overline{F} \cdot d\overline{r} = -\pi$$

Hence, Stokes' theorem is verified.

EXERCISE 1.7

(I) Evaluate the following integrals using Stokes' theorem:

1. $\iint_S \nabla \times \overline{F} \cdot \hat{n}\,dS$ where $\overline{F} = (x^2 + y + z)\hat{i} + 2xy\,\hat{j} - (3xyz + z^3)\hat{k}$ and S is the surface of the hemisphere $x^2 + y^2 + z^2 = 9$ above the xy-plane.

   [Ans.: -9π]

2. $\iint_S \nabla \times \overline{F} \times \hat{n}\,dS$ where $\overline{F} = 3y\hat{i} - xz\,\hat{j} + yz^2\hat{k}$ and S is the surface of the paraboloid $x^2 + y^2 = 2z$ bounded by the plane $z = 2$ and C is its boundary traversed in the clockwise direction.

   [Ans.: 20π]

3. $\int_C (y\,dx + z\,dy + x\,dz)$ where C is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane $x + z = a$.

$$\left[\text{Ans.}: \frac{-\pi a^2}{\sqrt{2}}\right]$$

(II) Verify Stokes' theorem for the following vector fields:

1. $\bar{F} = \left(x^3 + \frac{yz^2}{2}\right)\hat{i} + \left(\frac{xz^2}{2} + y^2\right)\hat{j} + (x\,y\,z)\hat{k}$ over the surface S of the cube $0 \le x \le 3,\ 0 \le y \le 3, 0 \le z \le 3.$

[Ans.: 0]

2. $\bar{F} = x\,z\,\hat{i} + y\,\hat{j} + y^2 x\,\hat{k}$ over the surface S of the tetrahedron bounded by the planes $y = 0$, $z = 0$ and $4x + y + 2z = 4$ above the yz–plane.

[Ans. : 0]

3. $\bar{F} = \left(x^3 + \frac{z^4}{4}\right)\hat{i} + 4x\,\hat{j} + (x\,z^3 + z^2)\hat{k}$ over the upper half surface S of the sphere $x^2 + y^2 + z^2 = 1.$

[Ans. : 4π]

4. $\bar{F} = (x^2 + y + 2)\hat{i} + 2xy\,\hat{j} + 4ze^x\,\hat{k}$ over the surface S of the paraboloid $z = 9 - (x^2 + y^2)$ above xy–plane.

[Ans. : -9π]

1.13 VOLUME INTEGRALS

If V be a region in space bounded by a closed surface S then the volume integral of a vector field $\bar{F}$ is $\iiint_V \bar{F}\,dV$.

Component form of volume Integral

If
$$\bar{F} = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k},$$
$$\iiint_V \bar{F}\,dV = \iiint_V (F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k})\,dx\,dy\,dz$$
$$= \hat{i}\iiint F_1\,dx\,dy\,dz + \hat{j}\iiint F_2\,dx\,dy\,dz + \hat{k}\iiint F_3\,dx\,dy\,dz$$

Another type of volume integral is $\iiint_V \phi\,dV$, where ϕ is a scalar function.

Example 1

Evaluate $\iiint\limits_V \bar{F}\,\mathrm{d}V$ *where* $\bar{F} = x\hat{i} + y\hat{j} + 2z\hat{k}$ *and V is the volume enclosed by the planes* $x = 0,\ y = 0,\ y = a,\ z = b^2$ *and the surface* $z = x^2$.

Solution

(i) V is the volume of the cylinder in positive octant with base as OAB and bounded between the planes $y = 0$ and $y = a$. y varies from 0 to a.

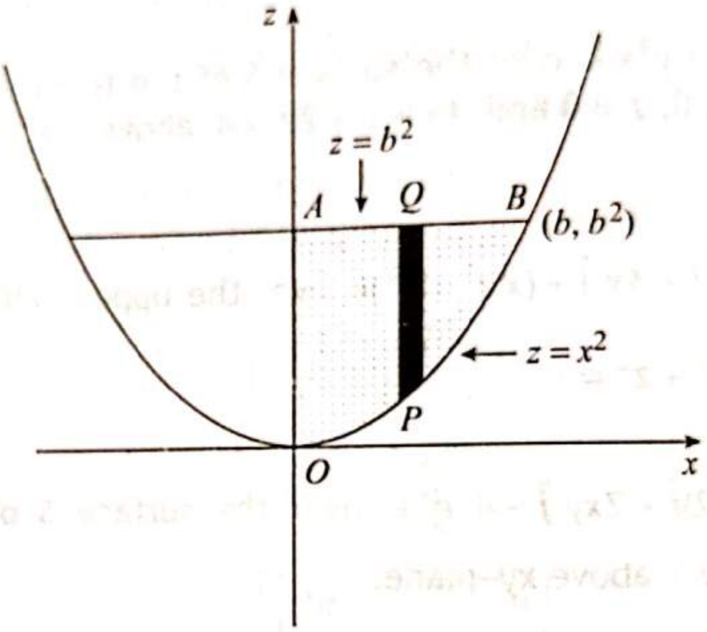

Fig. 1.57

(ii) Along the vertical strip PQ, z varies from x^2 to b^2 and in the region OAB x varies from 0 to b.

$$\iiint\limits_V \bar{F}\,\mathrm{d}V = \int_{x=0}^{b}\int_{z=x^2}^{b^2}\int_{y=0}^{a}(x\hat{i} + y\hat{j} + 2z\hat{k})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_0^b\int_{x^2}^{b^2}\left(x\hat{i}\,|y|_0^a + \hat{j}\left|\frac{y^2}{2}\right|_0^a + 2z\hat{k}\,|y|_0^a\right)\mathrm{d}z\,\mathrm{d}x$$

$$= \int_0^b\int_{x^2}^{b^2}\left(\hat{i}xa + \hat{j}\frac{a^2}{2} + \hat{k}\,2za\right)\mathrm{d}z\,\mathrm{d}x$$

$$= \int_0^b\left(\hat{i}xa|z|_{x^2}^{b^2} + \hat{j}\frac{a^2}{2}|z|_{x^2}^{b^2} + \hat{k}\,a|z^2|_{x^2}^{b^2}\right)\mathrm{d}x$$

$$= \int_0^b\left[\hat{i}\,xa(b^2 - x^2) + \hat{j}\frac{a^2}{2}(b^2 - x^2) + \hat{k}\,a(b^4 - x^4)\right]\mathrm{d}x$$

$$= \left| \hat{i}\, a\left(\frac{b^2x^2}{2} - \frac{x^4}{4}\right) + \hat{j}\,\frac{a^2}{2}\left(b^2x - \frac{x^3}{3}\right) + \hat{k}\, a\left(b^4x - \frac{x^5}{5}\right)\right|_0^b$$

$$= \hat{i}\, a\left(\frac{b^4}{2} - \frac{b^4}{4}\right) + \hat{j}\,\frac{a^2}{2}\left(b^3 - \frac{b^3}{3}\right) + \hat{k}\, a\left(b^5 - \frac{b^5}{5}\right)$$

$$= \frac{ab^4}{4}\hat{i} + \frac{a^2b^3}{3}\hat{j} + \frac{4ab^5}{5}\hat{k}.$$

Example 2

Evaluate $\iiint_V (\nabla \times \bar{F})\,dV$, *where* $\bar{F} = (2x^2 - 3z)\hat{i} - 2xy\,\hat{j} - 4x\,\hat{k}$ *and* V *is the closed region bounded by the planes* $x = 0$, $y = 0$, $z = 0$ *and* $2x + 2y + z = 4$.

Solution

$$\text{(i)}\quad \nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$= \hat{i}\,(0-0) - \hat{j}\,(-4+3) + \hat{k}\,(-2y-0)$$

$$= \hat{j} - 2y\,\hat{k}$$

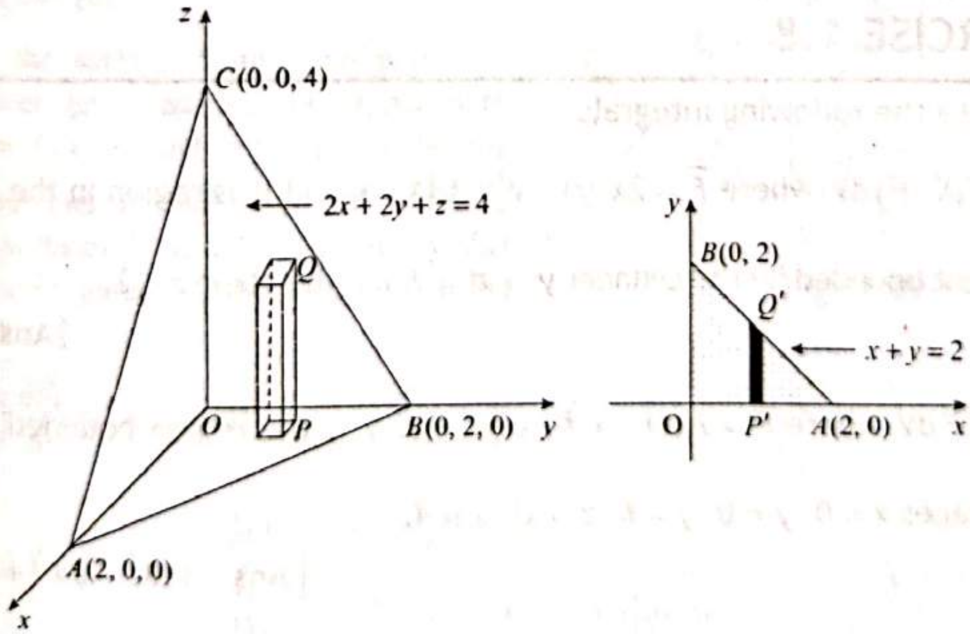

Fig. 1.58

(ii) Along the elementary volume PQ, z varies from 0 to $4-2x-2y$. Along the vertical strip $P'Q'$, y varies from 0 to $2-x$ and in the region OAB, x varies from 0 to 2.

$$\iiint_V (\nabla \times \overline{F})\, dV = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} (\hat{j} - 2y\hat{k})\, dx\, dy\, dz$$

$$= \int_0^2\int_0^{2-x} (\hat{j} - 2y\hat{k}) \left| z \right|_0^{4-2x-2y} dy\, dx$$

$$= \int_0^2\int_0^{2-x} (\hat{j} - 2y\hat{k})(4-2x-2y)\, dy\, dx$$

$$= \int_0^2\int_0^{2-x} \left[(4-2x-2y)\,\hat{j} - 2y\,(4-2x)\,\hat{k} + 4y^2\,\hat{k} \right] dy\, dx$$

$$= \int_0^2 \left[\left\{ (4-2x)\left| y \right|_0^{2-x} - \left| y^2 \right|_0^{2-x} \right\}\hat{j} - \left\{ 2(2-x)\left| y^2 \right|_0^{2-x} - 4\left| \frac{y^3}{3} \right|_0^{2-x} \right\}\hat{k} \right] dx$$

$$= \int_0^2 \left[\left\{ 2(2-x)(2-x) - (2-x)^2 \right\}\hat{j} - \left\{ 2(2-x)(2-x)^2 - \frac{4}{3}(2-x)^3 \right\}\hat{k} \right] dx$$

$$= \int_0^2 \left[(2-x)^2\,\hat{j} - \frac{2}{3}(2-x)^3\,\hat{k} \right] dx$$

$$= \left| \frac{(2-x)^3}{-3}\,\hat{j} - \frac{2}{3}\cdot\frac{(2-x)^4}{-4}\,\hat{k} \right|_0^2$$

$$= \frac{8}{3}\hat{j} - \frac{8}{3}\hat{k}$$

$$= \frac{8}{3}(\hat{j} - \hat{k})$$

EXERCISE 1.8

Evaluate the following integrals:

1. $\iiint_V (\nabla \cdot \overline{F})\, dV$ where $\overline{F} = 2x^2y\,\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k}$ and V is region in the first octant bounded by the cylinder $y^2 + z^2 = 9$ and the plane $z = 2$.

[Ans.: 180]

2. $\iiint_V \overline{F}\, dV$ where $\overline{F} = 2xz\,\hat{i} - x\,\hat{j} + y^2\,\hat{k}$ and V is the region bounded by the surfaces $x = 0$, $y = 0$, $y = 6$, $z = x^2$, $z = 4$.

[Ans.: $128\hat{i} - 24\hat{j} + 384\hat{k}$]

3. $\iiint_V f\,dV$ where $f = 45x^2y$ and V is the region bounded by the planes $4x + 2y + z = 8$, $x = 0$, $y = 0$, $z = 0$.

[Ans. : 128]

4. $\iiint_V \nabla \times \bar{F}\,dV$ where $\bar{F} = (x + 2y)\hat{i} - 3z\,\hat{j} + x\,\hat{k}$ and V is the closed region in the first octant bounded by the plane $2x + 2y + z = 4$.

$\left[\text{Ans. : } \frac{8}{3}(3\hat{i} - \hat{j} + 2\hat{k})\right]$

1.14 GAUSS'S DIVERGENCE THEOREM

If $\bar{F}$ be a vector field having continuous partial derivatives in the region bounded by a closed surface S then $\iiint_V \nabla \cdot \bar{F}\,dV = \iint_S \bar{F} \cdot \hat{n}\,dS$

where $\hat{n}$ is the unit outward normal at any point of the surface S.

Proof Let $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\iiint_V \nabla \cdot \bar{F}\,dV = \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\,dx\,dy\,dz$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz \qquad (1.13)$$

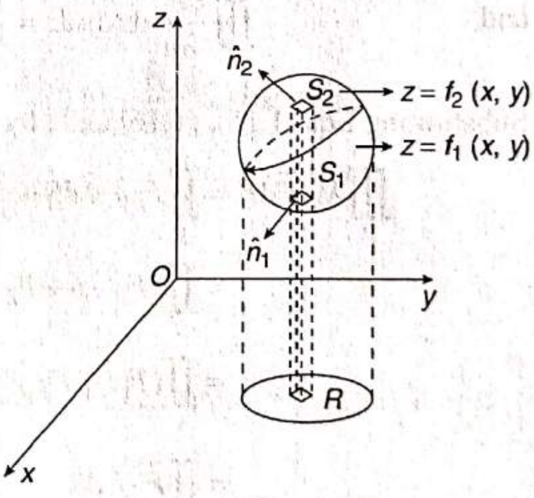

Fig. 1.59

Assume a closed surface S such that any line parallel to the coordinate axes intersects S at most at two points.

Divide the surface S into two parts: S_1, the lower part and S_2, the upper part. Let $z = f_1(x, y)$ and $z = f_2(x, y)$ be the equations and $\hat{n}_1$ and $\hat{n}_2$ be the normals to the surfaces S_1 and S_2 respectively. Let R be the projection of the surface S on the xy-plane.

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R \left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial F_3}{\partial z}\,dz\right] dx\,dy$$

$$= \iint_R F_3(x, y, z)\Big|_{f_1}^{f_2}\,dx\,dy$$

$$= \iint_R \left[F_3(x, y, f_2) - F_3(x, y, f_1)\right] dx\,dy$$

$$= \iint_R F_3(x, y, f_2)\,\mathrm{d}x\,\mathrm{d}y - \iint_R F_3(x, y, f_1)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{...(1.14)}$$

$\mathrm{d}x\,\mathrm{d}y$ = projection of $\mathrm{d}S$ on xy-plane = $\hat{n}\cdot\hat{k}\,\mathrm{d}S$

For surface S_2: $z = f_2(x, y)$

$$\mathrm{d}x\,\mathrm{d}y = \hat{n}_2\cdot\hat{k}\;\mathrm{d}S_2$$

For surface S_1: $z = f_1(x, y)$

$$\mathrm{d}x\,\mathrm{d}y = -\hat{n}_1\cdot\hat{k}\,\mathrm{d}S_1$$

Substituting in Eq. (1.14),

$$\iiint_V \frac{\partial F_3}{\partial z}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{S_2} F_3\,\hat{n}_2\cdot\hat{k}\,\mathrm{d}S_2 - \iint_{S_1} F_3\,(-\hat{n}_1\cdot\hat{k})\,\mathrm{d}S_1$$

$$= \iint_{S_2} F_3\,\hat{n}_2\cdot\hat{k}\,\mathrm{d}S_2 + \iint_{S_1} F_3\,\hat{n}_1\cdot\hat{k}\,\mathrm{d}S_1$$

$$= \iint_S F_3\,\hat{n}\cdot\hat{k}\,\mathrm{d}S \qquad \text{...(1.15)}$$

Similarly, projecting the surface S on yz and zx-planes,

$$\iiint_V \frac{\partial F_1}{\partial x}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_S F_1\,\hat{n}\cdot\hat{i}\,\mathrm{d}S \qquad \text{...(1.16)}$$

and

$$\iiint_V \frac{\partial F_2}{\partial y}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_S F_2\,\hat{n}\cdot\hat{j}\,\mathrm{d}S \qquad \text{...(1.17)}$$

Substituting Eqs (1.15), (1.16) and (1.17) in Eq. (1.13),

$$\iiint_V \nabla\cdot\overline{F}\mathrm{d}V = \iint_S F_1\,\hat{n}\cdot\hat{i}\,\mathrm{d}S + \iint_S F_2\,\hat{n}\cdot\hat{j}\,\mathrm{d}S + \iint_S F_3\,\hat{n}\cdot\hat{k}\,\mathrm{d}S$$

$$= \iint_S (F_1\,\hat{i}\cdot\hat{n} + F_2\,\hat{j}\cdot\hat{n} + F_3\,\hat{k}\cdot\hat{n})\mathrm{d}S$$

$$= \iint_S (F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k})\cdot\hat{n}\,\mathrm{d}S$$

$$= \iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S$$

Hence, $\displaystyle\iiint_V \nabla\cdot\overline{F}\mathrm{d}V = \iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S$

Note: Cartesian form of Gauss's divergence theorem is

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_2}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_S (F_1\,\mathrm{d}y\,\mathrm{d}z + F_2\,\mathrm{d}z\,\mathrm{d}x + F_3\,\mathrm{d}x\,\mathrm{d}y)$$

Example 1

If S is any closed surface, show that $\iint_S curl\,\bar{F}\cdot\hat{n}\,dS = 0$.

Solution

By Gauss's divergence theorem,

$$\iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\text{curl}\,\bar{F}\,dV$$
$$= \iiint_V \nabla\cdot(\nabla\times\bar{F})\,dV$$
$$= 0 \qquad [\because \nabla\cdot(\nabla\times\bar{F}) = 0]$$

Example 2

Using the scalar form of the Gauss's divergence theorem, evaluate $\iint_S (x\,dydz + 2y\,dzdx + 3z\,dxdy)$ *where S is the closed surface of the sphere* $x^2 + y^2 + z^2 = 1$.

Solution

By Gauss's Divergence theorem,

$$\iint_S x\,dy\,dz + 2y\,dz\,dx + 3z\,dx\,dy = \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z)\right] dx\,dy\,dz$$
$$= \iiint_V (1+2+3)\,dx\,dy\,dz$$
$$= 6\iiint_V dx\,dy\,dz$$
$$= 6\left(\frac{4}{3}\pi\right) \left[\because \text{Volume of sphere } x^2+y^2+z^2=1 \text{ is } \frac{4}{3}\pi\right]$$
$$= 8\pi$$

Example 3

If S is any closed surface enclosing volume V and $\bar{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$, *prove that* $\iint_S \bar{F}\cdot\hat{n}\,dS = (a+b+c)V = \frac{4}{3}\pi(a+b+c)$ *where V is the volume bounded by the sphere* $x^2 + y^2 + z^2 = 1$.

Solution

By Gauss's divergence theorem,

$$\iint_S \bar{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\bar{F}\,dV$$

(i) $\bar{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)$$

$$= a + b + c$$

(ii) $$\iint_S \bar{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\bar{F}\,dV$$

$$= \iiint_V (a+b+c)\,dV$$

$$= (a+b+c)V$$

$$= (a+b+c)\frac{4}{3}\pi(1)^3$$

$$= \frac{4}{3}\pi(a+b+c)$$

Example 4

Find $\iint_S \bar{r}\cdot d\bar{S}$ *where S is the surface of the tetrahedron whose vertices are* (1, 0, 0), (0, 1, 0), (0, 0, 1).

Solution

By Gauss's divergence theorem,

$$\iint_S \bar{r}\cdot d\bar{S} = \iiint_V \nabla\cdot\bar{r}\,dV$$

Let

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla\cdot\bar{r} = 1+1+1 = 3$$

$$\iint_S \bar{r}\cdot d\bar{S} = \iiint_V 3\,dV$$

$$= 3\iiint dV$$

$$= 3\int_0^1\int_0^1\int_0^1 dx\,dy\,dz$$

$$= 3\int_0^1\int_0^1 |x|_0^1\,dy\,dz$$

$$= 3\int_0^1\int_0^1 dy\,dz$$

$$= 3\int_0^1 |y|_0^1 \, dz$$

$$= 3\int_0^1 dz$$

$$= 3|z|_0^1$$

$$= 3$$

Example 5

Evaluate $\iint_S (yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}) \cdot d\bar{S}$, *where S is the surface of the sphere in the first octant.*

Solution

By Gauss's divergence theorem,

$$\iint_S \bar{F} \cdot d\bar{S} = \iiint_V \nabla \cdot \bar{F} \, dV \quad \ldots (1)$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) \right] dV$$

$$= 0$$

Example 6

Evaluate $\iint_S (x^3 dy\,dz + x^2 y\,dz\,dx + x^2 z\,dx\,dy)$ *where S is the closed surface consisting of the circular cylinder* $x^2 + y^2 = a^2$, $z = 0$ *and* $z = b$.

Solution

By Gauss's divergence theorem,

$$\iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\,dy\,dz \quad \ldots (1)$$

(i) $F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = x^3\,dy\,dz + x^2 y\,dz\,dx + x^2 z\,dx\,dy$

$$F_1 = x^3, \quad F_2 = x^2 y, \quad F_3 = x^2 z$$

(ii) $\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial z}(x^2 z)$

$$= 3x^2 + x^2 + x^2 = 5x^2$$

(iii) $\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\,dy\,dz = \iiint 5x^2\,dx\,dy\,dz$

(iii) $\nabla\cdot\overline{F}=\frac{\partial}{\partial x}x+\frac{\partial}{\partial y}y+\frac{\partial}{\partial z}z=3$

(iv) $\iiint_V \nabla\cdot\overline{F}\,dV=\iiint_V 3\,dV$

$= 3$ (Volume of the region bounded by the ellipsoid)

$$=3\cdot\frac{4}{3}\pi\frac{1}{\sqrt{a}}\cdot\frac{1}{\sqrt{b}}\cdot\frac{1}{\sqrt{c}}\quad\left[\because\ \frac{x^2}{\left(\frac{1}{\sqrt{a}}\right)^2}+\frac{y^2}{\left(\frac{1}{\sqrt{b}}\right)^2}+\frac{z^2}{\left(\frac{1}{\sqrt{c}}\right)^2}=1\right]$$

$$=\frac{4\pi}{\sqrt{abc}}$$

From Eq. (1),

$$\iint_S \frac{dS}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}=\frac{4\pi}{\sqrt{abc}}$$

Example 9

Evaluate $\iint_S \overline{F}\cdot d\overline{S}$ *using divergence theorem where* $\overline{F}=x^3\hat{i}+y^3\hat{j}+z^3\hat{k}$ *and S is the surface of the sphere* $x^2+y^2+z^2=a^2$.

Solution

By Gauss's divergence theorem,

$$\iint_S \overline{F}\cdot d\overline{S}=\iiint_V \nabla\cdot\overline{F}\,dV$$

(i) $\overline{F}=x^3\hat{i}+y^3\hat{j}+z^3\hat{k}$

$$\nabla\cdot\overline{F}=\frac{\partial}{\partial x}x^3+\frac{\partial}{\partial y}y^3+\frac{\partial}{\partial z}z^3$$

$$=3x^2+3y^2+3z^2$$

(ii) $\iiint_V \nabla\cdot\overline{F}\,dV=3\iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz$

Putting $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$ equation of the sphere $x^2+y^2+z^2$ reduces to $r=a$ and $dx\,dy\,dz=r^2\sin\theta\,dr\,d\theta\,d\phi$

For the complete sphere,

r varies from 0 to a
θ varies from 0 to π
ϕ varies from 0 to 2π

$$\iiint_V \nabla\cdot\overline{F}\,dV = 3\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^2\cdot r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 3\int_0^{2\pi} d\phi \cdot \int_0^{\pi} \sin\theta\, d\theta \cdot \int_0^{a} r^4\, dr$$

$$= 3|\phi|_0^{2\pi}\,|-\cos\theta|_0^{\pi}\left|\frac{r^5}{5}\right|_0^a$$

$$= 3\cdot 2\pi(-\cos\pi + \cos 0)\frac{a^5}{5}$$

$$= \frac{12}{5}\pi a^5$$

From Eq. (1),

$$\iint_S \overline{F}\cdot d\overline{S} = \frac{12}{5}\pi a^5$$

Example 10

Evaluate $\iint \overline{F}\cdot d\overline{S}$ *using Gauss's divergence theorem where* $\overline{F} = 2xy\hat{i} + yz^2\hat{j} + zx\hat{k}$ *and S is the surface of the region bounded by* $x = 0, y = 0, z = 0, y = 3, x + 2z = 6$.

Solution

By Gauss's divergence theorem,

$$\iint_S \overline{F}\cdot d\overline{S} = \iiint_V \nabla\cdot\overline{F}\,dV \qquad \text{... (1)}$$

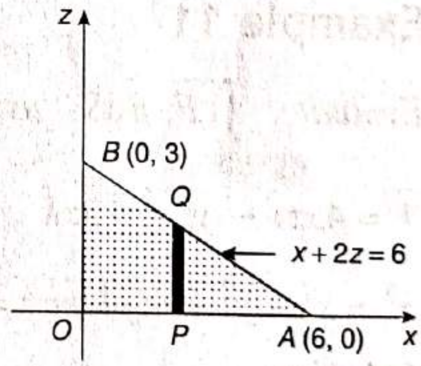

Fig. 1.61

(i) $\overline{F} = 2xy\hat{i} + yz^2\hat{j} + zx\hat{k}$

$$\nabla\cdot\overline{F} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(zx)$$

$$= 2y + z^2 + x$$

(ii) $\iiint_V \nabla\cdot\overline{F} = \iiint_V (2y + z^2 + x)\,dx\,dy\,dz$

In the given region, y varies from 0 to 3.

In xz-plane, region is bounded by the lines $x = 0, z = 0, x + 2z = 6$.

Along the vertical strip PQ, z varies from 0 to $\frac{6-x}{2}$ and in the region, x varies from 0 to 6.

$$\iiint_V \nabla\cdot\overline{F} = \int_{x=0}^{6}\int_{z=0}^{\frac{6-x}{2}}\int_{y=0}^{3}(2y + z^2 + x)\,dy\,dz\,dx$$

$$= \int_0^6\int_0^{\frac{6-x}{2}} \left|y^2 + z^2 y + xy\right|_0^3 dz\,dx$$

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$$= \int_0^6 \int_0^{\frac{6-x}{2}} (9 + 3z^2 + 3x)\,dz\,dx$$

$$= \int_0^6 \left| 9z + z^3 + 3xz \right|_0^{\frac{6-x}{2}} dx$$

$$= \int_0^6 \left[9\left(\frac{6-x}{2}\right) + \left(\frac{6-x}{2}\right)^3 + 3x\left(\frac{6-x}{2}\right) \right] dx$$

$$= \int_0^6 \left[27 + \frac{9x}{2} - \frac{3x^2}{2} + \left(\frac{6-x}{2}\right)^3 \right] dx$$

$$= \left| 27x + \frac{9}{2} \cdot \frac{x^2}{2} - \frac{x^3}{2} + \frac{1}{8} \cdot \frac{(6-x)^4}{-4} \right|_0^6$$

$$= 162 + 81 - 108 + \frac{6^4}{32}$$

$$= \frac{351}{2}$$

From Eq. (1),

$$\iint_S \overline{F} \cdot d\overline{S} = \frac{351}{2}$$

Example 11

Evaluate $\iint_S \overline{F} \cdot \hat{n}\,dS$ *using Gauss's divergence theorem whe[re]* $\overline{F} = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$ *over the region bounded by the con[e]* $z^2 = x^2 + y^2$ *and plane* $z = 4$, *above the xy plane.*

Solution

By Gauss's divergence theorem,

$$\iint_S \overline{F} \cdot \hat{n}\,dS = \iiint_V \nabla \cdot \overline{F}\,dV$$

(i) $\overline{F} = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$

$$\nabla \cdot \overline{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(3z)$$

$$= 4z + xz^2 + 3$$

(ii) $\iiint_V \nabla \cdot \overline{F}\,dV = \iiint (4z + xz^2 + 3)\,dx\,dy\,dz$

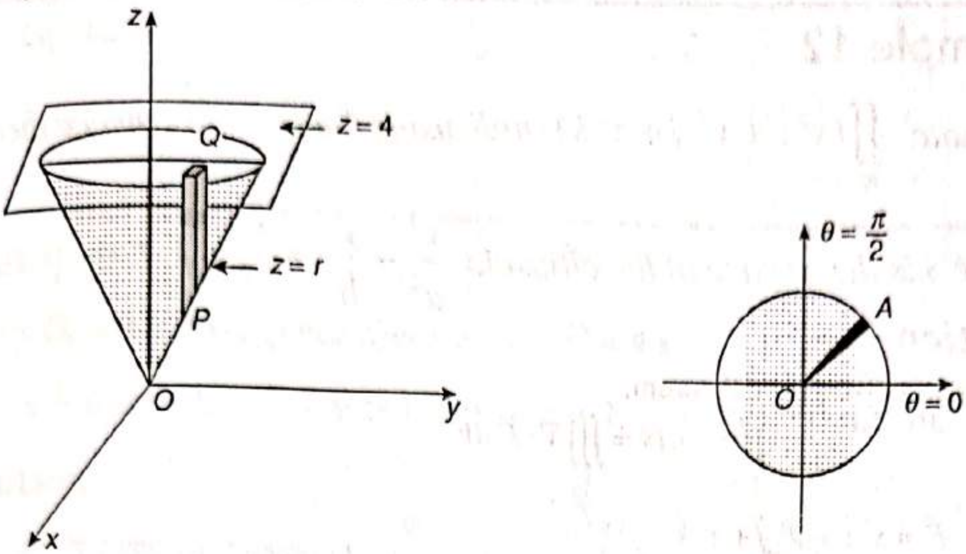


Fig. 1.62

Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, equation of the cone $z^2 = x^2 + y^2$ reduces to $z = r$, and $dx\,dy\,dz = r\,dr\,d\theta\,dz$. Along the elementary volume PQ, z varies from r to 4. Projection of the region in $r\theta$-plane is the curve of intersection of the cone $r = z$ and plane $z = 4$ which is a circle $r = 4$.

Along the radius vector OA, r varies from 0 to 4 and for the complete circle, θ varies from 0 to 2π.

$$\iiint_V \nabla\cdot\overline{F}\,dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{4}\int_{z=r}^{4}(4z + r\cos\theta\cdot z^2 + 3)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4 \left|2z^2 + r\cos\theta\cdot\frac{z^3}{3} + 3z\right|_r^4 r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4\left[2r(16-r^2) + \frac{r^2\cos\theta}{3}(64-r^3) + 3r(4-r)\right]dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4\left(44r + \frac{64}{3}r^2\cos\theta - 3r^2 - 2r^3 - \frac{r^5}{3}\cos\theta\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left|22r^2 + \frac{64}{3}\cos\theta\cdot\frac{r^3}{3} - r^3 - \frac{r^4}{2} - \frac{1}{3}\cdot\frac{r^6}{6}\cos\theta\right|_0^4 d\theta$$

$$= \int_0^{2\pi}\left(160 + \frac{2048}{9}\cos\theta\right)d\theta$$

$$= 160\left|\theta\right|_0^{2\pi} + \frac{2048}{9}\left|\sin\theta\right|_0^{2\pi}$$

$$= 160\cdot 2\pi + 0$$

$$= 320\pi$$

From Eq. (1),

$$\iint_S \overline{F}\cdot\hat{n}\,dS = 320\pi$$

Example 12

Evaluate $\iint\limits_S (x^2\,\hat{i} + y^2\,\hat{j} + z^2\,\hat{k}) \cdot \hat{n}\,dS$ *using Gauss's divergence theore*

where S is the surface of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$

Solution

By Gauss's divergence theorem,

$$\iint\limits_S \bar{F} \cdot \hat{n}\,dS = \iiint\limits_V \nabla \cdot \bar{F}\,dV$$

(i) $\bar{F} = x^2\,\hat{i} + y^2\,\hat{j} + z^2\,\hat{k}$

$$\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

(ii) $\iiint\limits_V \nabla \cdot \bar{F}\,\mathrm{d}V = \iiint\limits_V (2x + 2y + 2z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

Putting $x = ar\sin\theta\cos\phi$, $y = br\sin\theta\sin\phi$, $z = cr\cos\theta$, equation of the ellips

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ reduces to $r = 1$ and $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = abc\,r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$.

For the complete ellipsoid,

r varies from 0 to 1
θ varies from 0 to π
ϕ varies from 0 to 2π

$$\iiint\limits_V \nabla \cdot \bar{F}\,\mathrm{d}V = 2\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} [(ar\sin\theta\cos\phi + br\sin\theta\sin\phi + cr\cos\theta)\,abc\,r^2\sin\theta]\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 2\int_0^{2\pi}\int_0^{\pi} (a\sin^2\theta\cos\phi + b\sin^2\theta\sin\phi + c\cos\theta\sin\theta)\left|\frac{r^4}{4}\right|_0^1 abc\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= \frac{abc}{2}\left[\int_0^{\pi}\left(a\sin^2\theta\left|\sin\phi\right|_0^{2\pi} + b\sin^2\theta\left|-\cos\phi\right|_0^{2\pi} + c\cos\theta\sin\theta\left|\phi\right|_0^{2\pi}\right)\mathrm{d}\theta\right.$$

$$= \frac{abc}{2}\int_0^{\pi}(0 + 0 + c\cos\theta\sin\theta \cdot 2\pi)\,\mathrm{d}\theta$$

$$= \pi\frac{abc^2}{2}\int_0^{\pi}\sin 2\theta\,\mathrm{d}\theta$$

$$= \pi\frac{abc^2}{2}\left|-\frac{\cos 2\theta}{2}\right|_0^{\pi}$$

$$= \frac{-\pi abc^2}{4}(\cos 2\pi - \cos 0)$$

$$= 0$$

From Eq. (1),

$$\iint_S \overline{F}\cdot\hat{n}\,dS = 0$$

Example 13

Verify Gauss's divergence theorem for $\overline{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ *over the cube* $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.$ **[Summer 2012]**

Solution

By Gauss's divergence theorem,

$$\iiint_V \nabla\cdot\overline{F}\mathrm{d}V = \iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S$$

(i) $\overline{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

$$\nabla\cdot\overline{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y$$

$$= 4z - y$$

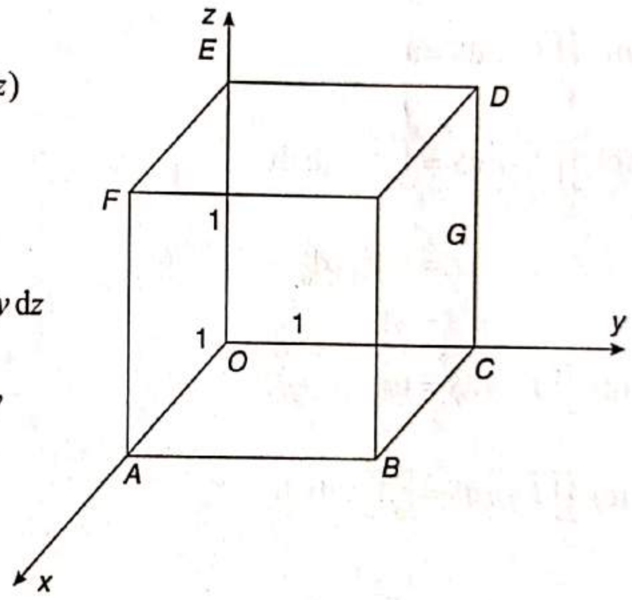


Fig. 1.63

(ii) $\iiint_V \nabla\cdot\overline{F}\mathrm{d}V = \int_0^1\int_0^1\int_0^1 (4z - y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

$$= \int_0^1\int_0^1 \left|2z^2 - yz\right|_0^1 \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 \mathrm{d}x \int_0^1 (2 - y)\mathrm{d}y$$

$$= |x|_0^1 \left|2y - \frac{y^2}{2}\right|_0^1$$

$$= 2 - \frac{1}{2}$$

$$= \frac{3}{2} \qquad \text{... (1)}$$

(iii) Surface S of the cube consists of 6 surfaces, S_1, S_2, S_3, S_4, S_5 and S_6.

$$\iint_S \overline{F}\cdot\hat{n}\,\mathrm{d}S = \iint_{S_1} \overline{F}\cdot\hat{n}\,\mathrm{d}S + \iint_{S_2} \overline{F}\cdot\hat{n}\,\mathrm{d}S + \iint_{S_3} \overline{F}\cdot\hat{n}\,\mathrm{d}S$$

$$+ \iint_{S_4} \overline{F}\cdot\hat{n}\,\mathrm{d}S + \iint_{S_5} \overline{F}\cdot\hat{n}\,\mathrm{d}S + \iint_{S_6} \overline{F}\cdot\hat{n}\,\mathrm{d}S \qquad \text{... (2)}$$

Face	Equation	Outward Normal $\hat{n}$	$\overline{F}\cdot\hat{n}$	dS
$S_1 : ABGF$	$x = 1$	$\hat{i}$	$4z$	$dy\,dz$
$S_2 : OCDE$	$x = 0$	$-\hat{i}$	0	$dy\,dz$
$S_3 : BCDG$	$y = 1$	$\hat{j}$	-1	$dz\,dx$
$S_4 : OAFE$	$y = 0$	$-\hat{j}$	0	$dz\,dx$
$S_5 : DEFG$	$z = 1$	$\hat{k}$	y	$dx\,dy$
$S_6 : OABC$	$z = 0$	$-\hat{k}$	0	$dx\,dy$

(a) $\iint_{S_1} \overline{F}\cdot\hat{n}\,dS = \int_0^1\int_0^1 4z\,dy\,dz$

$$= 4\left|y\right|_0^1\left|\frac{z^2}{2}\right|_0^1$$

$$= 2$$

(b) $\iint_{S_2} \overline{F}\cdot\hat{n}\,dS = 0$

(c) $\iint_{S_3} \overline{F}\cdot\hat{n}\,dS = \int_0^1\int_0^1 -dz\,dx$

$$= -\left|z\right|_0^1\left|x\right|_0^1$$

$$= -1$$

(d) $\iint_{S_4} \overline{F}\cdot\hat{n}\,dS = 0$

(e) $\iint_{S_5} \overline{F}\cdot\hat{n}\,dS = \int_0^1\int_0^1 y\,dx\,dy$

$$= \left|\frac{y^2}{2}\right|_0^1\left|x\right|_0^1$$

$$= \frac{1}{2}$$

(f) $\iint_{S_6} \overline{F}\cdot\hat{n}\,dS = 0$

Substituting in Eq. (2),

$$\iint_S \overline{F}\cdot\hat{n}\,dS = 2 + 0 + -1 + 0 + \frac{1}{2}$$

$$= \frac{3}{2} \qquad \text{...(3)}$$

From Eqs. (1) and (3),

$$\iiint_V \nabla \cdot \bar{F}\, dV = \iint_S \bar{F} \cdot \hat{n}\, dS$$

Hence, Gauss's divergence theorem is verified.

Example 14

Verify Gauss's divergence theorem for the vector function $\bar{F} = \left(x^3 - yz\right)\hat{i} - 2x^2 y\hat{j} + 2\hat{k}$ *over the cube bounded by* $x = 0, y = 0, z = 0$ *and* $x = a, y = a, z = a$.

Solution

By Gauss's divergence theorem,

$$\iiint_V \nabla \cdot \bar{F} dV = \iint_S \bar{F} \cdot \hat{n}\, dS$$

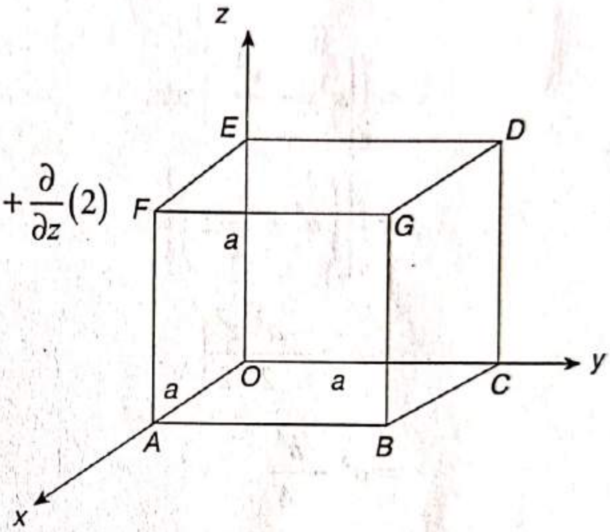


Fig. 1.64

(i) $\bar{F} = \left(x^3 - yz\right)\hat{i} - 2x^2 y\hat{j} + 2\hat{k}$

$$\nabla \cdot \bar{F} = \frac{\partial}{\partial x}\left(x^3 - yz\right) + \frac{\partial}{\partial y}\left(-2x^2 y\right) + \frac{\partial}{\partial z}(2)$$

$$= 3x^2 - 2x^2$$

$$= x^2$$

(ii) $$\iiint_V \nabla \cdot \bar{F}\, dV = \int_0^a \int_0^a \int_0^a x^2 dx\, dy\, dz$$

$$= \int_0^a \int_0^a \left|\frac{x^3}{3}\right|_0^a dy\, dz$$

$$= \int_0^a \int_0^a \frac{a^3}{3} dy\, dz$$

$$= \frac{a^3}{3} \int_0^a |y|_0^a\, dz$$

$$= \int_0^a \frac{a^4}{3} dz$$

$$= \frac{a^4}{3} |z|_0^a$$

$$= \frac{a^5}{3} \qquad \cdots(1)$$

(iii) Surfaces S of the cube consists of 6 surfaces S_1, S_2, S_3, S_4, S_5 and S_6.

$$\iint_S \bar{F} \cdot \hat{n}\, dS = \iint_{S_1} \bar{F} \cdot \hat{n}\, dS + \iint_{S_2} \bar{F} \cdot \hat{n}\, dS + \iint_{S_3} \bar{F} \cdot \hat{n}\, dS + \iint_{S_4} \bar{F} \cdot \hat{n}\, dS + \iint_{S_5} \bar{F} \cdot \hat{n}\, dS + \iint_{S_6} \bar{F} \cdot \hat{n}\, dS \qquad \cdots(2)$$

Face	Equation	Outward Normal $\hat{n}$	$\bar{F}\cdot\hat{n}$	dS
$S_1 : ABGF$	$x = a$	$\hat{i}$	$a^3 - 4z$	$dy\,dz$
$S_2 : OCDE$	$x = 0$	$-\hat{i}$	yz	$dy\,dz$
$S_3 : BCDG$	$y = a$	$\hat{j}$	$-2x^2a$	$dz\,dx$
$S_4 : OAFE$	$y = 0$	$-\hat{j}$	0	$dz\,dx$
$S_5 : DEFG$	$z = a$	$\hat{k}$	2	$dx\,dy$
$S_6 : OABC$	$z = 0$	$-\hat{k}$	-2	$dx\,dy$

(a) $$\iint_{S_1} \bar{F}\cdot\hat{n}\,dS = \int_0^a\int_0^a \left(a^3 - yz\right) dy\,dz$$

$$= \int_0^a \left| a^3y - \frac{y^2}{2}z \right|_0^a dz$$

$$= \int_0^a \left[\left(a^4 - \frac{a^2}{2}z\right)\right] dz$$

$$= \left| a^4z - \frac{a^2}{2}\frac{z^2}{2} \right|_0^a$$

$$= a^5 - \frac{a^4}{4}$$

(b) $$\iint_{S_2} \bar{F}\cdot\hat{n}\,dS = \int_0^a\int_0^a yz\,dy\,dz$$

$$= \int_0^a \left| \frac{y^2}{2}z \right|_0^a dz$$

$$= \int_0^a \frac{a^2}{2} z\,dz$$

$$= \frac{a^2}{2}\left| \frac{z^2}{2} \right|_0^a$$

$$= \frac{a^2}{2}\left(\frac{a^2}{2}\right)$$

$$= \frac{a^4}{4}$$

(c) $$\iint_{S_3} \bar{F}\cdot\hat{n}\,dS = \int_0^a\int_0^a -2x^2a\,dz\,dx$$

$$= -2a\int_0^a\int_0^a x^2 \, dz \, dx$$

$$= -2a\int_0^a \left|\frac{x^3}{3}\right|_0^a dx$$

$$= -2a\int_0^a \frac{a^3}{3} dx$$

$$= -\frac{2a^4}{3}\int_0^a dx$$

$$= -\frac{2a^4}{3}|x|_0^a$$

$$= -\frac{2a^5}{3}$$

(*d*) $\iint_{S_4} \bar{F}\cdot\hat{n}\, dS = 0$

(*e*) $\iint_{S_5} \bar{F}\cdot\hat{n}\, dS = \int_0^a\int_0^a 2 \, dx \, dy$

$$= 2\int_0^a |x|_0^a \, dy$$

$$= 2\int_0^a a \, dy$$

$$= 2a|y|_0^a$$

$$= 2a(a)$$

$$= 2a^2$$

(*f*) $\iint_{S_6} \bar{F}\cdot\hat{n}\, dS = \int_0^a\int_0^a (-2) \, dx \, dy$

$$= -2\int_0^a |x|_0^a \, dy$$

$$= -2\int_0^a a \, dy$$

$$= -2a|y|_0^a$$

$$= -2a(a)$$

$$= -2a^2$$

Substituting in Eq. (2),

$$\iint_S \bar{F}\cdot\hat{n}\,dS = \left(a^5 - \frac{a^4}{4}\right) + \frac{a^4}{4} + \left(-\frac{2a^5}{3}\right) + 0 + 2a^2 + \left(-2a^2\right)$$

$$= a^5 - \frac{2}{3}a^5$$

$$= \frac{a^5}{3} \quad \cdots(3)$$

From Eqs (1) and (3),

$$\iiint_V \nabla\cdot\bar{F}\,dV = \iint_S \bar{F}\cdot\hat{n}\,dS = \frac{a^5}{3}$$

Hence, Gauss's divergence theorem is verified.

Example 15

Verify Gauss's divergence theorem for $\bar{F} = \left(x^2 - yz\right)\hat{i} + \left(y^2 - zx\right)\hat{j} + \left(z^2 - xy\right)\hat{k}$ *taken over the rectangular parallelepiped bounded by* $0 \le x \le a, 0 \le y \le b, 0 \le z \le c.$

Solution

By Gauss's divergence theorem,

$$\iiint_V \nabla\cdot\bar{F}\,dV = \iint_S \bar{F}\cdot\hat{n}\,dS$$

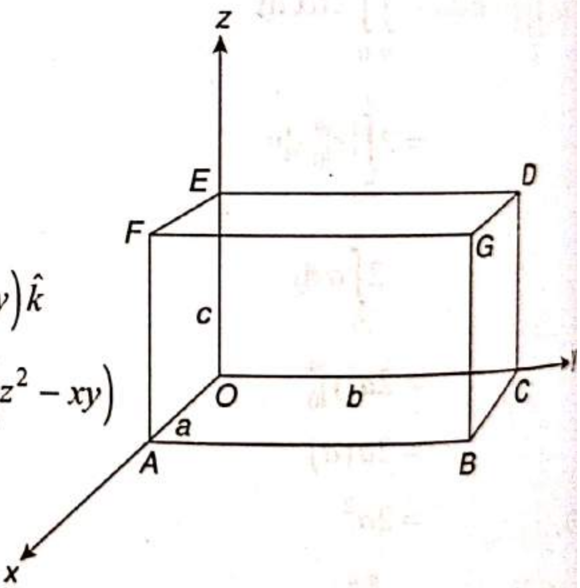


Fig. 1.65

(i) $\bar{F} = \left(x^2 - yz\right)\hat{i} + \left(y^2 - zx\right)\hat{j} + \left(z^2 - xy\right)\hat{k}$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}\left(x^2 - yz\right) + \frac{\partial}{\partial y}\left(y^2 - zx\right) + \frac{\partial}{\partial z}\left(z^2 - xy\right)$$

$$= 2x + 2y + 2z$$

$$= 2(x + y + z)$$

(ii) $$\iiint_V \nabla\cdot\bar{F}\,dV = \int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a} 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^c\int_0^b \left|\frac{x^2}{2} + (y+z)x\right|_0^a dy\,dz$$

$$= 2\int_0^c\int_0^b \left[\frac{a^2}{2} + (y+z)a\right] dy\,dz$$

$$= 2\int_0^c \left|\frac{a^2}{2}y + a\left(\frac{y^2}{2} + zy\right)\right|_0^b dz$$

$$= 2\int_0^c \left[\frac{a^2}{2}b + a\left(\frac{b^2}{2} + zb\right)\right] dz$$

$$= 2\int_0^c \left(\frac{a^2b + ab^2}{2} + abz\right) dz$$

$$= 2\left|\frac{a^2b + ab^2}{2}z + ab\frac{z^2}{2}\right|_0^c$$

$$= 2\left[\frac{a^2bc + ab^2c + abc^2}{2}\right]$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc(a + b + c) \qquad \cdots(1)$$

(iii) Surface S of the rectangular parallelepiped consists of 6 surfaces S_1, S_2, S_3, S_4, S_5 and S_6.

$$\iint_S \bar{F}\cdot\hat{n}\,dS = \iint_{S_1} \bar{F}\cdot\hat{n}\,dS + \iint_{S_2} \bar{F}\cdot\hat{n}\,dS + \iint_{S_3} \bar{F}\cdot\hat{n}\,dS + \iint_{S_4} \bar{F}\cdot\hat{n}\,dS + \iint_{S_5} \bar{F}\cdot\hat{n}\,dS + \iint_{S_6} \bar{F}\cdot\hat{n}\,dS \qquad \cdots(2)$$

Face	Equation	Outward Normal $\hat{n}$	$\bar{F}\cdot\hat{n}$	dS
$S_1 : ABGF$	$x = a$	$\hat{i}$	$a^2 - yz$	$dy\,dz$
$S_2 : OCDE$	$x = 0$	$-\hat{i}$	yz	$dy\,dz$
$S_3 : BCDG$	$y = b$	$\hat{j}$	$b^2 - zx$	$dz\,dx$
$S_4 : OAFE$	$y = 0$	$-\hat{j}$	zx	$dz\,dx$
$S_5 : DEFG$	$z = c$	$\hat{k}$	$c^2 - xy$	$dx\,dy$
$S_6 : OABC$	$z = 0$	$-\hat{k}$	xy	$dx\,dy$

(*a*) $$\iint_{S_1} \bar{F}\cdot\hat{n}\,dS = \int_0^c\int_0^b (a^2 - yz)\,dy\,dz$$

$$= \int_0^c \left|a^2y - \frac{y^2}{2}z\right|_0^b dz$$

$$= \int_0^c \left[a^2b - \frac{b^2}{2}z\right] dz$$

$$= \left|a^2bz - \frac{b^2}{2}\frac{z^2}{2}\right|_0^c$$

$$= a^2bc - \frac{b^2c^2}{4}$$

$$(b) \iint_{S_2} \bar{F}\cdot\hat{n}\,\mathrm{d}S = \int_0^c\int_0^b yz\,\mathrm{d}y\,\mathrm{d}z$$

$$= \left|\frac{y^2}{2}\right|_0^b \left|\frac{z^2}{2}\right|_0^c$$

$$= \frac{b^2}{2}\cdot\frac{c^2}{2}$$

$$= \frac{b^2c^2}{4}$$

$$(c) \iint_{S_3} \bar{F}\cdot\hat{n}\,\mathrm{d}S = \int_0^a\int_0^c \left(b^2 - zx\right)\mathrm{d}z\,\mathrm{d}x$$

$$= \int_0^a \left|b^2 z - \frac{z^2}{2}x\right|_0^c \mathrm{d}x$$

$$= \int_0^a \left[b^2 c - \frac{c^2}{2}x\right]\mathrm{d}x$$

$$= \left|b^2 cx - \frac{c^2}{2}\frac{x^2}{2}\right|_0^a$$

$$= b^2 ca - \frac{c^2a^2}{4}$$

$$(d) \iint_{S_4} \bar{F}\cdot\hat{n}\,\mathrm{d}S = \int_0^a\int_0^c zx\,\mathrm{d}z\,\mathrm{d}x$$

$$= \left|\frac{z^2}{2}\right|_0^c \left|\frac{x^2}{2}\right|_0^a$$

$$= \frac{c^2}{2}\cdot\frac{a^2}{2}$$

$$= \frac{c^2a^2}{4}$$

$$(e) \iint_{S_5} \bar{F}\cdot\hat{n}\,\mathrm{d}S = \int_0^b\int_0^a \left(c^2 - xy\right)\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^b \left|c^2 x - \frac{x^2}{2}y\right|_0^a \mathrm{d}y$$

$$= \int_0^b \left(ac^2 - \frac{a^2}{2}y\right)\mathrm{d}y$$

$$= \left| ac^2y - \frac{a^2}{2}\frac{y^2}{2} \right|_0^b$$

$$= ac^2b - \frac{a^2b^2}{4}$$

(f) $$\iint_{S_6} \overline{F} \cdot \hat{n}\, dS = \int_0^b \int_0^a xy\, dx\, dy$$

$$= \left| \frac{x^2}{2} \right|_0^a \left| \frac{y^2}{2} \right|_0^2$$

$$= \frac{a^2}{2} \cdot \frac{b^2}{2}$$

$$= \frac{a^2b^2}{4}$$

Substituting in Eq. (2),

$$\iint_S \overline{F} \cdot \hat{n}\, dS = a^2bc - \frac{b^2c^2}{4} + \frac{b^2c^2}{4} + b^2ca - \frac{c^2a^2}{4} + \frac{a^2c^2}{4} + ac^2b - \frac{a^2b^2}{4} + \frac{a^2b^2}{4}$$

$$= a^2bc + b^2ca + ac^2b$$

$$= abc(a+b+c) \qquad \cdots(3)$$

From Eqs (1) and (3),

$$\iiint_V \nabla \cdot \overline{F}\, dV = \iint_S \overline{F} \cdot \hat{n}\, dS = abc(a+b+c)$$

Hence, Gauss's divergence theorem is verified.

Example 16

Verify Gauss's divergence theorem for $\overline{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ *where S is the surface of the cuboid formed by the planes* $x = 0, x = a, y = 0, y = b, z = 0$ *and* $z = c$.

Solution

By Gauss's divergence theorem,

$$\iiint_V \nabla \cdot \overline{F}\, dV = \iint_S \overline{F} \cdot \hat{n}\, dS$$

(i) $\overline{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

$$\nabla \cdot \overline{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$
$$= 2(x + y + z)$$

(ii) $\iiint_V \nabla \cdot \bar{F}\, dV = \int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} 2(x+y+z)\, dx\, dy\, dz$

$$= 2\int_0^c \int_0^b \left| \frac{x^2}{2} + (y+z)x \right|_0^a dy\, dz$$

$$= 2\int_0^c \int_0^b \left[\frac{a^2}{2} + (y+z)a \right] dy\, dz$$

$$= 2\int_0^c \left| \frac{a^2}{2}y + a\left(\frac{y^2}{2} + zy\right) \right|_0^b dz$$

$$= 2\int_0^c \left[\frac{a^2}{2}b + a\left(\frac{b^2}{2} + zb\right) \right] dz$$

$$= 2\int_0^c \left(\frac{a^2b + ab^2}{2} + abz \right) dz$$

$$= 2\left| \frac{a^2b + ab^2}{2} z + ab\frac{z^2}{2} \right|_0^c$$

$$= 2\left[\frac{a^2bc + ab^2c + abc^2}{2} \right]$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc(a+b+c) \quad \cdots(1)$$

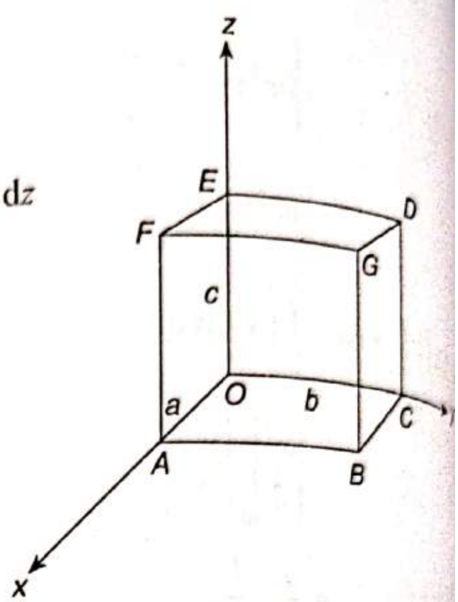


Fig. 1.66

(iii) Surface S of the cuboid consists of 6 surfaces S_1, S_2, S_3, S_4, S_5 and S_6.

$$\iint_S \bar{F}\cdot\hat{n}\, dS = \iint_{S_1} \bar{F}\cdot\hat{n}\, dS + \iint_{S_2} \bar{F}\cdot\hat{n}\, dS + \iint_{S_3} \bar{F}\cdot\hat{n}\, dS + \iint_{S_4} \bar{F}\cdot\hat{n}\, dS + \iint_{S_5} \bar{F}\cdot\hat{n}\, dS + \iint_{S_6} \bar{F}\cdot\hat{n}\, dS \quad \cdots(2)$$

Face	Equation	Outward Normal $\hat{n}$	$\bar{F}\cdot\hat{n}$	dS
$S_1 : ABGF$	$x = a$	$\hat{i}$	a^2	dy dz
$S_2 : OCDE$	$x = 0$	$-\hat{i}$	0	dy dz
$S_3 : BCDG$	$y = b$	$\hat{j}$	b^2	dz dx
$S_4 : OAFE$	$y = 0$	$-\hat{j}$	0	dz dx
$S_5 : DEFG$	$z = c$	$\hat{k}$	c^2	dx dy
$S_6 : OABC$	$z = 0$	$-\hat{k}$	0	dx dy

$$(a)\iint_{S_1} \bar{F}\cdot\hat{n}\,dS = \int_0^c\int_0^b a^2\,dy\,dz$$

$$= a^2\int_0^c |y|_0^b\,dz$$

$$= a^2 b\int_0^c dz$$

$$= a^2 b|z|_0^c$$

$$= a^2 bc$$

$$(b)\iint_{S_2} \bar{F}\cdot\hat{n}\,dS = 0$$

$$(c)\iint_{S_3} \bar{F}\cdot\hat{n}\,dS = \int_0^a\int_0^c b^2\,dz\,dx$$

$$= b^2\int_0^a |z|_0^c\,dx$$

$$= b^2\int_0^a c\,dx$$

$$= b^2 c|x|_0^a$$

$$= ab^2c$$

$$(d)\iint_{S_4} \bar{F}\cdot\hat{n}\,dS = 0$$

$$(e)\iint_{S_5} \bar{F}\cdot\hat{n}\,dS = \int_0^b\int_0^a c^2\,dx\,dy$$

$$= c^2\int_0^b |x|_0^a\,dy$$

$$= c^2 a\int_0^b dy$$

$$= c^2 a|y|_0^b$$

$$= abc^2$$

$$(f)\iint_{S_6} \bar{F}\cdot\hat{n}\,dS = 0$$

Substituting in Eq. (2),

$$\iint_S \overline{F}\cdot\hat{n}\,dS = a^2bc + 0 + ab^2c + 0 + abc^2 + 0$$
$$= abc(a+b+c) \quad \ldots(3)$$

From Eqs (1) and (3),

$$\iiint_V \nabla\cdot\overline{F}\,dV = \iint_S \overline{F}\cdot\hat{n}\,dS = abc(a+b+c)$$

Hence, Gauss's divergence theorem is verified.

Example 17

Verify Gauss's divergence theorem for $\overline{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ *over the region bounded by the cylinder* $y^2+z^2=9$ *and the plane* $x=2$ *in the first octant.*

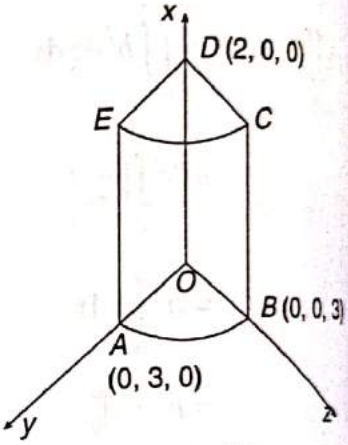


Fig. 1.67

Solution

By Gauss's divergence theorem,

$$\iiint_V \nabla\cdot\overline{F}\,dV = \iint_S \overline{F}\cdot\hat{n}\,dS$$

(i) $\overline{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$

$$\nabla\cdot\overline{F} = \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)$$
$$= 4xy - 2y + 8xz$$

(ii) $\iiint_V \nabla\cdot\overline{F}\,dV = \iiint (4xy - 2y + 8xz)\,dx\,dy\,dz$

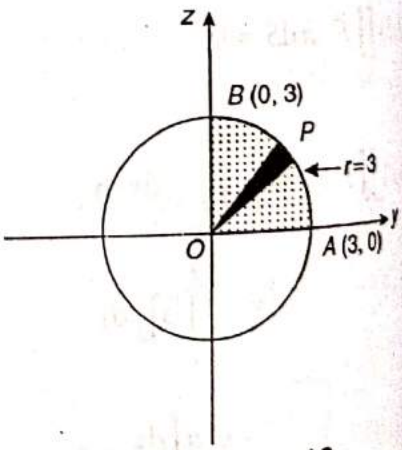


Fig. 1.68

For the given region, x varies from 0 to 2. Putting $y = r\cos\theta$, $z = r\sin\theta$, the equation of the cylinder $y^2+z^2=9$ reduces to $r=3$ and $dy\,dz = r\,dr\,d\theta$. Along the radius vector OP, r varies from 0 to 3 and for the region in the first octant, θ varies from 0 to $\frac{\pi}{2}$.

$$\iiint_V \nabla\cdot\overline{F}\,dV = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{3}\int_{x=0}^{2}(4x\cdot r\cos\theta - 2\cdot r\cos\theta + 8x\cdot r\sin\theta)\,dx\cdot r\,dr\,d\theta$$
$$= \int_0^{\frac{\pi}{2}}\int_0^3\left[4r^2\cos\theta\left|\frac{x^2}{2}\right|_0^2 - (2r^2\cos\theta)|x|_0^2 + (8r^2\sin\theta)\left|\frac{x^2}{2}\right|_0^2\right]dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^3 (4r^2 \cos\theta + 16r^2 \sin\theta)\,dr\,d\theta$$

$$= 4\left|\frac{r^3}{3}\right|_0^3 \left|\sin\theta\right|_0^{\frac{\pi}{2}} + 16\left|\frac{r^3}{3}\right|_0^3 \left|-\cos\theta\right|_0^{\frac{\pi}{2}}$$

$$= 36 + 144$$

$$= 180 \qquad \text{... (1)}$$

(iii) The surface S consists of 5 surfaces, S_1, S_2, S_3, S_4, S_5.

$$\iint_S \overline{F}\cdot\hat{n}\,dS = \iint_{S_1} \overline{F}\cdot\hat{n}\,dS + \iint_{S_2} \overline{F}\cdot\hat{n}\,dS + \iint_{S_3} \overline{F}\cdot\hat{n}\,dS + \iint_{S_4} \overline{F}\cdot\hat{n}\,dS + \iint_{S_5} \overline{F}\cdot\hat{n}\,dS \qquad \text{... (2)}$$

(a) On $S_1(OAED) : z = 0,\ \hat{n} = -\hat{k},\ dS = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = dx\,dy$

x varies from 0 to 2 and y varies from 0 to 3.

$$\iint_{S_1} \overline{F}\cdot\hat{n}\,dS = \iint_{S_1} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot(-\hat{k})\,dx\,dy$$

$$= 0$$

(b) On $S_2\,(OBCD) : y = 0,\ \hat{n} = -\hat{j},\ dS = \dfrac{dz\,dx}{|\hat{n}\cdot\hat{j}|} = dz\,dx$

x varies from 0 to 2 and y varies from 0 to 3.

$$\iint_{S_2} \overline{F}\cdot\hat{n}\,dS = \iint_{S_2} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot(-\hat{j})\,dz\,dx = 0$$

(c) On $S_3\,(OAB) : x = 0,\ \hat{n} = -\hat{i},\ dS = \dfrac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = dy\,dz$

y and z varies from 0 to 3.

$$\iint_{S_3} \overline{F}\cdot\hat{n}\,dS = \iint_{S_3} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot(-\hat{i})\,dy\,dz = 0$$

(d) On $S_4\,(DEC) : x = 2,\ \hat{n} = \hat{i},\ dS = \dfrac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = dy\,dz$

y and z varies from 0 to 3.

$$\iint_{S_4} \overline{F}\cdot\hat{n}\,dS = \iint_{S_4} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot\hat{i}\,dy\,dz = \iint 8y\,dy\,dz$$

Putting $y = r\cos\theta$, $z = r\sin\theta$, equation of the cylinder $y^2 + z^2 = 9$ reduces to $r = 3$ and $dy\,dz = r\,dr\,d\theta$.

$$\iint_{S_4} \overline{F}\cdot\hat{n}\,dS = 8\int_0^{\frac{\pi}{2}} \int_0^3 r\sin\theta\cdot r\,dr\,d\theta$$

$$= 8\int_0^{\frac{\pi}{2}} \sin\theta\,d\theta\cdot\int_0^3 r^2\,dr$$

$$= 8\left|-\cos\theta\right|_0^{\frac{\pi}{2}} \left|\frac{r^3}{3}\right|_0^3$$

$$= 72$$

(e) On S_5 $(ABCE)$: This is the curved surface of the cylinder $y^2 + z^2 = 9$ bounded between $x = 0$ and $x = 2$.

Let $\phi = y^2 + z^2$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2y\,\hat{j} + 2z\,\hat{k}}{\sqrt{4y^2 + 4z^2}}$$

$$= \frac{y\hat{i} + z\hat{k}}{3} \qquad [\because y^2 + z^2 = 9]$$

$$dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{3\,dx\,dy}{z}$$

$$\iint_{S_5} \overline{F}\cdot\hat{n}\,dS = \iint_{S_5} (2x^2y\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k})\cdot\left(\frac{y\,\hat{j} + z\,\hat{k}}{3}\right)\frac{3\,dx\,dy}{z}$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} (-y^3 + 4xz^3)\frac{dx\,dy}{z}$$

The parametric equation of the cylinder $y^2 + z^2 = 9$ is

$$y = 3\cos\theta, \quad z = 3\sin\theta$$

$$dy = -3\sin\theta\,d\theta = -z\,d\theta, \quad \frac{dy}{z} = -d\theta$$

When $\qquad y = 0, \qquad \theta = \frac{\pi}{2}$

$\qquad\qquad y = 3, \qquad \theta = 0$

$$\iint_{S_5} \overline{F}\cdot\hat{n}\,dS = \int_{\frac{\pi}{2}}^{0}\int_0^2 (-27\cos^3\theta + x108\sin^3\theta)(-d\theta)dx$$

$$= \int_0^{\frac{\pi}{2}}\left(-27\cos^3\theta\left|x\right|_0^2 + 108\sin^3\theta\left|\frac{x^2}{2}\right|_0^2\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left(-54\cos^3\theta + 216\sin^3\theta\right)d\theta$$

$$= -54\cdot\frac{1}{2}B\left(2, \frac{1}{2}\right) + 216\cdot\frac{1}{2}B\left(2, \frac{1}{2}\right)$$

$$= B\left(2, \frac{1}{2}\right)(-27+108) \quad \left[\because \int \sin^p\theta \cos^q\theta \, d\theta = B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)\right]$$

$$= \frac{\left|\overline{2}\right|\overline{\frac{1}{2}}}{\left|\overline{\frac{5}{2}}\right.} 81$$

$$= \frac{1 \cdot \left|\overline{\frac{1}{2}}\right.}{\frac{3}{2} \cdot \frac{1}{2} \left|\overline{\frac{1}{2}}\right.} 81$$

$$= 108$$

Substituting in Eq. (2),

$$\iint_S \overline{F} \cdot \hat{n} \, dS = 0+0+0+72+108 = 180 \qquad \text{... (3)}$$

From Eqs. (1) and (3),

$$\iiint_V \nabla \cdot \overline{F} \, dV = \iint_S \overline{F} \cdot \hat{n} \, dS = 180$$

Hence, Gauss's divergence theorem is verified.

Example 18

Verify Gauss's divergence theorem for $\overline{F} = 2xz\hat{i} + yz\hat{j} + z^2\hat{k}$ *over the upper half of the sphere* $x^2 + y^2 + z^2 = a^2$.

Solution

By Gauss's divergence theorem,

$$\iiint_V \nabla \cdot \overline{F} \, dV = \iint_S \overline{F} \cdot \hat{n} \, dS$$

(i) $\overline{F} = 2xz\hat{i} + yz\hat{j} + z^2\hat{k}$

$$\nabla \cdot \overline{F} = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(z^2) = 2z + z + 2z = 5z$$

(ii) $\iiint_V \nabla \cdot \overline{F} \, dV = \iiint_V 5z \, dx \, dy \, dz$

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, equation of the sphere $x^2 + y^2 + z^2 = a^2$ reduces to $r = a$ and $dx\,dy\,dz = r^2\sin\theta \, dr \, d\theta \, d\phi$.

For upper half of the sphere (hemisphere),

r varies from 0 to a

θ varies from 0 to $\frac{\pi}{2}$

ϕ varies from 0 to 2π

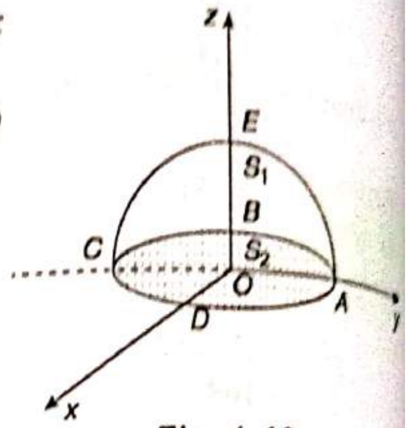


Fig. 1.69

$$\iiint_V \nabla\cdot\overline{F}\,dV = 5\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a} r\cos\theta\cdot r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 5\int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\, d\theta \left|\frac{r^4}{4}\right|_0^a$$

$$= 5\frac{a^4}{4}\int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{2}\, d\theta$$

$$= 5\left|\phi\right|_0^{2\pi}\cdot\frac{1}{2}\left|-\frac{\cos 2\theta}{2}\right|_0^{\frac{\pi}{2}}\cdot\frac{a^4}{4}$$

$$= -\frac{5a^4}{16}\cdot 2\pi(\cos\pi - \cos 0)$$

$$= \frac{5}{4}\pi a^4 \quad \text{...(1)}$$

(iii) Given surface is not closed. We close this surface from below by the circular surface S_2 in xy-plane.

Thus, the surface S consists of two surfaces S_1 and S_2.

$$\iint_S \overline{F}\cdot\hat{n}\, dS = \iint_{S_1} \overline{F}\cdot\hat{n}\, dS + \iint_{S_2} \overline{F}\cdot\hat{n}\, dS \quad \text{...(2)}$$

(a) Surface $S_1(ABCEA)$: This is the curved surface of the upper half of the sphere.

Let $\phi = x^2 + y^2 + z^2$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} \qquad [\because x^2 + y^2 + z^2 = a^2]$$

Let R be the projection of S_1 on the xy-plane, which is a circle $x^2 + y^2 = a^2$.

$$dS = \frac{dx\, dy}{|\hat{n}\cdot\hat{k}|} = \frac{a\, dx\, dy}{z}$$

$$\iint_{S_1} \overline{F}\cdot\hat{n}\, dS = \iint_R (2xz\,\hat{i} + yz\,\hat{j} + z^2\,\hat{k})\cdot\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\right)\frac{a\, dx\, dy}{z}$$

$$= \iint_R (2x^2 + y^2 + z^2)\,dx\,dy$$

$$= \iint_R (2x^2 + y^2 + a^2 - x^2 - y^2)\,dx\,dy \qquad [\because z^2 = a^2 - x^2 - y^2]$$

$$= \iint_R (x^2 + a^2)\,dx\,dy$$

Putting $x = r\cos\theta$, $y = r\sin\theta$, equation of the circle $x^2 + y^2 = a^2$ reduces to $r = a$ and $dx\,dy = r\,dr\,d\theta$. Along the radius vector OP, r varies from 0 to a and for the complete circle, θ varies from 0 to 2π.

$$\iint_{S_1} \nabla \times \overline{F} \cdot \hat{n}\,dS = \int_0^{2\pi}\int_0^a (r^2\cos^2\theta + a^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\left|\frac{r^4}{4}\right|_0^a \cos^2\theta + a^2\left|\frac{r^2}{2}\right|_0^a\right]d\theta$$

$$= \int_0^{2\pi}\left[\frac{a^4}{4}\left(\frac{1+\cos 2\theta}{2}\right) + \frac{a^4}{2}\right]d\theta$$

$$= a^4\left|\frac{5}{8}\theta + \frac{1}{8}\frac{\sin 2\theta}{2}\right|_0^{2\pi}$$

$$= \frac{5}{4}\pi a^4$$

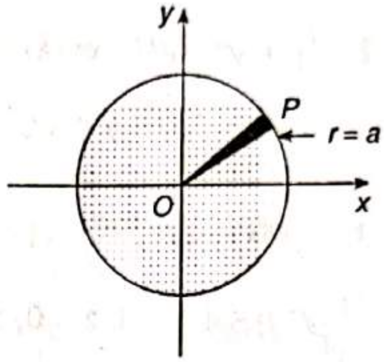


Fig. 1.70

(b) Surface S_2 $(ABCDA)$: This is the circle $x^2 + y^2 = a^2$ in xy-plane where $z = 0$, $\hat{n} = -\hat{k}$

$$dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = dx\,dy$$

$$\iint_{S_2} \overline{F}\cdot\hat{n}\,dS = \iint_{S_2} (2xz\,\hat{i} + yz\,\hat{j} + z^2\,\hat{k})\cdot(-\hat{k})\,dx\,dy$$

$$= 0 \qquad [\because z = 0]$$

Substituting in Eq. (2),

$$\iint_S \overline{F}\cdot\hat{n}\,dS = \frac{5}{4}\pi a^4 \qquad \text{... (3)}$$

From Eqs. (1) and (3),

$$\iiint_V \nabla\cdot\overline{F}\,dV = \iint_S \overline{F}\cdot\hat{n}\,dS = \frac{5}{4}\pi a^4$$

Hence, Gauss's divergence theorem is verified.

EXERCISE 1.9

(I) Evaluate the following integrals using Gauss's divergence theorem:

1. $\iint_S (y^2z^2\,\hat{i} + z^2x^2\,\hat{j} + z^2y^2\,\hat{k}) \cdot \hat{n}\, dS$, where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane.

   $\left[\text{Ans.: } \frac{\pi}{12}\right]$

2. $\iint_S (x^2y\,\hat{i} + y^3\hat{j} + xz^2\,\hat{k}) \cdot \hat{n}\, dS$, where S is the surface of the parallelepiped $0 \le x \le 2, 0 \le y \le 3, 0 \le z \le 4$.

   [Ans.: 384]

3. $\iint_S (4x\,\hat{i} - 2y^2\,\hat{j} + z^2\,\hat{k}) \cdot \hat{n}\, dS$, where S is the surface of the region bounded by $y^2 = 4x$, $x = 1$, $z = 0$, $z = 3$.

   [Ans.: 56]

4. $\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$, where S is the part of the plane $x + 2y + 3z = 6$ which lies in the first octant.

   [Ans.: 18]

5. $\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$, where S is the surface of the sphere $(x-2)^2 + (y-2)^2 + (z-2)^2 = 4$.

   [Ans.: 32π]

6. $\iint_S (2xy^2\,\hat{i} + x^2y\,\hat{j} + x^3\,\hat{k})$, where S is the surface of the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 4$.

   $\left[\text{Ans.: } \frac{3072\pi}{5}\right]$

7. $\iint_S (x^3\,\hat{i} + y^3\,\hat{j} + z^3\,\hat{k})$, where S is the surface of the region bounded within $z = \sqrt{16 - x^2 - y^2}$ and $x^2 + y^2 = 4$.

   $\left[\text{Ans.: } \frac{2\pi}{5}(2188 - 1056\sqrt{3})\right]$

(II) Verify Gauss's divergence theorem for the following:

1. $\bar{F}=(x^2-yz)\hat{i}+(y^2-zx)\hat{j}+(z^2-xy)\hat{k}$ over the region R bounded by the parallelepiped $0\le x\le a, 0\le y\le b, 0\le z\le c$.

   [Ans.: $abc(a+b+c)$]

2. $\bar{F}=x\hat{i}+y\hat{j}+z\hat{k}$ over the region R bounded by the sphere $x^2+y^2+z^2=16$.

   [Ans.: 256π]

3. $\bar{F}=4x\hat{i}-2y^2\hat{j}+z^2\hat{k}$ over the region bounded by the cylinder $x^2+y^2=4$ and the planes $z=0$, $z=3$.

   [Ans.: 84π]

4. $\bar{F}=2xy\hat{i}+6yz\hat{j}+3zx\hat{k}$ over the region bounded by the coordinate planes and the plane $x+y+z=2$.

   $\left[\text{Ans.: } \frac{22}{3}\right]$

Points to Remember

Parameterization of Curves and Surfaces

When a curve is parameterized by taking values of t from some interval $[a, b]$, the position vector $\bar{r}(t)$ of any point t on the curve can be written as

$$\bar{r}(t)=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}$$

When a surface is parameterized by taking points (u, v), out of some two dimensional space, the position vector $\bar{r}(u, v)$ can be written as

$$\bar{r}(u,v)=x(u,v)\hat{i}+y(u,v)\hat{j}+z(u,v)\hat{k}$$

Arc Length of Curves in Space

When a curve is parameterized by taking values of t from some interval $[a, b]$, the position vector $\bar{r}(t)$ of any point t on the curve can be written as,

$$\bar{r}(t)=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}$$

The tangent vector $\bar{r}'(t)$ is

$$\bar{r}'(t)=x'(t)\hat{i}+y'(t)\hat{j}+z'(t)\hat{k}$$

$$|\bar{r}'(t)|=\sqrt{[x'(t)]^2+[y'(t)^2]+[z'(t)]^2}$$

Hence, the arc length of the parameterized curve is

$$s=\int_a^b|\bar{r}'(t)|\,dt$$

Gradient

The gradient of a scalar field ϕ is denoted by grad ϕ or $\nabla\phi$ and is defined as

$$\text{grad } \phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial x} + \hat{k}\frac{\partial\phi}{\partial x}$$

Directional Derivative

The directional derivative of a scalar field $\phi(x, y, z)$ in the *direction of vector* $\bar{a}$ is the component of $\nabla\phi$ in the direction of $\bar{a}$ If $\hat{a}$ is the unit vector in the direction of $\bar{a}$ then directional derivatives of ϕ in the direction of $\bar{a} = \nabla\phi\cdot\hat{a} = \nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|}$

Divergence

The divergence of a vector field $\bar{F}$ is denoted by div $\bar{F}$ or $\nabla\cdot\bar{F}$ and is defined as

$$\text{div } \bar{F} = \nabla\cdot\bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\bar{F}$$

Solenoidal Vector Fields

A vector field $\bar{F}$ is said to be solenoidal if div $\bar{F} = 0$ at all points of the function.

Curl

The curl of a vector field $\bar{F}$ is denoted by curl $\bar{F}$ or $\nabla\times\bar{F}$ and is defined as

$$\text{curl } \bar{F} = \nabla\times\bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Irrotational Vector Fields

A vector field $\bar{F}$ is said to be *irrotational*, if curl $\bar{F} = 0$ at all points of the function, otherwise it is said to be rotational.

Conservative Fields and Scalar Potential Function

The vector field $\bar{F}$ is conservative if there is a scalar potential function ϕ such that $\bar{F} = \nabla\phi$.

Component Test for Conservative Fields

Let $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector field where F_1, F_2 and F_3 have continuous first order partial derivatives and domain of $\bar{F}$ is connected and simply connected. Then vector field $\bar{F}$ is conservative if the following conditions are satisfied.

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$
$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$
$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$$

This test is called component test for conservative fields.

Exact Differential Forms

Any expression $F_1(x, y, z)\mathrm{d}x + F_2(x, y, z)\mathrm{d}y + F_3(x, y, z)\mathrm{d}z$ is a differential form. A differential form is exact on a domain D in space if

$$F_1\mathrm{d}x + F_2\mathrm{d}y + F_3\mathrm{d}z = \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y + \frac{\partial \phi}{\partial z}\mathrm{d}z = \mathrm{d}\phi$$

for some scalar function ϕ throughout domain D.

Line Integrals

Let $\overline{F}(\overline{r})$ be a vector field defined at every point of a curve C. If $\overline{r}$ is the position vector of a point $P(x, y, z)$ on the curve C then the line integral of $\overline{F}(\overline{r})$ over a curve C is defined by

$$\int_C \overline{F}(\overline{r})\cdot \mathrm{d}\overline{r} = \int_C (F_1\mathrm{d}x + F_2\mathrm{d}y + F_3\mathrm{d}z)$$

where $\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

If the curve C is represented by a parametric representation

$$\overline{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k},$$

then the line integral along the curve C from $t = a$ to $t = b$ is

$$\int_C \overline{F}(\overline{r})\cdot \mathrm{d}\overline{r} = \int_a^b \overline{F}\cdot\frac{\mathrm{d}\overline{r}}{\mathrm{d}t}\mathrm{d}t$$
$$= \int_a^b \left(F_1\frac{\mathrm{d}x}{\mathrm{d}t} + F_2\frac{\mathrm{d}y}{\mathrm{d}t} + F_3\frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t$$

Work Done by a Force

If vector field $\overline{F}$ is the force acting on a particle moving along the arc AB of the curve C, then the line integral $\int_A^B \overline{F}\cdot \mathrm{d}\overline{r}$ represents the work done in displacing (moving) the particle from the point A to the point B.

Circulation

If vector field $\overline{F}$ is the velocity of a fluid particle and C is a closed curve, then the line integral $\oint_C \overline{F}\cdot \mathrm{d}\overline{r}$ represents the circulation of $\overline{F}$ around the curve C.

Path Independence of Line Integral

Line integral depends only on the start and end values and therefore is independent of the path.

Fundamental Theorem of Line Integrals

Let $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector field whose component are continuous throughout an open connected region D in space. Then there exists a differentiate function ϕ such that

$$\bar{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

if and only if for all points A and B in D, the value of $\int_A^B \bar{F}\cdot d\bar{r}$ is independent of the path joining A to B in D.

If the integral is independent of the path from A to B, its value is

$$\int_A^B \bar{F}\cdot d\bar{r} = \phi(B) - \phi(A)$$

Flux

If the vector field $\bar{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ is some kind of fluid flow and C is a curve, then the line integral $\int_C \bar{F}\cdot\hat{n}\,dS$ represents the flux of $\bar{F}$ through the curve C, where $\hat{n}$ is unit normal vector to C.

Green's Theorem in the Plane

If $M(x, y)$, $N(x, y)$ and their partial derivatives $\frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are continuous in some region R of xy-plane bounded by a closed curve C then

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Stokes' Theorem

If S be an open surface bounded by a closed curve C and $\bar{F}$ be a continuous and differentiable vector function then

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS$$

where $\hat{n}$ is the unit outward normal at any point of the surface S.

Gauss's Divergence Theorem

If $\overline{F}$ be a vector field having continuous partial derivatives in the region bounded by a closed surface S then $\iiint_V \nabla \cdot \overline{F} dV = \iint_S \overline{F} \cdot \hat{n} dS$

where $\hat{n}$ is the unit outward normal at any point of the surface S.

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. If $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ then div $\overline{r}$ is **[Summer 2014, Winter 2015]**
   (a) 0 (b) 1 (c) 2 (d) 3
2. If $\overline{F}$ is conservative then **[Winter 2014]**
   (a) $\nabla \times \overline{F} = 0$ (b) $\nabla \times \overline{F} \neq 0$
   (c) $\nabla \overline{F} = 0$ (d) $\nabla \cdot \overline{F} = 0$
3. If $\overline{F}$ is a conservative field then curl $\overline{F}$ is
   (a) $\hat{i}$ (b) $\hat{j}$ (c) $\hat{k}$ (d) 0
4. The divergence of $\overline{F} = xyz\hat{i} + zx^2 y\hat{j} + (3z^2 - y^2 z)\hat{k}$ is **[Summer 2015]**
   (a) $yz + zx + 2xz$ (b) $yz + 3x^2 + (2xz - y^2)$
   (c) $yz + xy$ (d) $xy - yz$
5. $\overline{F}$ is solenoidal vector, if div($\overline{F}$) is **[Winter 2015]**
   (a) $\overline{F}$ (b) 1 (c) 0 (d) –1
6. If $\overline{r} = x\hat{i} + y\hat{j} - z\hat{k}$ then curl ($\overline{r}$) is **[Summer 2016]**
   (a) 1 (b) 2 (c) 0 (d) none of these
7. If $\phi = \frac{x^2}{2} + \frac{y^2}{3}$, then the value of the $|\text{grad}\ \phi|$ at (1, 3) is
   (a) $\sqrt{\frac{13}{9}}$ (b) $\sqrt{\frac{9}{2}}$ (c) $\sqrt{5}$ (d) $\frac{9}{2}$
8. If $\phi = xyz$, then the value of $|\text{grad}\ \phi|$ at (1, 2, –1) is **[Summer 2016]**
   (a) 0 (b) 1 (c) 2 (d) 3
9. If $\phi = ax^2 + by^2 + cz^2$ satisfies Laplacian equation, then $a + b + c =$
   (a) 0 (b) 1 (c) 2 (d) none
10. If $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and if $(r^n \overline{r})$ is solenoidal then $n =$
    (a) 3 (b) –3 (c) 1 (d) none
11. If $\phi(x, y, z) = c$ is a surface the $\nabla\phi$ is
    (a) normal to $\phi = c$ (b) tangent to $\phi = c$
    (c) binormal to $\phi = c$ (d) none

12. If div $\bar{A} = 0$ then $\bar{A}$ is called
(a) irrotational (b) solenoidal
(c) constant vector (d) none

13. If $f = \tan^{-1}\left(\frac{y}{x}\right)$ then div (grad f) is equal to
(a) 1 (b) −1 (c) 0 (d) 2

14. The value of λ so that the vector $(x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + \lambda z)\hat{k}$ is a solen
vector is
(a) −2 (b) 3 (c) 1 (d) none

15. If the value of line integral $\oint_C \bar{F} \cdot d\bar{r}$ does not depend on path C then $\bar{F}$ is
[Summer 2014, Winter 20
(a) solenoidal (b) incompressible
(c) irrotational (d) none of these

16. Stoke's theorem connects
(a) a line integral and a surface integral
(b) a surface integral and a volume integral
(c) a line integral and a volume integral
(d) gradient of a function and its surface integral

17. The Gauss divergence theorem relates certain
(a) surface integrals to volume integrals
(b) surface integrals to line integrals
(c) line integrals to volume integrals
(d) vector quantities to other vector quantities

18. If $\bar{F}$ is conservative and ϕ is scalar potential then
(a) $\bar{F} = \nabla \cdot \phi$ (b) $\bar{F} = \nabla\phi$ (c) $\nabla \times \phi$ (d) none of these

19. If M, N and their partial derivatives are continuous in some region R of xy-pl
bounded by a closed curve C, then $\oint_C (M\,dx + N\,dy)$ is

(a) $\iint_R \left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right) dx\,dy$ (b) $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

(c) $\iint_R \left(\frac{\partial N}{\partial y} + \frac{\partial M}{\partial x}\right) dx\,dy$ (d) $\iint_R \left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}\right) dx\,dy$

20. For any closed surface S, $\iint_R \text{curl}\,\bar{F} \cdot dS =$

(a) 0 (b) $2\bar{F}$ (c) $\hat{n}$ (d) $\oint \bar{F} \cdot d\bar{r}$

21. If $\iint_S \overline{F}\cdot\hat{n}\,ds = 0$, then $\overline{F}$ is

(a) irrotational (b) solenoidal
(c) incompressible (d) none of these

22. If S is any closed surface enclosing a volume V and $\overline{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$ then $\iint_S \overline{F}\cdot\hat{n}\,ds =$

(a) V (b) $3V$ (c) $6V$ (d) none

23. If $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then $\oint_C \overline{r}\cdot d\overline{r} =$

(a) 0 (b) $\overline{r}$ (c) x (d) none

24. If $\hat{n}$ is unit outward drawn normal to any closed surface S, then $\iiint_V \text{div}\,\hat{n}\,dV =$

(a) S (b) $2S$ (c) $3S$ (d) none

25. The value of the line integral $\int \text{grad}\,(x + y - z)\,d\overline{r}$ from (0, 1, –1) to (1, 2, 0) is

(a) –1 (b) 0 (c) 2 (d) 3

26. A necessary and sufficient condition that the line integral $\int_C \overline{A}\cdot d\overline{r} = 0$ for every closed curve C is that

(a) div $\overline{A} = 0$ (b) div $\overline{A} \neq 0$ (c) curl $\overline{A} = 0$ (d) curl $\overline{A} \neq 0$

27. If $\overline{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$ where a, b, c are constants then $\iint \overline{F}\cdot\hat{n}\,dS$ where S is the surface of the unit sphere is

(a) 0 (b) $\frac{4}{3}\pi(a+b+c)$

(c) $\frac{4}{3}\pi(a+b+c)^2$ (d) none

28. The value of the line integral $\int_C (y^2dx + x^2dy)$ where C is the boundary of the square $-1 \le x \le 1, -1 \le y \le 1$ is

(a) 0 (b) $2(x+y)$ (c) 4 (d) $\frac{4}{3}$

29. The value of $\iint_S (yz\,dydz + zx\,dzdx + xy\,dxdy)$ where S is the surface of unit sphere $x^2 + y^2 + z^2 = 1$ is

(a) 0 (b) 4π (c) $\frac{4}{3}\pi$ (d) 10π

30. The work done by the force $\overline{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ in moving a particle from the point (1, 1, 1) to the point (3, 3, 2) along the path C is

(a) 17 (b) 10 (c) 0 (d) can not be found

31. The value of $\iint_S (x dy\, dx + y dz\, dx + z dx\, dy)$ where S is the surface $x^2 + y^2 + z^2 = a^2$ is

(a) 4π (b) $\frac{4}{3}\pi a^3$ (c) $4\pi a^3$ (d) 4π

Answers

1. (d)	2. (b)	3. (d)	4. (b)	5. (c)	6. (c)	7. (c)	8. (d)
9. (a)	10. (b)	11. (a)	12. (b)	13. (c)	14. (a)	15. (d)	16. (a)
17. (a)	18. (a)	19. (b)	20. (d)	21. (b)	22. (c)	23. (a)	24. (a)
25. (d)	26. (c)	27. (b)	28. (a)	29. (a)	30. (a)	31. (c)	

CHAPTER 2

Laplace Transform and Inverse Laplace Transform

Chapter Outline

2.1 INTRODUCTION

Laplace transform is the most widely used integral transform. It is a powerful mathematical technique which enables us to solve linear differential equations by using algebraic methods. It can also be used to solve systems of simultaneous differential equations, partial differential equations, and integral equations. It is applicable to continuous functions, piecewise continuous functions, periodic functions, step functions, and impulse functions. It has many important applications in mathematics, physics, optics, electrical engineering, control engineering, signal processing, and probability theory.

2.2 LAPLACE TRANSFORM

[Winter 2016]

If $f(t)$ is a function of t defined for all $t \geq 0$ then $\int_0^\infty e^{-st} f(t)\, dt$ is defined as the Laplace transform of $f(t)$, provided the integral exists and is denoted by $L\{f(t)\}$.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

The integral is a function of the parameter s and is denoted by $F(s), \bar{f}(s)$ or $\phi(s)$.

Sufficient Conditions for Existence of Laplace Transforms

The Laplace transform of the function $f(t)$ exists when the following sufficient conditions are satisfied:

(i) $f(t)$ is piecewise continuous, i.e., $f(t)$ is continuous in every sub-interval and $f(t)$ has finite limits at the end points of each sub-interval.

(ii) $f(t)$ is of exponential order of α, i.e., there exists M, α such that $|f(t)| \leq Me^{\alpha t}$, for all $t \geq 0$. In other words,

$$\lim_{t \to \infty} \{e^{-\alpha t} f(t)\} = \text{finite quantity}$$

e.g.,

- $L\{\tan t\}$ does not exist since $\tan t$ is not piecewise continuous.
- $L\{e^{t^2}\}$ does not exist since e^{t^2} is not of any exponential order.

2.3 LAPLACE TRANSFORM OF ELEMENTARY FUNCTIONS

[Winter 2012]

(i) $f(t) = 1$

Proof: $L\{1\} = \int_0^\infty e^{-st}\, dt$

$$= \left| \frac{e^{-st}}{-s} \right|_0^\infty$$

$$= \frac{1}{s}$$

(ii) $f(t) = t^n$ [Winter 2014, 2013; Summer 2013]

Proof: $L\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt$

Putting $st = x$, $dt = \frac{dx}{s}$

$$L\{t^n\} = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} \, x^n \, dx$$

$$= \frac{\overline{|n+1}}{s^{n+1}} \qquad s > 0, \; n+1 > 0$$

If n is a positive integer, $\overline{|n+1} = n!$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

(iii) $f(t) = e^{-at}$ **[Winter 2014; Summer 2015, 2013]**

Proof: $L\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at} \, dt$

$$= \int_0^\infty e^{-(s+a)t} \, dt$$

$$= \left| \frac{e^{-(s+a)t}}{-(s+a)} \right|_0^\infty$$

$$= \frac{1}{s+a}$$

Similarly, $L\{e^{at}\} = \dfrac{1}{s-a}$

(iv) $f(t) = \sin at$

Proof: $L\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt$

$$= \left| \frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right|_0^\infty$$

$$= 0 - \frac{1}{s^2 + a^2}(-a)$$

$$= \frac{a}{s^2 + a^2}$$

(v) $f(t) = \cos at$

Proof: $L\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt$

$$= \left| \frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \right|_0^\infty$$

$$= 0 - \frac{1}{s^2 + a^2}(-s)$$

$$= \frac{s}{s^2 + a^2}$$

[Winter 2014, 2012; Summer 2015, 2014]

(vi) $f(t) = \sinh at$

Proof: $L\{\sinh at\} = \int_0^{\infty} e^{-st} \sinh at \, dt$

$$= \int_0^{\infty} e^{-st} \left(\frac{e^{at} - e^{-at}}{2} \right) dt$$

$$= \frac{1}{2}\left[\int_0^{\infty} e^{-(s-a)t} dt - \int_0^{\infty} e^{-(s+a)t} dt \right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a} \right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2 - a^2} \right]$$

$$= \frac{a}{s^2 - a^2}$$

[Winter 2014]

(vii) $f(t) = \cosh at$

Proof: $L\{\cosh at\} = \int_0^{\infty} e^{-st} \cosh at \, dt$

$$= \int_0^{\infty} e^{-st} \left(\frac{e^{at} + e^{-at}}{2} \right) dt$$

$$= \frac{1}{2}\left[\int_0^{\infty} e^{-(s-a)t} dt + \int_0^{\infty} e^{-(s+a)t} dt \right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a} \right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2 - a^2} \right]$$

$$= \frac{s}{s^2 - a^2}$$

Example 1

Find the Laplace transform of

$$f(t) = 0 \qquad 0 \le t < 3$$
$$= 4 \qquad t \ge 3$$

[Winter 2014]

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^3 e^{-st} \cdot 0\, dt + \int_3^\infty e^{-st} \cdot 4\, dt$$

$$= 0 + 4\left|\frac{e^{-st}}{-s}\right|_3^\infty$$

$$= 4\left|\frac{0}{-s} - \frac{e^{-3s}}{-s}\right|$$

$$= \frac{4}{s} e^{-3s}$$

Example 2

Find the Laplace transform of $f(t) = \begin{cases} t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st} t\, dt + \int_1^\infty e^{-st} \cdot 0\, dt$$

$$= \left| t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right|_0^1 + 0$$

$$= \left| -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right|_0^1$$

$$= \left(-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)$$

$$= -e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right) + \frac{1}{s^2}$$

$$= -e^{-s}\left(\frac{s+1}{s^2}\right) + \frac{1}{s^2}$$

$$= \frac{1}{s^2}[1 - e^{-s}(s+1)]$$

Example 3

Find the Laplace transform of $f(t) = (t-2)^2 \quad t>2$

$\qquad = 0 \quad 0<t<2$

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st}\cdot 0\,dt + \int_2^\infty e^{-st}(t-2)^2\,dt$$

$$= 0 + \left|\frac{e^{-st}}{-s}(t-2)^2 - \frac{e^{-st}}{s^2}2(t-2) + \frac{e^{-st}}{-s^3}2\right|_2^\infty$$

$$= 0 - \frac{e^{-2s}}{-s^3}2$$

$$= \frac{2}{s^3}e^{-2s}$$

Example 4

Find the Laplace transform of $f(t) = 1 \quad 0<t<1$

$\qquad = e^t \quad 1<t<4$

$\qquad = 0 \quad t>4$

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\cdot 1\,dt + \int_1^4 e^{-st}e^t\,dt + \int_4^\infty e^{-st}\cdot 0\,dt$$

$$= \left|\frac{e^{-st}}{-s}\right|^1 + \left|\frac{e^{t(1-s)}}{1-s}\right|_1^4 + 0$$

$$= \frac{e^{-s}-1}{-s} + \frac{e^{4(1-s)} - e^{(1-s)}}{1-s}$$

$$= \frac{1-e^{-s}}{s} + \frac{e^{(1-s)} - e^{4(1-s)}}{s-1}$$

Example 5

Find the Laplace transform of $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & t > \pi \end{cases}$

Solution

$$\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)\, dt \\
&= \int_0^\pi e^{-st} \sin t\, dt + \int_\pi^\infty e^{-st}(0)\, dt \\
&= \left| \frac{e^{-st}}{s^2+1}(-s\sin t - \cos t) \right|_0^\pi + 0 \\
&= \frac{e^{-\pi s}}{s^2+1}(-s\sin\pi - \cos\pi) - \left[\frac{1}{s^2+1}(0-1)\right] \\
&= \frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1} \\
&= \frac{1+e^{-\pi s}}{s^2+1}
\end{aligned}$$

Example 6

Find the Laplace transform of $f(t) = \begin{cases} 0 & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$

[Summer 2015]

Solution

$$\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)\, dt \\
&= \int_0^\pi e^{-st}(0)\, dt + \int_\pi^\infty e^{-st} \sin t\, dt \\
&= 0 + \left| \frac{e^{-st}}{s^2+1}(-s\sin t - \cos t) \right|_\pi^\infty \\
&= 0 - \frac{e^{-\pi s}}{s^2+1}(-s\sin\pi - \cos\pi) \\
&= -\frac{e^{-\pi s}}{s^2+1}(0+1) \\
&= -\frac{e^{-\pi s}}{s^2+1}
\end{aligned}$$

Example 7

Find the Laplace transform of $f(t)=\begin{cases}\cos t & 0<t<2\pi \\ 0 & t>2\pi\end{cases}$

Solution

$$L\{f(t)\}=\int_0^{\infty} e^{-st} f(t)\,dt$$

$$=\int_0^{2\pi} e^{-st}\cos t\,dt+\int_{2\pi}^{\infty} e^{-st}\cdot 0\,dt$$

$$=\left|\frac{e^{-st}}{s^2+1}(-s\cos t+\sin t)\right|_0^{2\pi}+0$$

$$=\left[\frac{e^{-2\pi s}}{s^2+1}(-s\cos 2\pi+\sin 2\pi)\right]-\left[\frac{1}{s^2+1}(-s+0)\right]$$

$$=\frac{e^{-2\pi s}}{s^2+1}(-s+0)+\frac{s}{s^2+1}$$

$$=\frac{s}{s^2+1}(1-e^{-2\pi s})$$

Example 8

Find the Laplace transform of $f(t)=\cos\left(t-\frac{2\pi}{3}\right) \quad t>\frac{2\pi}{3}$

$$=0 \quad t<\frac{2\pi}{3}$$

Solution

$$L\{f(t)\}=\int_0^{\infty} e^{-st} f(t)\,dt$$

$$=\int_0^{\frac{2\pi}{3}} e^{-st}\cdot 0\,dt+\int_{\frac{2\pi}{3}}^{\infty} e^{-st}\cos\left(t-\frac{2\pi}{3}\right)dt$$

$$=\int_{\frac{2\pi}{3}}^{\infty} e^{-st}\cos\left(t-\frac{2\pi}{3}\right)dt$$

Putting $t-\frac{2\pi}{3}=x,\qquad dt=dx$

When $t = \frac{2\pi}{3}, \qquad x = 0$

When $t \to \infty, \qquad x \to \infty$

$$L\{f(t)\} = \int_0^\infty e^{-s\left(x+\frac{2\pi}{3}\right)} \cos x \, dx$$

$$= e^{-\frac{2\pi}{3}s} \int_0^\infty e^{-xs} \cos x \, dx$$

$$= e^{-\frac{2\pi}{3}s} \left| \frac{e^{-xs}}{s^2+1}(-s\cos x + \sin x) \right|_0^\infty$$

$$= \frac{e^{-\frac{2\pi}{3}s}}{s^2+1}(0+s)$$

$$= \frac{se^{-\frac{2\pi}{3}s}}{s^2+1}$$

Example 9

Find the Laplace transform of $\begin{aligned} f(t) &= \cos t \qquad 0 < t < \pi \\ &= \sin t \qquad t > \pi \end{aligned}$

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt$$

$$= \int_0^\pi e^{-st} \cos t \, dt + \int_\pi^\infty e^{-st} \sin t \, dt$$

$$= \left| \frac{e^{-st}}{s^2+1}(-s\cos t + \sin t) \right|_0^\pi + \left| \frac{e^{-st}}{s^2+1}(-s\sin t - \cos t) \right|_\pi^\infty$$

$$= \frac{1}{s^2+1}\left[e^{-\pi s}(-s\cos\pi) - (-s\cos 0) + 0 - e^{-\pi s}(-\cos\pi) \right]$$

$$= \frac{1}{s^2+1}\left[e^{-\pi s}(s-1) + s \right]$$

Example 10

Find the Laplace transform of $f(t) = t \qquad 0 < t < \frac{1}{2}$

$$= t - 1 \qquad \frac{1}{2} < t < 1$$

$$= 0 \qquad t > 1$$

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^{\frac{1}{2}} e^{-st} t\, dt + \int_{\frac{1}{2}}^{1} e^{-st}(t-1)\, dt + \int_1^\infty e^{-st} \cdot 0\, dt$$

$$= \left| \frac{e^{-st}}{-s} t - \frac{e^{-st}}{s^2} \cdot 1 \right|_0^{\frac{1}{2}} + \left| \frac{e^{-st}}{-s}(t-1) - \frac{e^{-st}}{s^2} \cdot 1 \right|_{\frac{1}{2}}^{1} + 0$$

$$= e^{-\frac{s}{2}}\left(-\frac{1}{2s} - \frac{1}{s^2}\right) - e^0\left(0 - \frac{1}{s^2}\right) - \frac{e^{-s}}{s^2} - e^{-\frac{s}{2}}\left(\frac{1}{2s} - \frac{1}{s^2}\right)$$

$$= e^{-\frac{s}{2}}\left(-\frac{1}{s}\right) + \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-\frac{s}{2}}}{s}$$

Example 11

Find the Laplace transform of $f(t) = 0 \qquad 0 < t < \pi$

$$= \sin^2(t - \pi) \qquad t > \pi$$

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\pi e^{-st} \cdot 0\, dt + \int_\pi^\infty e^{-st} \sin^2(t-\pi)\, dt$$

$$= 0 + \int_\pi^\infty e^{-st}\left[\frac{1-\cos 2(t-\pi)}{2}\right] dt$$

$$= \frac{1}{2}\int_\pi^\infty e^{-st}[1 - \cos(2\pi - 2t)]\, dt$$

$$=\frac{1}{2}\int_{\pi}^{\infty} e^{-st}(1-\cos 2t)\mathrm{d}t$$

$$=\frac{1}{2}\left[\int_{\pi}^{\infty} e^{-st}\mathrm{d}t-\int_{\pi}^{\infty} e^{-st}\cos 2t\ \mathrm{d}t\right]$$

$$=\frac{1}{2}\left[\left|\frac{e^{-st}}{-s}\right|_{\pi}^{\infty}-\left|\frac{e^{-st}}{s^2+4}(-s\cos 2t+2\sin 2t)\right|_{\pi}^{\infty}\right]$$

$$=\frac{1}{2}\left[\left(0+\frac{e^{-\pi s}}{s}\right)-\left\{0-\frac{e^{-\pi s}}{s^2+4}(-s\cos 2\pi+2\sin 2\pi)\right\}\right]$$

$$=\frac{e^{-\pi s}}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right]$$

Example 12

Find the Laplace transform of $\frac{1}{\sqrt{t}}$. **[Winter 2016]**

Solution

$$L\{f(t)\}=\int_0^{\infty} e^{-st}f(t)\mathrm{d}t$$

$$=\int_0^{\infty} e^{-st}\frac{1}{\sqrt{t}}\mathrm{d}t$$

$$=\int_0^{\infty} e^{-st}t^{-\frac{1}{2}}\mathrm{d}t$$

Putting $st=x$, $\mathrm{d}t=\frac{\mathrm{d}x}{s}$

When $t=0$, $x=0$

When $t\to\infty$, $x\to\infty$

$$L\{f(t)\}=\int_0^{\infty} e^{-x}\left(\frac{x}{s}\right)^{-\frac{1}{2}}\frac{\mathrm{d}x}{s}$$

$$=\frac{1}{\sqrt{s}}\int_0^{\infty} e^{-x}x^{-\frac{1}{2}}\mathrm{d}x$$

$$=\frac{1}{\sqrt{s}}\int_0^{\infty} e^{-x}x^{\frac{1}{2}-1}\mathrm{d}x$$

$$= \frac{1}{\sqrt{s}} \left\lceil \frac{1}{2} \right.$$

$$= \sqrt{\frac{\pi}{s}} \qquad \left[\because \left\lceil \frac{1}{2} \right. = \sqrt{\pi} \right]$$

EXERCISE 2.1

Find the Laplace transforms of the following functions:

1. $f(t) = t \quad 0 < t < 3$
   $= 6 \quad t > 3$

$$\left[\textbf{Ans.}: \frac{1}{s^2} + \left(\frac{3}{s} - \frac{1}{s^2} \right) e^{-3s} \right]$$

2. $f(t) = t^2 \quad 0 < t < 1$
   $= 1 \quad t > 1$

$$\left[\textbf{Ans.}: \frac{1}{s}(1 - e^{-s}) - \frac{2e^{-s}}{s^2} + \frac{2}{s^3}(1 - e^{-s}) \right]$$

3. $f(t) = (t - a)^3 \quad t > a$
   $= 0 \quad t < a$

$$\left[\textbf{Ans.}: \frac{6}{s^4} e^{-as} \right]$$

4. $f(t) = 0 \quad 0 \le t \le 1$
   $= t \quad 1 < t < 2$
   $= 0 \quad t > 2$

$$\left[\textbf{Ans.}: \left(\frac{1}{s^2} + \frac{1}{s} \right) e^{-s} - \left(\frac{1}{s^2} + \frac{2}{s} \right) e^{-2s} \right]$$

5. $f(t) = t^2 \quad 0 < t < 2$
   $= t - 1 \quad 2 < t < 3$
   $= 7 \quad t > 3$

$$\left[\textbf{Ans.}: \frac{2}{s^3} - \frac{e^{-2s}}{s^3}(2 + 3s + 3s^2) + \frac{e^{-3s}}{s^2}(5s - 1) \right]$$

6. $f(t) = e^t \quad 0 < t < 1$

$\quad = 0 \quad t > 1$

$$\left[\text{Ans.}: \frac{1}{1-s}(e^{1-s} - 1)\right]$$

7. $f(t) = \cos\left(t - \frac{2\pi}{3}\right) \quad t > \frac{2\pi}{3}$

$\quad = 0 \quad t < \frac{2\pi}{3}$

$$\left[\text{Ans.}: e^{-\frac{2\pi s}{3}} \frac{s}{s^2+1}\right]$$

8. $f(t) = \sin 2t \quad 0 < t < \pi$

$\quad = 0 \quad t > \pi$

$$\left[\text{Ans.}: \frac{2(1-e^{-\pi s})}{s^2+4}\right]$$

2.4 BASIC PROPERTIES OF LAPLACE TRANSFORM

2.4.1 Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$ then

$$L\{a f_1(t) + b f_2(t)\} = a F_1(s) + b F_2(s)$$

where a and b are constants.

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$L\{af_1(t) + bf_2(t)\} = \int_0^\infty e^{-st} \{af_1(t) + bf_2(t)\}\, \mathrm{d}t$$

$$= a\int_0^\infty e^{-st} f_1(t)\mathrm{d}t + b\int_0^\infty e^{-st} f_2(t)\mathrm{d}t$$

$$= a F_1(s) + b F_2(s)$$

Example 1

Find the Laplace transform of $(\sqrt{t} - 1)^2$.

Solution

$$L\left\{(\sqrt{t}-1)^2\right\} = L\left\{t - 2\sqrt{t} + 1\right\}$$

$$= L\{t\} - 2L\left\{\sqrt{t}\right\} + L\{1\}$$

$$= \frac{1}{s^2} - \frac{2\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{s}$$

$$= \frac{1}{s^2} - \frac{2\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{s}$$

$$= \frac{1}{s^2} - \frac{\sqrt{\pi}}{s^{\frac{3}{2}}} + \frac{1}{s}$$

Example 2

Find the Laplace transform of $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3$.

Solution

$$L\left\{\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3\right\} = L\left\{t^{\frac{3}{2}} - 3t^{\frac{1}{2}} + 3t^{-\frac{1}{2}} - t^{-\frac{3}{2}}\right\}$$

$$= L\left\{t^{\frac{3}{2}}\right\} - 3L\left\{t^{\frac{1}{2}}\right\} + 3L\left\{t^{-\frac{1}{2}}\right\} - L\left\{t^{-\frac{3}{2}}\right\}$$

$$= \frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} - \frac{3\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + \frac{3\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-\frac{1}{2}}}$$

$$= \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}} - \frac{3\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} + \frac{3\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}s^{-\frac{1}{2}}} \qquad \left[\because \Gamma(n+1) = n\Gamma(n),\quad \Gamma(n) = \frac{\Gamma(n+1)}{n}\right]$$

$$= \sqrt{\frac{\pi}{s}}\left(\frac{3}{4s^2} - \frac{3}{2s} + 3 + 2s\right)$$

Example 3

Find the Laplace transform of $t^2 + \sin 2t$.

Solution

$$L\{t^2 + \sin 2t\} = L\{t^2\} + L\{\sin 2t\}$$
$$= \frac{2}{s^3} + \frac{2}{s^2+4}$$

Example 4

Find the Laplace transform of $4t^2 + \sin 3t + e^{2t}$.

Solution

$$L\{4t^2 + \sin 3t + e^{2t}\} = 4L\{t^2\} + L\{\sin 3t\} + L\{e^{2t}\}$$
$$= 4 \cdot \frac{2}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2}$$
$$= \frac{8}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2}$$

Example 5

Find the Laplace transform of $\sin 2t \sin 3t$. **[Summer 2014]**

Solution

$$L\{\sin 2t \sin 3t\} = L\left\{\frac{\cos t - \cos 5t}{2}\right\}$$
$$= \frac{1}{2}L\{\cos t\} - \frac{1}{2}L\{\cos 5t\}$$
$$= \frac{s}{2(s^2+1)} - \frac{s}{2(s^2+25)}$$

Example 6

Find the Laplace transform of $\sin^2 3t$. **[Winter 2014]**

Solution

$$L\{\sin^2 3t\} = L\left\{\frac{1-\cos 6t}{2}\right\}$$
$$= \frac{1}{2}\left[L\{1\} - L\{\cos 6t\}\right]$$
$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]$$
$$= \frac{1}{2}\left[\frac{s^2+36-s^2}{s(s^2+36)}\right]$$
$$= \frac{18}{s(s^2+36)}$$

Example 7

Find the Laplace transform of $\sin^3 2t$.

Solution

$$L\{\sin^3 2t\} = L\left\{\frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t\right\}$$
$$= \frac{3}{4}L\{\sin 2t\} - \frac{1}{4}L\{\sin 6t\}$$
$$= \frac{3}{4}\left(\frac{2}{s^2+4}\right) - \frac{1}{4}\left(\frac{6}{s^2+36}\right)$$
$$= \frac{3}{2(s^2+4)} - \frac{3}{2(s^2+36)}$$

Example 8

Find the Laplace transform of $\cos^2 t$. **[Summer 2018]**

Solution

$$L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\}$$
$$= \frac{1}{2}\left[L\{1\} + L\{\cos 2t\}\right]$$
$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$
$$= \frac{1}{2}\left[\frac{s^2+4+s^2}{s(s^2+4)}\right]$$
$$= \frac{s^2+2}{s(s^2+4)}$$

Example 9

Find the Laplace transform of $t^2 - e^{-2t} + \cosh^2 3t$.

Solution

$$L\{t^2 - e^{-2t} + \cosh^2 3t\} = L\{t^2\} - L\{e^{-2t}\} + L\{\cosh^2 3t\}$$

$$= L\{t^2\} - L\{e^{-2t}\} + \frac{1}{2}L\{1 + \cosh 6t\}$$

$$= \frac{2}{s^3} - \frac{1}{s+2} + \frac{1}{2s} + \frac{s}{2(s^2 - 36)}$$

Example 10

Find the Laplace transform of $(\sin 2t - \cos 2t)^2$.

Solution

$$L\{(\sin 2t - \cos 2t)^2\} = L\{\sin^2 2t + \cos^2 2t - 2\cos 2t \sin 2t\}$$

$$= L\{1 - \sin 4t\}$$

$$= L\{1\} - L\{\sin 4t\}$$

$$= \frac{1}{s} - \frac{4}{s^2 + 16}$$

Example 11

Find the Laplace transform of $\cos t \cos 2t \cos 3t$.

Solution

$$L\{\cos t \cos 2t \cos 3t\} = L\left\{\frac{1}{2}(\cos 3t + \cos t)\cos 3t\right\}$$

$$= \frac{1}{2}L\{\cos^2 3t + \cos t \cos 3t\}$$

$$= \frac{1}{2}L\left\{\frac{1 + \cos 6t}{2} + \frac{\cos 4t + \cos 2t}{2}\right\}$$

$$= L\left\{\frac{1}{4} + \frac{1}{4}\cos 6t + \frac{1}{4}\cos 4t + \frac{1}{4}\cos 2t\right\}$$

$$= L\left\{\frac{1}{4}\right\} + \frac{1}{4}L\{\cos 6t\} + \frac{1}{4}L\{\cos 4t\} + \frac{1}{4}L\{\cos 2t\}$$

$$= \frac{1}{4s} + \frac{s}{4(s^2+36)} + \frac{s}{4(s^2+16)} + \frac{s}{4(s^2+4)}$$

Example 12

Find the Laplace transform of $\cosh^5 t$.

Solution

$$L\{\cosh^5 t\} = L\left\{\left(\frac{e^t+e^{-t}}{2}\right)^5\right\}$$

$$= L\left\{\frac{1}{2^5}(e^{5t} + 5e^{4t}e^{-t} + 10e^{3t}e^{-2t} + 10e^{2t}e^{-3t} + 5e^{t}e^{-4t} + e^{-5t})\right\}$$

$$= \frac{1}{32}L\{(e^{5t}+e^{-5t}) + 5(e^{3t}+e^{-3t}) + 10(e^{t}+e^{-t})\}$$

$$= \frac{1}{16}L\{\cosh 5t + 5\cosh 3t + 10\cosh t\}$$

$$= \frac{1}{16}[L\{\cosh 5t\} + 5L\{\cosh 3t\} + 10L\{\cosh t\}]$$

$$= \frac{1}{16}\left[\frac{s}{s^2-25} + \frac{5s}{s^2-9} + \frac{10s}{s^2-1}\right]$$

Example 13

Find the Laplace transform of $\sin\sqrt{t}$.

Solution

We know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\sin\sqrt{t} = t^{\frac{1}{2}} - \frac{t^{\frac{3}{2}}}{3!} + \frac{t^{\frac{5}{2}}}{5!} - \cdots$$

$$L\{\sin\sqrt{t}\} = L\left\{t^{\frac{1}{2}}\right\} - \frac{1}{3!}L\left\{t^{\frac{3}{2}}\right\} + \frac{1}{5!}L\left\{t^{\frac{5}{2}}\right\} - \cdots$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} - \frac{1}{3!}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} + \frac{1}{5!}\frac{\Gamma\left(\frac{7}{2}\right)}{s^{\frac{7}{2}}} - \cdots$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}} + \frac{1}{5!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{7}{2}}} - \cdots$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{2s^{\frac{3}{2}}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}} e^{-\frac{1}{(4s)}}$$

Example 14

Find the Laplace transform of $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$.

Solution

We know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \cdots$$

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-\frac{1}{2}} - \frac{t^{\frac{1}{2}}}{2!} + \frac{t^{\frac{3}{2}}}{4!} - \cdots$$

$$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = L\left\{t^{-\frac{1}{2}}\right\} - \frac{1}{2!}L\left\{t^{\frac{1}{2}}\right\} + \frac{1}{4!}L\left\{t^{\frac{3}{2}}\right\} - \cdots$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{1}{2!}\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{4!}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} - \cdots$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{1}{2!}\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{4!}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}} - \cdots$$

$$= \sqrt{\frac{\pi}{s}}\left[1 - \frac{1}{4s} + \frac{1}{2!(4s)^2} - \cdots\right]$$

$$= \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{(4s)}}$$

EXERCISE 2.2

Find the Laplace transforms of the following functions:

1. $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

$$\left[\text{Ans.}: \frac{1}{s-2} + \frac{24}{s^4} + \frac{3(s-2)}{s^2+9}\right]$$

2. $e^{2t} + 4t^3 - \sin 2t \cos 3t$

$$\left[\text{Ans.}: \frac{1}{s-2} + \frac{24}{s^4} - \frac{5}{2}\cdot\frac{1}{s^2+25} + \frac{1}{2(s^2+1)}\right]$$

3. $3t^2 + e^{-t} + \sin^3 2t$

$$\left[\text{Ans.}: \frac{6}{s^3} + \frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s^2+4} - \frac{3}{2}\cdot\frac{1}{s^2+36}\right]$$

4. $(t^2 + a)^2$

$$\left[\text{Ans.}: \frac{a^2s^4 + 4as^2 + 24}{s^5}\right]$$

5. $\sin(\omega t + \alpha)$

$$\left[\text{Ans.}: \cos\alpha\cdot\frac{\omega}{s^2+\omega^2} + \sin\alpha\cdot\frac{s}{s^2+\omega^2}\right]$$

6. $\sin 2t \cos 3t$

$$\left[\text{Ans.}: \frac{2(s^2-5)}{(s^2+1)(s^2+25)}\right]$$

7. $\cos^3 2t$

$$\left[\text{Ans.}: \frac{s(s^2+28)}{(s^2+36)(s^2+4)}\right]$$

8. $\sinh^3 3t$

$$\left[\text{Ans.}: \frac{162}{(s^2-81)(s^2-8)}\right]$$

9. $\dfrac{1+2t}{\sqrt{t}}$

$$\left[\text{Ans.}: \sqrt{\frac{\pi}{s}}\left(1+\frac{1}{s}\right)\right]$$

10. $\sin(t+\alpha)\cos(t-\alpha)$

$$\left[\text{Ans.}: \frac{1}{s^2+4}+\frac{\sin 2\alpha}{s}\right]$$

2.4.2 First Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{e^{-at} f(t)\} = F(s+a)$.

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$L\{e^{-at} f(t)\} = \int_0^\infty e^{-st} e^{-at} f(t)\mathrm{d}t$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\mathrm{d}t$$

$$= F(s+a)$$

Example 1

Find the Laplace transform of $e^{-3t}t^4$.

Solution

$$L\{t^4\} = \frac{4!}{s^5}$$

By the first shifting theorem,

$$L\{e^{-3t}t^4\} = \frac{4!}{(s+3)^5}$$

Example 2

Find the Laplace transform of $e^t t^{-\frac{1}{2}}$.

Solution

$$L\left\{t^{-\frac{1}{2}}\right\} = \frac{\left\lceil\frac{1}{2}\right.}{s^{\frac{1}{2}}}$$

$$= \sqrt{\frac{\pi}{s}}$$

By the first shifting theorem,

$$L\left\{e^t t^{-\frac{1}{2}}\right\} = \sqrt{\frac{\pi}{s-1}}$$

Example 3

Find the Laplace transform of $(t+1)^2 e^t$.

Solution

$$L\{t+1)^2\} = L\{t^2 + 2t + 1\}$$
$$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

By the first shifting theorem,

$$L\{(t+1)^2 e^t\} = \frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{1}{s-1}$$

Example 4

Find the Laplace transform of $e^t (1+\sqrt{t})^4$.

Solution

$$L\{(1+\sqrt{t})^4\} = L\{1 + 4\sqrt{t} + 6(\sqrt{t})^2 + 4(\sqrt{t})^3 + (\sqrt{t})^4\}$$
$$= L\left\{1 + 4t^{\frac{1}{2}} + 6t + 4t^{\frac{3}{2}} + t^2\right\}$$
$$= \frac{1}{s} + \frac{4\left\lceil\frac{3}{2}\right.}{s^{\frac{3}{2}}} + \frac{6\lceil 2}{s^2} + \frac{4\left\lceil\frac{5}{2}\right.}{s^{\frac{5}{2}}} + \frac{\lceil 3}{s^3}$$
$$= \frac{1}{s} + \frac{4 \cdot \frac{1}{2}\left\lceil\frac{1}{2}\right.}{s^{\frac{3}{2}}} + \frac{6}{s^2} + \frac{4 \cdot \frac{3}{2} \cdot \frac{1}{2}\left\lceil\frac{1}{2}\right.}{s^{\frac{5}{2}}} + \frac{2}{s^3}$$
$$= \frac{1}{s} + \frac{2\sqrt{\pi}}{s^{\frac{3}{2}}} + \frac{6}{s^2} + \frac{3\sqrt{\pi}}{s^{\frac{5}{2}}} + \frac{2}{s^3}$$

By the first shifting theorem,

$$L\{e^t (1+\sqrt{t})^4\} = \frac{1}{s-1} + \frac{2\sqrt{\pi}}{(s-1)^{\frac{3}{2}}} + \frac{6}{(s-1)^2} + \frac{3\sqrt{\pi}}{(s-1)^{\frac{5}{2}}} + \frac{2}{(s-1)^3}$$

Example 5

Find the Laplace transform of $e^{2t}\sin 3t$. **[Summer 2018]**

Solution

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

By the first shifting theorem,

$$L\{e^{2t}\sin 3t\} = \frac{3}{(s-2)^2+9}$$

$$= \frac{3}{s^2-4s+13}$$

Example 6

Find the Laplace transform of $e^{-3t}(2\cos 5t - 3\sin 5t)$. **[Summer 2014]**

Solution

$$L\{2\cos 5t - 3\sin 5t\} = 2L\{\cos 5t\} - 3L\{\sin 5t\}$$

$$= \frac{2s}{s^2+25} - \frac{3(5)}{s^2+25}$$

$$= \frac{2s}{s^2+25} - \frac{15}{s^2+25}$$

By the first shifting theorem,

$$L\{e^{-3t}(2\cos 5t - 3\sin 5t)\} = \frac{2(s+3)}{(s+3)^2+25} - \frac{15}{(s+3)^2+25}$$

$$= \frac{2s+6-15}{s^2+6s+9+25}$$

$$= \frac{2s-9}{s^2+6s+34}$$

Example 7

Find the Laplace transform of $e^{2t}\sin^2 t$. **[Summer 2017]**

Solution

$$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$$

$$= \frac{1}{2}[L\{1\} - L\{\cos 2t\}]$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= \frac{1}{2}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right]$$

$$= \frac{1}{2}\left[\frac{4}{s(s^2+4)}\right]$$

$$= \frac{2}{s(s^2+4)}$$

By the first shifting theorem,

$$L(e^{2t}\sin^2 t) = \frac{2}{(s-2)[(s-2)^2+4]}$$

$$= \frac{2}{(s-2)(s^2-4s+8)}$$

Example 8

Find the Laplace transform of $e^{4t}\sin^3 t$.

Solution

$$L\{\sin^3 t\} = \frac{1}{4}L\{3\sin t - \sin 3t\}$$

$$= \frac{3}{4(s^2+1)} - \frac{3}{4(s^2+9)}$$

By the first shifting theorem,

$$L\{e^{4t}\sin^3 t\} = \frac{3}{4[(s-4)^2+1]} - \frac{3}{4[(s-4)^2+9]}$$

$$= \frac{3}{4(s^2-8s+17)} - \frac{3}{4(s^2-8s+25)}$$

$$= \frac{3[(s^2-8s+25)-(s^2-8s+17)]}{4(s^2-8s+17)(s^2-8s+25)}$$

$$= \frac{6}{(s^2-8s+7)(s^2-8s+25)}$$

Example 9

Find the Laplace transform of $e^{-2t}(\sin 4t + t^2)$. **[Winter 2014]**

Solution

$$L\{\sin 4t + t^2\} = \frac{4}{s^2+16} + \frac{2}{s^3}$$

By the first shifting theorem,

$$L\{e^{-2t}(\sin 4t + t^2)\} = \frac{4}{(s+2)^2+16} + \frac{2}{(s+2)^3}$$

Example 10

Find the Laplace transform of cosh *at* cos *at.*

Solution

$$\cosh at \cos at = \left(\frac{e^{at}+e^{-at}}{2}\right)\cos at$$

$$= \frac{1}{2}(e^{at}\cos at + e^{-at}\cos at)$$

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{\cosh at \cos at\} = \frac{1}{2}L\ \{e^{at}\cos at + e^{-at}\cos at\}$$

By the first shifting theorem,

$$L\{\cosh at \cos at\} = \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{s^2+2a^2-2as} + \frac{s+a}{s^2+2a^2+2as}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)(s^2+2a^2+2as)+(s+a)(s^2+2a^2-2as)}{(s^2+2a^2)^2-4a^2s^2}\right]$$

$$= \frac{s^3}{s^4+4a^4}$$

Example 11

Find the Laplace transform of $\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t.$

Solution

$$\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t=\left(\frac{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}{2}\right)\sin\frac{\sqrt{3}}{2}t$$

$$=\frac{1}{2}\left(e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t-e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right)$$

$$L\left\{\sin\frac{\sqrt{3}}{2}t\right\}=\frac{\frac{\sqrt{3}}{2}}{s^2+\frac{3}{4}}$$

$$L\left\{\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t\right\}=\frac{1}{2}L\left\{e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t-e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right\}$$

By the first shifting theorem,

$$L\left\{\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t\right\}=\frac{1}{2}\left[\frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$=\frac{\sqrt{3}}{4}\left[\frac{1}{s^2+1-s}-\frac{1}{s^2+1+s}\right]$$

$$=\frac{\sqrt{3}}{2}\frac{s}{s^4+s^2+1}$$

Example 12

Find the Laplace transform of $e^{-3t}\cosh 4t\sin 3t.$

Solution

$$e^{-3t}\cosh 4t\sin 3t=e^{-3t}\left(\frac{e^{4t}+e^{-4t}}{2}\right)\sin 3t$$

$$=\frac{1}{2}(e^{t}\sin 3t+e^{-7t}\sin 3t)$$

$$L\{\sin 3t\}=\frac{3}{s^2+9}$$

$$L\{e^{-3t}\cosh 4t\sin 3t\} = \frac{1}{2}L\{e^{t}\sin 3t + e^{-7t}\sin 3t\}$$

By the first shifting theorem,

$$L\{e^{-3t}\cosh 4t\sin 3t\} = \frac{1}{2}\left[\frac{3}{(s-1)^2+9} + \frac{3}{(s+7)^2+9}\right]$$

$$= \frac{3(s^2+6s+34)}{(s^2-2s+10)(s^2+14s+58)}$$

Example 13

Find the Laplace transform of $\sin 2t\cos t\cosh 2t$.

Solution

$$\sin 2t\cos t\cosh 2t = \left(\frac{\sin 3t+\sin t}{2}\right)\left(\frac{e^{2t}+e^{-2t}}{2}\right)$$

$$= \frac{1}{4}(e^{2t}\sin 3t + e^{2t}\sin t + e^{-2t}\sin 3t + e^{-2t}\sin t)$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{\sin 2t\cos t\cosh 2t\} = \frac{1}{4}L\{e^{2t}\sin 3t + e^{2t}\sin t + e^{-2t}\sin 3t + e^{-2t}\sin t\}$$

By the first shifting theorem,

$$L\{\sin 2t\cos t\cosh 2t\} = \frac{1}{4}\left[\frac{3}{(s-2)^2+9} + \frac{1}{(s-2)^2+1} + \frac{3}{(s+2)^2+9} + \frac{1}{(s+2)^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{3(s^2+13)}{(s^2-4s+13)(s^2+4s+13)} + \frac{s^2+5}{(s^2-4s+5)(s^2+4s+5)}\right]$$

$$= \frac{1}{2}\left[\frac{3(s^2+13)}{s^4+10s^2+169} + \frac{s^2+5}{s^4-6s^2+25}\right]$$

Example 14

Find the Laplace transform of $\dfrac{\cos 2t\sin t}{e^{t}}$.

Solution

$$\frac{\cos 2t \sin t}{e^t} = e^{-t}\left(\frac{\sin 3t - \sin t}{2}\right)$$

$$= \frac{1}{2}(e^{-t}\sin 3t - e^{-t}\sin t)$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\left\{\frac{\cos 2t \sin t}{e^t}\right\} = \frac{1}{2}L\{e^{-t}\sin 3t - e^{-t}\sin t\}$$

By the first shifting theorem,

$$L\left\{\frac{\cos 2t \sin t}{e^t}\right\} = \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{2}\frac{2s^2+4s-4}{(s^2+2s+10)(s^2+2s+2)}$$

$$= \frac{s^2+2s-2}{(s^2+2s+10)(s^2+2s+2)}$$

Example 15

Find the Laplace transform of $e^{-4t}\sinh t \sin t$.

Solution

$$e^{-4t}\sinh t \sin t = e^{-4t}\left(\frac{e^t - e^{-t}}{2}\right)\sin t$$

$$= \frac{1}{2}(e^{-3t}\sin t - e^{-5t}\sin t)$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{e^{-4t}\sinh t \sin t\} = \frac{1}{2}L\{e^{-3t}\sin t - e^{-5t}\sin t\}$$

By the first shifting theorem,

$$L\{e^{-4t}\sinh t \sin t\} = \frac{1}{2}\left[\frac{1}{(s+3)^2+1} - \frac{1}{(s+5)^2+1}\right]$$

$$=\frac{1}{2}\frac{4s+16}{(s^2+6s+10)(s^2+10s+26)}$$

$$=\frac{2(s+4)}{(s^2+6s+10)(s^2+10s+26)}$$

EXERCISE 2.3

Find the Laplace transforms of the following functions:

1. t^3e^{-3t}

$$\left[\text{Ans.}: \frac{6}{(s+3)^4}\right]$$

2. $e^{-t}\cos 2t$

$$\left[\text{Ans.}: \frac{s+1}{s^2+2s+5}\right]$$

3. $2e^{3t}\sin 4t$

$$\left[\text{Ans.}: \frac{8}{s^2-6s+25}\right]$$

4. $(t+2)^2 e^t$

$$\left[\text{Ans.}: \frac{4s^2-4s+2}{(s-1)^3}\right]$$

5. $e^{2t}(3\sin 4t - 4\cos 4t)$

$$\left[\text{Ans.}: \frac{20-4s}{s^2-4s+20}\right]$$

6. $e^{-4t}\cosh 2t$

$$\left[\text{Ans.}: \frac{s+4}{s^2+8s+12}\right]$$

7. $(1+te^{-t})^3$

$$\left[\text{Ans.}: \frac{1}{s}+\frac{3}{(s+1)^2}+\frac{6}{(s+2)^3}+\frac{6}{(s+3)^4}\right]$$

8. $e^{-t}(3\sinh 2t - 5\cos 2t)$

$$\left[\text{Ans.}: \frac{1-5s}{s^2+2s-3}\right]$$

9. $e^{t}\sin 2t\sin 3t$ $\left[\text{Ans.}: \dfrac{12(s-1)}{(s^2-2s+2)(s^2-2s+26)}\right]$

10. $e^{-3t}\cosh 5t\sin 4t$ $\left[\text{Ans.}: \dfrac{4(s^2+6s+50)}{(s^2-4s+20)(s^2+16s+20)}\right]$

11. $e^{-4t}\cosh t\sin t$ $\left[\text{Ans.}: \dfrac{s^2+8s+18}{(s^2+6s+10)(s^2+10s+26)}\right]$

12. $e^{2t}\sin^4 t$ $\left[\text{Ans.}: \dfrac{3}{8(s-2)}-\dfrac{s-2}{2(s^2-4s+8)}+\dfrac{s-4}{8(s^2-8s+32)}\right]$

2.4.3 Second Shifting Theorem

[Summer 2013]

If $L\{f(t)\}=F(s)$

and $g(t)=f(t-a) \quad t>a$

$=0 \quad t<a$

then $L\{g(t)\}=e^{-as}F(s)$

Proof: $L\{f(t)\}=\int_0^\infty e^{-st}f(t)\mathrm{d}t$

$$L\{g(t)\}=\int_0^\infty e^{-st}g(t)\mathrm{d}t$$

$$=\int_0^a e^{-st}\cdot 0\,\mathrm{d}t+\int_a^\infty e^{-st}f(t-a)\mathrm{d}t$$

Putting $t-a=x,\quad \mathrm{d}t=\mathrm{d}x$

When $t=a,\quad x=0$

When $t\to\infty,\ x\to\infty$

$$L\{g(t)\}=\int_0^\infty e^{-s(a+x)}f(x)\mathrm{d}x$$

$$=e^{-as}\int_0^\infty e^{-sx}f(x)\mathrm{d}x$$

$$=e^{-as}\int_0^\infty e^{-st}f(t)\mathrm{d}t$$

$$=e^{-as}F(s)$$

Example 1

Find the Laplace transform of $g(t) = e^{t-a} \quad t > a$

$= 0 \quad t < a$

Solution

Let $f(t) = e^t$

$$L\{f(t)\} = F(s) = \frac{1}{s-1}$$

By the second shifting theorem,

$$L\{g(t)\} = e^{-as}\frac{1}{s-1}$$

Example 2

Find the Laplace transform of $g(t) = \cos(t-a) \quad t > a$

$= 0 \quad t < a$

Solution

Let $f(t) = \cos t$

$$L\{f(t)\} = F(s) = \frac{s}{s^2+1}$$

By the second shifting theorem,

$$L\{g(t)\} = e^{-as}\frac{s}{s^2+1}$$

Example 3

Find the Laplace transform of $g(t) = \sin\left(t - \frac{\pi}{4}\right) \quad t > \frac{\pi}{4}$

$= 0 \quad t < \frac{\pi}{4}$

Solution

Let $f(t) = \sin t$

$$L\{f(t)\} = F(s) = \frac{1}{s^2+1}$$

By the second shifting theorem,

$$L\{g(t)\} = e^{-\frac{\pi s}{4}}\frac{1}{s^2+1}$$

Example 4

Find the Laplace transform of $g(t) = (t-1)^3 \quad t > 1$

$$= 0 \quad t < 1$$

Solution

Let $f(t) = t^3$

$$L\{f(t)\} = F(s) = \frac{3!}{s^4}$$

By the second shifting theorem,

$$L\{g(t)\} = e^{-s}\frac{3!}{s^4}$$

EXERCISE 2.4

Find the Laplace transforms of the following functions:

1. $f(t) = \cos\left(t - \frac{2\pi}{3}\right) \quad t > \frac{2\pi}{3}$

   $= 0 \quad t < \frac{2\pi}{3}$

   $\left[\textbf{Ans. : } e^{-\frac{2\pi s}{3}} \frac{s}{s^2+1}\right]$

2. $f(t) = (t-2)^2 \quad t > 2$

   $= 0 \quad t < 2$

   $\left[\textbf{Ans. : } e^{-2s} \frac{2}{s^3}\right]$

3. $f(t) = 5\sin 3\left(t - \frac{\pi}{4}\right) \quad t > \frac{\pi}{4}$

   $= 0 \quad t < \frac{\pi}{4}$

   $\left[\textbf{Ans. : } e^{-\frac{\pi s}{4}} \frac{1}{s^2+9}\right]$

2.5 DIFFERENTIATION OF LAPLACE TRANSFORMS (MULTIPLICATION BY *t*)

If $L\{f(t)\} = F(s)$ then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$. **[Winter 2014, 2013]**

Proof: $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$

Differentiating both the sides w.r.t. s using DUIS,

$$\frac{\mathrm{d}}{\mathrm{d}s} F(s) = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$= \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\mathrm{d}t$$

$$= \int_0^\infty (-t\,e^{-st}) f(t)\mathrm{d}t$$

$$= \int_0^\infty e^{-st}\{-t\,f(t)\}\,\mathrm{d}t$$

$$= -L\{t\,f(t)\}$$

$$L\{t\,f(t)\} = (-1)\frac{\mathrm{d}}{\mathrm{d}s} F(s)$$

Similarly, $$L\{t^2 f(t)\} = (-1)^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} F(s)$$

In general, $$L\{t^n\, f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s)$$

Example 1

Find the Laplace transform of te^{-t}.

Solution

$$L\{e^{-t}\} = \frac{1}{s+1}$$

$$L\{te^{-t}\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\{e^{-t}\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s+1}\right)$$

$$= -\left[\frac{-1}{(s+1)^2}\right]$$

$$= \frac{1}{(s+1)^2}$$

Example 2

Find the Laplace transform of t cos at.

Solution

$$L\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$L\{t\cos at\} = -\frac{d}{ds} L\{\cos at\}$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2 + a^2}\right]$$

$$= -\left[\frac{(s^2 + a^2)(1) - s(2s)}{(s^2 + a^2)^2}\right]$$

$$= -\left[\frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2}\right]$$

$$= -\left[\frac{-s^2 + a^2}{(s^2 + a^2)^2}\right]$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

Example 3

Find the Laplace transform of t sin at. [Summer 2016]

Solution

$$L\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$L\{t\sin at\} = -\frac{d}{ds} L\{\sin at\}$$

$$= -\frac{d}{ds}\left(\frac{a}{s^2 + a^2}\right)$$

$$= -\left[\frac{(s^2 + a^2)(0) - a(2s)}{(s^2 + a^2)^2}\right]$$

$$= \frac{2as}{(s^2 + a^2)^2}$$

Example 4

Find the Laplace transform of t sin 2*t*. **[Summer 2015]**

Solution

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\{t\sin 2t\} = -\frac{d}{ds}L\{\sin 2t\}$$

$$= -\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= -\left[\frac{(s^2+4)(0)-2(2s)}{(s^2+4)^2}\right]$$

$$= \frac{4s}{(s^2+4)^2}$$

Example 5

Find the Laplace transform of t cosh *at.*

Solution

$$L\{\cosh at\} = \frac{s}{s^2-a^2}$$

$$L\{t\cosh at\} = -\frac{d}{ds}L\{\cosh at\}$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2-a^2}\right)$$

$$= -\left[\frac{(s^2-a^2)(1)-s(2s)}{(s^2-a^2)^2}\right]$$

$$= -\left[\frac{s^2-a^2-2s^2}{(s^2-a^2)^2}\right]$$

$$= -\left[\frac{-s^2-a^2}{(s^2-a^2)^2}\right]$$

$$= \frac{s^2+a^2}{(s^2-a^2)^2}$$

Example 6

Find the Laplace transform of $t\cos^2 t$.

Solution

$$L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\}$$

$$= \frac{1}{2}L\{1+\cos 2t\}$$

$$= \frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+4}\right)$$

$$L\{t\cos^2 t\} = -\frac{d}{ds}L\{\cos^2 t\}$$

$$= -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s}+\frac{s}{s^2+4}\right)$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2}+\frac{(s^2+4)(1)-s(2s)}{(s^2+4)^2}\right]$$

$$= \frac{1}{2s^2}+\frac{s^2-4}{2(s^2+4)^2}$$

Example 7

Find the Laplace transform of $t\sin^2 3t$. **[Winter 2017]**

Solution

$$L\{\sin^2 3t\} = L\left\{\frac{1-\cos 6t}{2}\right\}$$

$$= \frac{1}{2}L\{1-\cos 6t\}$$

$$= \frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+36}\right)$$

$$L\{t\sin^2 3t\} = -\frac{d}{ds}L\{\sin^2 3t\}$$

$$= -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s}-\frac{s}{s^2+36}\right)$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} - \frac{-s^2+36}{(s^2+36)^2}\right]$$

$$= \frac{1}{2s^2} + \frac{-s^2+36}{2(s^2+36)^2}$$

Example 8

Find the Laplace transform of t $\sin^3 t$.

Solution

$$L\{\sin^3 t\} = L\left\{\frac{3\sin t - \sin 3t}{4}\right\}$$

$$= \frac{1}{4}\left(\frac{3}{s^2+1} - \frac{3}{s^2+9}\right)$$

$$= \frac{3}{4}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

$$L\{t\sin^3 t\} = -\frac{d}{ds}L\{\sin^3 t\}$$

$$= -\frac{3}{4}\frac{d}{ds}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

$$= -\frac{3}{4}\left[\frac{-2s}{(s^2+1)^2} + \frac{2s}{(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{(s^2+9)^2 - (s^2+1)^2}{(s^2+1)^2(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{s^4+18s^2+81-s^4-2s^2-1}{(s^2+1)^2(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\frac{16(s^2+5)}{(s^2+1)^2(s^2+9)^2}$$

$$= \frac{24s(s^2+5)}{(s^2+1)^2(s^2+9)^2}$$

Example 9

Find the Laplace transform of $t \sin 2t \cosh t$.

Solution

$$L\{\sin 2t \cosh t\} = L\left\{\sin 2t\left(\frac{e^{t}+e^{-t}}{2}\right)\right\}$$

$$= \frac{1}{2}L\{e^{t}\sin 2t + e^{-t}\sin 2t\}$$

$$= \frac{1}{2}\left[\frac{2}{(s-1)^2+4}+\frac{2}{(s+1)^2+4}\right]$$

$$= \frac{1}{s^2-2s+5}+\frac{1}{s^2+2s+5}$$

$$L\{t\sin 2t\cosh t\} = -\frac{d}{ds}L\{\sin 2t\cosh t\}$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2-2s+5}+\frac{1}{s^2+2s+5}\right)$$

$$= \frac{2s-2}{(s^2-2s+5)^2}+\frac{2s+2}{(s^2+2s+5)^2}$$

Example 10

Find the Laplace transform of $t \sin 3t \cos 2t$. **[Winter 2016]**

Solution

$$L\{\sin 3t \cos 2t\} = L\left\{\frac{\sin 5t + \sin t}{2}\right\}$$

$$= \frac{1}{2}L\{\sin 5t\} + \frac{1}{2}L\{\sin t\}$$

$$= \frac{5}{2(s^2+25)}+\frac{1}{2(s^2+1)}$$

$$L\{t\sin 3t\cos 2t\} = -\frac{d}{ds}L\{\sin 3t\cos 2t\}$$

$$= -\frac{d}{ds}\left[\frac{5}{2(s^2+25)} + \frac{1}{2(s^2+1)}\right]$$

$$= -\frac{1}{2}\left[\frac{-5(2s)}{(s^2+25)^2} - \frac{1(2s)}{(s^2+1)^2}\right]$$

$$= \frac{5s}{(s^2+25)} + \frac{s}{(s^2+1)^2}$$

Example 11

Find the Laplace transform of $t\sqrt{1+\sin t}$.

Solution

$$L\{\sqrt{1+\sin t}\} = L\left\{\sin\frac{t}{2} + \cos\frac{t}{2}\right\}$$

$$= \frac{\frac{1}{2}}{s^2+\frac{1}{4}} + \frac{s}{s^2+\frac{1}{4}}$$

$$= \frac{1}{2}\cdot\frac{4}{4s^2+1} + \frac{4s}{4s^2+1}$$

$$= \frac{4s+2}{4s^2+1}$$

$$L\left\{t\sqrt{1+\sin t}\right\} = -\frac{d}{ds}L\left\{\sqrt{1+\sin t}\right\}$$

$$= -\frac{d}{ds}\left(\frac{4s+2}{4s^2+1}\right)$$

$$= -\left[\frac{(4s^2+1)4 - (4s+2)8s}{(4s^2+1)^2}\right]$$

$$= \frac{-16s^2 - 4 + 32s^2 + 16s}{(4s^2+1)^2}$$

$$= \frac{16s^2+16s-4}{(4s^2+1)^2}$$

$$= \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$$

Example 12

[Summer 2017]

Find the Laplace transform of $te^{4t}\cos 2t$.

Solution

$$L\{\cos 2t\} = \frac{s}{s^2+4}$$

By the first shifting theorem,

$$L\{e^{4t}\cos 2t\} = \frac{s-4}{(s-4)^2+4}$$

$$= \frac{s-4}{s^2-8s+20}$$

$$L\{t\,e^{4t}\cos 2t\} = -\frac{d}{ds}L\{e^{4t}\cos 2t\}$$

$$= -\frac{d}{ds}\left(\frac{s-4}{s^2-8s+20}\right)$$

$$= -\left[\frac{(s^2-8s+20)\,(1)-(s-4)\,(2s-8)}{(s^2-8s+20)^2}\right]$$

$$= -\left[\frac{s^2-8s+20-2s^2+8s+8s-32}{(s^2-8s+20)^2}\right]$$

$$= -\left[\frac{-s^2+8s-12}{(s^2-8s+20)^2}\right]$$

$$= \frac{s^2-8s+12}{(s^2-8s+20)^2}$$

$$= \frac{(s-4)^2-4}{(s-4)^2+4}$$

Example 13

Find the Laplace transform of $te^{at}\sin at$. **[Winter 2013]**

Solution

$$L\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$L\{e^{at}\sin at\} = \frac{a}{(s-a)^2 + a^2}$$

$$= \frac{a}{s^2 - 2as + 2a^2}$$

$$L\{te^{at}\sin at\} = -\frac{d}{ds}L\{e^{at}\sin at\}$$

$$= -\frac{d}{ds}\left[\frac{a}{(s-a)^2 + a^2}\right]$$

$$= -\frac{d}{ds}\left(\frac{a}{s^2 - 2as + 2a^2}\right).$$

$$= \frac{a}{(s^2 - 2as + 2a^2)^2}(2s - 2a)$$

$$= \frac{2a(s-a)}{(s^2 - 2as + 2a^2)^2}$$

Example 14

Find the Laplace transform of $t\left(\frac{\sin t}{e^t}\right)^2$.

Solution

$$t\left(\frac{\sin t}{e^t}\right)^2 = t\,e^{-2t}\sin^2 t$$

$$= t\,e^{-2t}\left(\frac{1-\cos 2t}{2}\right)$$

$$= \frac{1}{2}\,t\,e^{-2t}(1-\cos 2t)$$

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\{t(1-\cos 2t)\} = -\frac{d}{ds}L(1-\cos 2t)$$

$$= -\frac{d}{ds}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= -\left[-\frac{1}{s^2} - \frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right]$$

$$= \frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}$$

By the first shifting theorem,

$$L\left\{\frac{1}{2}te^{-2t}(1-\cos 2t)\right\} = \frac{1}{2}\left[\frac{1}{(s+2)^2} + \frac{4-(s+2)^2}{\{(s+2)^2+4\}^2}\right]$$

Example 15

Find the Laplace transform of $\sin 2t - 2t\cos 2t$.

Solution

$$L\{\sin 2t - 2t\cos 2t\} = L\{\sin 2t\} - 2L\{t\cos 2t\}$$

$$= \frac{2}{s^2+4} - 2\left[-\frac{d}{ds}L\{\cos 2t\}\right]$$

$$= \frac{2}{s^2+4} + 2\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= \frac{2}{s^2+4} + 2\left[\frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right]$$

$$= \frac{2}{s^2+4} + 2\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right]$$

$$= \frac{2}{s^2+4} + 2\left[\frac{4-s^2}{(s^2+4)^2}\right]$$

$$= \frac{2}{s^2+4}\left[1 + \frac{4-s^2}{s^2+4}\right]$$

$$= \frac{2}{s^2+4}\left[\frac{s^2+4+4-s^2}{s^2+4}\right]$$

$$= \frac{2}{s^2+4}\left[\frac{8}{s^2+4}\right]$$

$$= \frac{16}{(s^2+4)^2}$$

Example 16

Find the Laplace transform of $t(\sin t - t\cos t)$. **[Winter 2015]**

Solution

$$L\{t(\sin t - t\cos t)\} = L\{t\sin t - t^2\cos t\}$$

$$L\{t\sin t\} = -\frac{d}{ds}L\{\sin t\}$$

$$= -\frac{d}{ds}\frac{1}{(s^2+1)}$$

$$= -\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$= \frac{2s}{(s^2+1)^2}$$

$$L\{t^2\cos t\} = (-1)^2\frac{d^2}{ds^2}L\{\cos t\}$$

$$= (-1)^2\frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{1-s^2}{(s^2+1)^2}\right]$$

$$= \frac{(s^2+1)^2(-2s)-(1-s^2)\cdot 2(s^2+1)2s}{(s^2+1)^4}$$

$$= \frac{(s^2+1)(-2s)-4s(1-s^2)}{(s^2+1)^3}$$

$$= \frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}$$

$$= \frac{2s^3 - 6s}{(s^2+1)^3}$$

$$L\{t \sin t - t^2 \cos t\} = \frac{2s}{(s^2+1)^2} - \frac{2s^3 - 6s}{(s^2+1)^3}$$

$$= \frac{2s(s^2+1) - (2s^3 - 6s)}{(s^2+1)^3}$$

$$= \frac{2s^3 + 2s - 2s^3 + 6s}{(s^2+1)^3}$$

$$= \frac{8s}{(s^2+1)^3}$$

Example 17

Find the Laplace transform of $t^2 \sin \omega t$. **[Winter 2014]**

Solution

$$L\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}$$

$$L\{t^2 \sin \omega t\} = (-1)^2 \frac{d^2}{ds^2} L\{\sin \omega t\}$$

$$= \frac{d^2}{ds^2}\left(\frac{\omega}{s^2 + \omega^2}\right)$$

$$= \frac{d}{ds}\left[-\frac{\omega(2s)}{(s^2+\omega^2)^2}\right]$$

$$= -2\omega\left[\frac{1}{(s^2+\omega^2)^2} - \frac{2s}{(s^2+\omega^2)^3} \cdot 2s\right]$$

$$= -2\omega\left[\frac{s^2 + \omega^2 - 4s^2}{(s^2+\omega^2)^3}\right]$$

$$= \frac{2\omega(3s^2 - \omega^2)}{(s^2+\omega^2)^3}$$

Example 18

Find the Laplace transform of $t^2 \cosh 3t$. **[Summer 2016]**

Solution

$$L\{\cosh 3t\} = \frac{s}{s^2-9}$$

$$L\{t^2 \cosh 3t\} = (-1)^2 \frac{d^2}{ds^2} L\{\cosh 3t\}$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2-9}\right)$$

$$= \frac{d}{ds}\left[\frac{1}{s^2-9} - \frac{s}{(s^2-9)^2}(2s)\right]$$

$$= -\frac{1}{(s^2-9)^2}(2s) - \frac{4s}{(s^2-9)^2} + \frac{4s^2}{(s^2-9)^3}(2s)$$

$$= \frac{-(s^2-9)2s - 4s(s^2-9) + 8s^3}{(s^2-9)^3}$$

$$= \frac{-2s^3 + 18s - 4s^3 + 36s + 8s^3}{(s^2-9)^3}$$

$$= \frac{2s^3 + 54s}{(s^2-9)^3}$$

$$= \frac{2s(s^2+27)}{(s^2-9)^3}$$

Example 19

Find the Laplace transform of $t^2 \cosh \pi t$. **[Winter 2014]**

Solution

$$L\{\cosh \pi t\} = \frac{s}{s^2-\pi^2}$$

$$L\{t^2 \cosh \pi t\} = (-1)^2 \frac{d^2}{ds^2} L\{\cosh \pi t\}$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2-\pi^2}\right)$$

$$= \frac{d}{ds}\left[\frac{1}{s^2-\pi^2} - \frac{s}{(s^2-\pi^2)^2}(2s)\right]$$

$$= -\frac{1}{(s^2-\pi^2)^2}(2s) - \frac{4s}{(s^2-\pi^2)^2} + \frac{4s^2}{(s^2-\pi^2)^3}(2s)$$

$$= \frac{-2s^3 + 2\pi^2 s - 4s^3 + 4\pi^2 s + 8s^3}{(s^2-\pi^2)^3}$$

$$= \frac{2s^3 + 6\pi^2 s}{(s^2-\pi^2)^3}$$

$$= \frac{2s(s^2+3\pi^2)}{(s^2-\pi^2)^3}$$

Example 20

Find the Laplace transform of $t^2 e^t \sin 4t$.

Solution

$$L\{\sin 4t\} = \frac{4}{s^2+16}$$

$$L\{t^2 \sin 4t\} = (-1)^2 \frac{d^2}{ds^2} L\{\sin 4t\}$$

$$= \frac{d^2}{ds^2}\left(\frac{4}{s^2+16}\right)$$

$$= -\frac{d}{ds}\left[\frac{4(2s)}{(s^2+16)^2}\right]$$

$$= -\frac{d}{ds}\left[\frac{8s}{(s^2+16)^2}\right]$$

$$= -\left[\frac{(s^2+16)^2(8) - 8s \cdot 2(s^2+16)(2s)}{(s^2+16)^4}\right]$$

$$= \frac{-8s^2 - 128 + 32s^2}{(s^2+16)^3}$$

$$= \frac{24s^2 - 128}{(s^2+16)^3}$$

$$= \frac{8(3s^2-16)}{(s^2+16)^3}$$

By the first shifting theorem,

$$L\{t^2 e^t \sin 4t\} = \frac{8\left[3(s-1)^2-16\right]}{\left[(s-1)^2+16\right]^3}$$

$$= \frac{8(3s^2-6s-13)}{(s^2-2s+17)^3}$$

EXERCISE 2.5

Find the Laplace transforms of the following functions:

1. $t\cos^3 t$

$$\left[\text{Ans.}: \frac{1}{4}\left[\frac{-s^2+9}{(s^2+9)^2}+\frac{s^2+3}{(s^2+1)^2}\right]\right]$$

2. $t\cos(\omega t-\alpha)$

$$\left[\text{Ans.}: \frac{(s^2-\omega^2)\cos\alpha+2\omega s\sin\alpha}{(s^2+\omega^2)^2}\right]$$

3. $t\sqrt{1-\sin t}$

$$\left[\text{Ans.}: \frac{4(4s^2-4s-1)}{(4s^2+1)^2}\right]$$

4. $t\cosh 3t$

$$\left[\text{Ans.}: \frac{s^2+9}{(s^2-9)^2}\right]$$

5. $t\sinh 2t\sin 3t$

$$\left[\text{Ans.}: 3\left[\frac{s-2}{(s^2-4s+13)^2}-\frac{s-2}{(s^2+4s+13)^2}\right]\right]$$

6. $t(3\sin 2t-2\cos 2t)$

$$\left[\text{Ans.}: \frac{8+12s-2s^2}{(s^2+4)^2}\right]$$

7. $t\,e^{3t}\sin 2t$

$$\left[\text{Ans.}: \frac{4(s-3)}{(s^2-6s+13)^2}\right]$$

8. $t\sqrt{1+\sin 2t}$

$$\left[\textbf{Ans.}: \frac{s^2+2s-1}{(s^2+1)^2}\right]$$

9. $t\,e^{2t}(\cos t-\sin t)$

$$\left[\textbf{Ans.}: \frac{s^2-6s+7}{(s^2-4s+5)^2}\right]$$

10. $(t^2-3t+2)\sin 3t$

$$\left[\textbf{Ans.}: \frac{6s^4-18s^3+126s^2-162s+432}{(s^2+9)^3}\right]$$

11. $(t+\sin 2t)^2$

$$\left[\textbf{Ans.}: \frac{2}{s^3}+\frac{s}{(s^2+1)^2}+\frac{1}{2s}-\frac{s}{2(s^2+4)}\right]$$

12. $(t\sinh 2t)^2$

$$\left[\textbf{Ans.}: \frac{1}{2}\left[\frac{1}{(s-4)^3}+\frac{1}{(s+4)^3}\right]\right]$$

13. $t^2e^{-3t}\cosh 2t$

$$\left[\textbf{Ans.}: \frac{1}{(s+1)^3}+\frac{1}{(s+5)^3}\right]$$

14. $t^2e^{-2t}\sin 3t$

$$\left[\textbf{Ans.}: \frac{18(s^2+4s+1)}{(s^2+4s+13)^2}\right]$$

15. $(t\cos 2t)^2$

$$\left[\textbf{Ans.}: \frac{1}{s^3}-\frac{s(48-s^2)}{(s^2+16)^3}\right]$$

16. $t^2\sin t\cos 2t$

$$\left[\textbf{Ans.}: \frac{9(s^2-3)}{(s^2+9)^3}+\frac{1-3s^2}{(s^2+1)^3}\right]$$

17. $t^3 \cos t$

$$\left[\text{Ans.: } \frac{6s^4 - 36s^2 + 6}{(s^2+9)^3}\right]$$

2.6 INTEGRATION OF LAPLACE TRANSFORMS (DIVISION BY *t*)

If $L\{f(t)\} = F(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\text{d}s$. **[Winter 2014; Summer 2014]**

Proof: $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\text{d}t$

Integrating both the sides w.r.t. s from s to ∞,

$$\int_s^\infty F(s)\text{d}s = \int_s^\infty \int_0^\infty e^{-st} f(t)\text{d}t\,\text{d}s$$

Since s and t are independent variables, interchanging the order of integration,

$$\int_s^\infty F(s)\text{d}s = \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\,\text{d}s\right]\text{d}t$$

$$= \int_0^\infty \left|\frac{e^{-st}}{-t} f(t)\right|_s^\infty \text{d}t$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t}\text{d}t$$

$$= L\left\{\frac{f(t)}{t}\right\}$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\text{d}s$$

Example 1

Find the Laplace transform of $\dfrac{1-e^{-t}}{t}$.

Solution

$$L\{1-e^{-t}\} = \frac{1}{s} - \frac{1}{s+1}$$

$$L\left\{\frac{1-e^{-t}}{t}\right\}=\int_s^\infty L\{1-e^{-t}\}\,ds$$

$$=\int_s^\infty\left(\frac{1}{s}-\frac{1}{s+1}\right)ds$$

$$=\left|\log s-\log(s+1)\right|_s^\infty$$

$$=\left|\log\frac{s}{s+1}\right|_s^\infty$$

$$=\log\left|\frac{1}{1+\frac{1}{s}}\right|_s^\infty$$

$$=\log 1-\log\left(\frac{1}{1+\frac{1}{s}}\right)$$

$$=-\log\left(\frac{s}{s+1}\right)$$

$$=\log\left(\frac{s+1}{s}\right)$$

Example 2

Find the Laplace transform of $\dfrac{e^{-at}-e^{-bt}}{t}$.

Solution

$$L\{e^{-at}-e^{-bt}\}=\frac{1}{s+a}-\frac{1}{s+b}$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}=\int_s^\infty L\{e^{-at}-e^{-bt}\}\,ds$$

$$=\int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$$

$$=\left|\log(s+a)-\log(s+b)\right|_s^\infty$$

$$= \left|\log \frac{s+a}{s+b}\right|_s^{\infty}$$

$$= \left|\log \frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right|_s^{\infty}$$

$$= \log 1 - \log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)$$

$$= -\log\left(\frac{s+a}{s+b}\right)$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

Example 3

Find the Laplace transform of $\frac{\sinh t}{t}$.

Solution

$$L\{\sinh t\} = L\left\{\frac{e^t - e^{-t}}{2}\right\}$$

$$= \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

$$L\left\{\frac{\sinh t}{t}\right\} = \int_s^{\infty} L\{\sinh t\}\,ds$$

$$= \frac{1}{2}\int_s^{\infty}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)ds$$

$$= \frac{1}{2}\left|\log(s-1) - \log(s+1)\right|_s^{\infty}$$

$$= \frac{1}{2}\left|\log \frac{s-1}{s+1}\right|_s^{\infty}$$

$$= \frac{1}{2}\left|\log \frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right|_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right)\right]$$

$$= -\frac{1}{2}\log\left(\frac{s-1}{s+1}\right)$$

$$= \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$$

Example 4

Find the Laplace transform of $\frac{\sin 2t}{t}$. **[Winter 2014, 2012]**

Solution

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty L\{\sin 2t\}\,ds$$

$$= \int_s^\infty \frac{2}{s^2+4}\,ds$$

$$= 2\left|\frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \cot^{-1}\left(\frac{s}{2}\right)$$

$$= \tan^{-1}\left(\frac{2}{s}\right)$$

Example 5

Find the Laplace transform of $\frac{1-\cos 2t}{t}$. **[Winter 2017]**

Solution

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty L\{1-\cos 2t\}\mathrm{d}s$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)\mathrm{d}s$$

$$= \left|\log s - \frac{1}{2}\log(s^2+4)\right|_s^\infty$$

$$= -\frac{1}{2}\left|\log(s^2+4) - \log s^2\right|_s^\infty$$

$$= -\frac{1}{2}\left|\log\left(\frac{s^2+4}{s^2}\right)\right|_s^\infty$$

$$= -\frac{1}{2}\left|\log\left(1+\frac{4}{s^2}\right)\right|_s^\infty$$

$$= -\frac{1}{2}\log 1 + \frac{1}{2}\log\left(1+\frac{4}{s^2}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$$

Example 6

Find the Laplace transform of $\dfrac{\cos at - \cos bt}{t}$. **[Summer 2016]**

Solution

$$L\{\cos at - \cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty L\{\cos at - \cos bt\}\,\mathrm{d}s$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)\mathrm{d}s$$

$$= \left|\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right|_s^\infty$$

$$= \frac{1}{2}\left|\log \frac{s^2+a^2}{s^2+b^2}\right|_s^\infty$$

$$= \frac{1}{2}\left|\log \frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right|_s^\infty$$

$$= \frac{1}{2}\log 1 - \frac{1}{2}\log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)$$

$$= -\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Example 7

Find the Laplace transform of $\frac{e^{-t}\sin t}{t}$.

Solution

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{e^{-t}\sin t\} = \frac{1}{(s+1)^2+1}$$

$$L\left\{\frac{e^{-t}\sin t}{t}\right\} = \int_s^\infty L\{e^{-t}\sin t\}\, \mathrm{d}s$$

$$= \int_s^\infty \frac{1}{(s+1)^2+1}\mathrm{d}s$$

$$= \left|\tan^{-1}(s+1)\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1)$$

Example 8

Find the Laplace transform of $\dfrac{\cosh 2t \sin 2t}{t}$.

Solution

$$L\left\{\frac{\cosh 2t \sin 2t}{t}\right\} = L\left\{\left(\frac{e^{2t}+e^{-2t}}{2t}\right)\sin 2t\right\}$$

$$= \frac{1}{2}\left[L\left\{\frac{e^{2t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-2t}\sin 2t}{t}\right\}\right]$$

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty L\{\sin 2t\}\,\mathrm{d}s$$

$$= \int_s^\infty \frac{2}{s^2+4}\,\mathrm{d}s$$

$$= \left|\tan^{-1}\frac{s}{2}\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \cot^{-1}\left(\frac{s}{2}\right)$$

By the first shifting theorem,

$$L\left\{\frac{\cosh 2t \sin 2t}{t}\right\} = \frac{1}{2}\left[L\left\{e^{2t}\frac{\sin 2t}{t}\right\} + L\left\{e^{-2t}\frac{\sin 2t}{t}\right\}\right]$$

$$= \frac{1}{2}\left[\cot^{-1}\left(\frac{s-2}{2}\right) + \cot^{-1}\left(\frac{s+2}{2}\right)\right]$$

Example 9

Find the Laplace transform of $\dfrac{e^{-2t}\sin 2t \cosh t}{t}$.

Solution

$$L\left\{\frac{e^{-2t}\sin 2t \cosh t}{t}\right\} = L\left\{\frac{e^{-2t}\sin 2t}{t}\frac{(e^{t}+e^{-t})}{2}\right\}$$

$$= \frac{1}{2}\left[L\left\{\frac{e^{-t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-3t}\sin 2t}{t}\right\}\right]$$

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty L\{\sin 2t\}\,ds$$

$$= \int_s^\infty \frac{2}{s^2+4}\,ds$$

$$= 2\cdot\frac{1}{2}\left|\tan^{-1}\frac{s}{2}\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \cot^{-1}\left(\frac{s}{2}\right)$$

$$L\left\{\frac{e^{-2t}\sin 2t\cosh t}{t}\right\} = \frac{1}{2}\left[L\left\{\frac{e^{-t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-3t}\sin 2t}{t}\right\}\right]$$

$$= \frac{1}{2}\left[\cot^{-1}\left(\frac{s+1}{2}\right) + \cot^{-1}\left(\frac{s+3}{2}\right)\right]$$

Example 10

Find the Laplace transform of $\frac{1-\cos t}{t^2}$.

Solution

$$L\{1-\cos t\} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left\{\frac{1-\cos t}{t^2}\right\} = \int_s^\infty\int_s^\infty L\{1-\cos t\}\,ds\,ds$$

$$= \int_s^\infty\int_s^\infty\left[\frac{1}{s} - \frac{s}{s^2+1}\right]ds\,ds$$

$$= \int_s^\infty\left|\log s - \frac{1}{2}\log(s^2+1)\right|_s^\infty ds$$

$$= \int_s^\infty \left| \log \frac{s}{\sqrt{s^2+1}} \right|_s^\infty ds$$

$$= \int_s^\infty \left[0 - \log \frac{s}{\sqrt{s^2+1}} \right] ds$$

$$= -\int_s^\infty \log \frac{s}{\sqrt{s^2+1}} ds$$

$$= \int_s^\infty \log \frac{\sqrt{s^2+1}}{s} ds$$

$$= \frac{1}{2}\int_s^\infty \log\left(\frac{s^2+1}{s^2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \log\left(1+\frac{1}{s^2}\right) ds$$

$$= \frac{1}{2}\left[\left| s\log\left(1+\frac{1}{s^2}\right) \right|_s^\infty - \int_s^\infty s \frac{1}{\left(1+\frac{1}{s^2}\right)}\left(-\frac{2}{s^3}\right) ds \right]$$

$$= \frac{1}{2}\left[0 - s\log\left(1+\frac{1}{s^2}\right) + 2\int_s^\infty \frac{1}{s^2+1} ds \right]$$

$$= -\frac{1}{2} s\log\left(1+\frac{1}{s^2}\right) + \left|\tan^{-1} s\right|_s^\infty$$

$$= -\frac{s}{2}\log\left(\frac{s^2+1}{s^2}\right) + \frac{\pi}{2} - \tan^{-1} s$$

$$= -\frac{s}{2}\log\left(\frac{s^2+1}{s^2}\right) + \cot^{-1} s$$

Example 11

Find the Laplace transform of $\frac{\sin^2 t}{t^2}$.

Solution

$$L\left\{\frac{\sin^2 t}{t^2}\right\} = L\left\{\frac{1-\cos 2t}{2t^2}\right\}$$

$$= \frac{1}{2} L\left\{\frac{1-\cos 2t}{t^2}\right\}$$

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty L\{1-\cos 2t\}\,ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \left|\log s - \frac{1}{2}\log(s^2+4)\right|_s^\infty$$

$$= \left|\log \frac{s}{\sqrt{s^2+4}}\right|_s^\infty$$

$$= \left|\log \frac{1}{\sqrt{1+\frac{4}{s^2}}}\right|_s^\infty$$

$$= \log 1 - \log\left(\frac{1}{\sqrt{1+\frac{4}{s^2}}}\right)$$

$$= -\log\left(\frac{s}{\sqrt{s^2+4}}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$$

$$L\left\{\frac{1-\cos 2t}{t^2}\right\} = \int_s^\infty L\left\{\frac{1-\cos 2t}{t}\right\} ds$$

$$= \frac{1}{2}\int_s^\infty \log\left\{\frac{s^2+4}{s^2}\right\} ds$$

$$= \frac{1}{2}\left[\left|s\log\left(\frac{s^2+4}{s^2}\right)\right|_s^\infty - \int_s^\infty s\frac{s^2}{s^2+4}\left\{\frac{2s(s^2)-2s(s^2+4)}{s^4}\right\} ds\right]$$

$$= \frac{1}{2}\left[-s\log\left(\frac{s^2+4}{s^2}\right) - \int_s^\infty -\frac{8}{s^2+4} ds\right]$$

$$= \frac{1}{2}\left[-s \log\left(\frac{s^2+4}{s^2}\right) + 8 \cdot \frac{1}{2}\left|\tan^{-1}\frac{s}{2}\right|_s^\infty\right]$$

$$= \frac{1}{2}\left[-s \log\left(\frac{s^2+4}{s^2}\right) + 4\left\{\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)\right\}\right]$$

$$= \frac{1}{2}\left[-s \log\left(\frac{s^2+4}{s^2}\right) + 4\cot^{-1}\left(\frac{s}{2}\right)\right]$$

$$L\left\{\frac{\sin^2 t}{t^2}\right\} = \frac{1}{2} L\left\{\frac{1-\cos 2t}{t^2}\right\}$$

$$= \frac{1}{4}\left[-s \log\left(\frac{s^2+4}{s^2}\right) + 4 \cot^{-1}\left(\frac{s}{2}\right)\right]$$

EXERCISE 2.6

Find the Laplace transforms of the following functions:

1. $\frac{\sin^2 t}{t}$

$$\left[\textbf{Ans.}: \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)\right]$$

2. $\left(\frac{\sin 2t}{\sqrt{t}}\right)^2$

$$\left[\textbf{Ans.}: \frac{1}{4}\log\left(\frac{s^2+16}{s^2}\right)\right]$$

3. $\frac{\sin^3 t}{t}$

$$\left[\textbf{Ans.}: \frac{1}{4}\left(3\cot^{-1} s - \cot^{-1}\frac{s}{3}\right)\right]$$

4. $\frac{1-\cos at}{t}$

$$\left[\textbf{Ans.}: \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2}\right)\right]$$

5. $\dfrac{\sin t \sin 5t}{t}$ $\left[\text{Ans.: } \dfrac{1}{2}\log\left(\dfrac{s^2+36}{s^2+16}\right)\right]$

6. $\dfrac{2\sin t \sin 2t}{t}$ $\left[\text{Ans.: } \dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2+1}\right)\right]$

7. $\dfrac{e^{2t}\sin t}{t}$ $\left[\text{Ans.: } \cot^{-1}(s-2)\right]$

8. $\dfrac{e^{2t}\sin^3 t}{t}$ $\left[\text{Ans.: } \dfrac{3}{4}\cot^{-1}(s-2)-\dfrac{1}{4}\cot^{-1}\left(\dfrac{s-2}{3}\right)\right]$

2.7 LAPLACE TRANSFORMS OF DERIVATIVES

If $L\{f(t)\}=F(s)$ then $L\{f'(t)\}=sF(s)-f(0)$

$$L\{f''(t)\}=s^2F(s)-sf(0)-f'(0)$$

In general,

$$L\{f^n(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\ldots-f^{(n-1)}(0)$$

Proof: $L\{f'(t)\}=\int_0^\infty e^{-st}f'(t)dt$

Integrating by parts,

$$L\{f'(t)\}=\left|e^{-st}f(t)\right|_0^\infty-\int_0^\infty(-se^{-st})f(t)\mathrm{d}t$$

$$=-f(0)+s\int_0^\infty e^{-st}f(t)\mathrm{d}t$$

$$=-f(0)+sL\{f(t)\}$$

Similarly, $L\{f''(t)\}=-f'(0)+s\,L\{f'(t)\}$

$$=-f'(0)+s\left[-f(0)+sL\{f(t)\}\right]$$

$$=-f'(0)-sf(0)+s^2L\{f(t)\}$$

In general, $L\{f^n(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\ldots-f^{n-1}(0)$

Example 1

Find $L\{f(t)\}$ *and* $L\{f'(t)\}$ *of* $f(t)=\dfrac{\sin t}{t}$.

Solution

$$
\begin{aligned}
L\{f(t)\} = F(s) &= L\left\{\frac{\sin t}{t}\right\} \\
&= \int_s^\infty L\{\sin t\}\mathrm{d}s \\
&= \int_s^\infty \frac{1}{s^2+1}\mathrm{d}s \\
&= \left|\tan^{-1} s\right|_s^\infty \\
&= \frac{\pi}{2} - \tan^{-1} s \\
&= \cot^{-1} s
\end{aligned}
$$

$$
\begin{aligned}
L\{f'(t)\} &= sF(s) - f(0) \\
&= s\cot^{-1} s - \lim_{t\to 0}\frac{\sin t}{t} \\
&= s\cot^{-1} s - 1
\end{aligned}
$$

Example 2

Find $L\{f(t)\}$ *and* $L\{f'(t)\}$ *of* $f(t) = 3 \quad 0 \le t < 5$

$$= 0 \quad t > 5$$

Solution

$$
\begin{aligned}
L\{f(t)\} = F(s) &= \int_0^\infty e^{-st} f(t)\mathrm{d}t \\
&= \int_0^5 e^{-st}\cdot 3\mathrm{d}t + \int_5^\infty 0\cdot \mathrm{d}t \\
&= 3\left|\frac{e^{-st}}{-s}\right|_0^5 + 0 \\
&= \frac{-3}{s}(e^{-5s}-1)
\end{aligned}
$$

$$= \frac{3}{s}(1 - e^{-5s})$$

$$L\{f'(t)\} = sF(s) - f(0)$$

$$= s\frac{3}{s}(1 - e^{-5s}) - 3$$

$$= -3e^{-5s}$$

Example 3

Find $L\{f(t)\}$ *and* $L\{f'(t)\}$ *of* $f(t) = e^{-5t}\sin t$.

Solution

$$L\{f(t)\} = F(s) = L\{e^{-5t}\sin t\}$$

$$= \frac{1}{(s+5)^2 + 1}$$

$$L\{f'(t)\} = sF(s) - f(0)$$

$$= s\left(\frac{1}{s^2 + 10s + 26}\right) - e^0 \sin 0$$

$$= \frac{s}{s^2 + 10s + 26}$$

Example 4

Find $L\{f(t)\}$ *and* $L\{f'(t)\}$ *of* $f(t) = t \qquad 0 \le t < 3$

$$= 6 \qquad t > 3$$

Solution

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^3 e^{-st} t\, dt + \int_3^\infty e^{-st} \cdot 6 dt$$

$$= \left|\frac{e^{-st}}{-s} \cdot t\right|_0^3 - \left|\frac{e^{-st}}{s^2}\right|_0^3 + 6\left|\frac{e^{-st}}{-s}\right|_3^\infty$$

$$= -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} + \frac{6}{s} e^{-3s}$$

$$= \frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)$$

$$L\{f'(t)\} = s\,F(s) - f(0)$$

$$= s\left[\frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)\right] - 0$$

$$= \frac{1}{s} + e^{-3s}\left(3 - \frac{1}{s}\right)$$

EXERCISE 2.7

Find $L\{f'(t)\}$ of the following functions:

1. $f(t) = \left(\dfrac{1-\cos 2t}{t}\right)$

$$\left[\textbf{Ans.: } s\log\left(\frac{\sqrt{s^2+4}}{s}\right)\right]$$

2. $f(t) = t + 1 \quad 0 \le t \le 2$

   $\quad = 3 \qquad t > 2$

$$\left[\textbf{Ans.: } \frac{1}{s}(1 - e^{-2s})\right]$$

2.8 LAPLACE TRANSFORMS OF INTEGRALS

If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \dfrac{F(s)}{s}$.

Proof: $L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \int_0^\infty e^{-st}\left\{\int_0^t f(t)\mathrm{d}t\right\}\mathrm{d}t$

Integrating by parts,

$$L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \left|\int_0^t f(t)\mathrm{d}t\left(\frac{e^{-st}}{-s}\right)\right|_0^\infty - \int_0^\infty\left[\left(\frac{e^{-st}}{-s}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(t)\mathrm{d}t\right)\right]\mathrm{d}t$$

$$= \int_0^\infty \frac{1}{s}e^{-st} f(t)\mathrm{d}t$$

$$= \frac{1}{s}L\{f(t)\}$$

$$= \frac{F(s)}{s}$$

Example 1

Find the Laplace transform of $\int_0^t e^{-t}\mathrm{d}t$.

Solution

$$L\{e^{-t}\} = \frac{1}{s+1}$$

$$L\left\{\int_0^t e^{-t}\mathrm{d}t\right\} = \frac{1}{s}L\{e^{-t}\}$$

$$= \frac{1}{s(s+1)}$$

Example 2

Find the Laplace transform of $\int_0^t e^{-2t}t^3\,\mathrm{d}t$.

Solution

$$L\{e^{-2t}t^3\} = \frac{3!}{(s+2)^4}$$

$$= \frac{6}{(s+2)^4}$$

$$L\left\{\int_0^t e^{-2t}t^3\mathrm{d}t\right\} = \frac{1}{s}L\left\{e^{-2t}t^3\right\}$$

$$= \frac{6}{s(s+2)^4}$$

Example 3

Find the Laplace transform of $\int\limits_0^t e^{-t}\cos t\,\mathrm{d}t$. **[Summer 2013]**

Solution

$$L\{\cos t\} = \frac{s}{s^2+1}$$

$$L\{e^{-t}\cos t\} = \frac{s+1}{(s+1)^2+1}$$

$$= \frac{s+1}{s^2+2s+2}$$

$$L\left\{\int_0^t e^{-t}\cos t\,\mathrm{d}t\right\} = \frac{1}{s}L\{e^{-t}\cos t\}$$

$$= \frac{s+1}{s^3+2s^2+2s}$$

Example 4

Find the Laplace transform of $\int_0^t e^u(u+\sin u)\mathrm{d}u$. **[Winter 2015]**

Solution

$$L\{t+\sin t\} = L\{t\}+L\{\sin t\}$$

$$= \frac{1}{s^2}+\frac{1}{s^2+1}$$

$$L\{e^t(t+\sin t)\} = \frac{1}{(s-1)^2}+\frac{1}{(s-1)^2+1}$$

$$L\left\{\int_0^t e^u(u+\sin u)\,\mathrm{d}u\right\} = \frac{1}{s}L\{e^t(t+\sin t)\}$$

$$= \frac{1}{s}\left[\frac{1}{s^2-2s+1}+\frac{1}{s^2-2s+2}\right]$$

Example 5

Find the Laplace transform of $\int_0^t t\cosh t\,\mathrm{d}t$.

Solution

$$L\{t\cosh t\} = L\left\{t\left(\frac{e^t+e^{-t}}{2}\right)\right\}$$

$$= \frac{1}{2}L\{t\,e^t+t\,e^{-t}\}$$

$$= \frac{1}{2}\left[\frac{1}{(s-1)^2}+\frac{1}{(s+1)^2}\right]$$

$$= \frac{1}{2}\cdot\frac{2(s^2+1)}{(s^2-1)^2}$$

$$= \frac{s^2+1}{(s^2-1)^2}$$

$$L\left\{\int_0^t t\cosh t\,dt\right\} = \frac{1}{s}L\{t\cosh t\}$$

$$= \frac{s^2+1}{s(s^2-1)^2}$$

Example 6

Find the Laplace transform of $\int_0^t t\,e^{-4t}\sin 3t\,dt$.

Solution

$$L\{t\sin 3t\} = -\frac{d}{ds}L\{\sin 3t\}$$

$$= -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{6s}{(s^2+9)^2}$$

$$L\{t\,e^{-4t}\sin 3t\} = \frac{6(s+4)}{\left[(s+4)^2+9\right]^2}$$

$$= \frac{6(s+4)}{(s^2+8s+25)^2}$$

$$L\left\{\int_0^t t\,e^{-4t}\sin 3t\,dt\right\} = \frac{1}{s}L\{t\,e^{-4t}\sin 3t\}$$

$$= \frac{6(s+4)}{s(s^2+8s+25)^2}$$

Example 7

Find the Laplace transform of $e^{-4t}\int_0^t t\sin 3t\,dt$.

Solution

$$L\{t\sin 3t\} = -\frac{d}{ds}L\{\sin 3t\}$$

$$= -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{6s}{(s^2+9)^2}$$

$$L\left\{\int_0^t t\sin 3t\right\} = \frac{1}{s}L\{t\sin 3t\}$$

$$= \frac{6}{(s^2+9)^2}$$

$$L\left\{e^{-4t}\int_0^t t\sin 3t\right\} = \frac{6}{\left[(s+4)^2+9\right]^2}$$

$$= \frac{6}{(s^2+8s+25)^2}$$

Example 8

Find the Laplace transform of $\cosh t \int_0^t e^t \cosh t \, dt$.

Solution

$$L\{\cosh t\} = \frac{s}{s^2-1}$$

$$L\{e^t\cosh t\} = \frac{s-1}{(s-1)^2-1}$$

$$= \frac{s-1}{s^2-2s+1-1}$$

$$= \frac{s-1}{s(s-2)}$$

$$L\left\{\int_0^t e^t\cosh t \, dt\right\} = \frac{1}{s}L\{e^t\cosh t\}$$

$$= \frac{s-1}{s^2(s-2)}$$

$$L\left\{\cosh t\int_0^t e^t\cosh t \, dt\right\} = L\left\{\left(\frac{e^t+e^{-t}}{2}\right)\int_0^t e^t\cosh t \, dt\right\}$$

$$= \frac{1}{2}\left[L\left\{e^t\int_0^t e^t\cosh t \, dt\right\} + L\left\{e^{-t}\int_0^t e^t\cosh t \, dt\right\}\right]$$

$$= \frac{1}{2}\left[\frac{(s-1)-1}{(s-1)^2(s-1-2)} + \frac{(s+1)-1}{(s+1)^2(s+1-2)}\right]$$

$$= \frac{1}{2}\left[\frac{s-2}{(s-1)^2(s-3)} + \frac{s}{(s+1)^2(s-1)}\right]$$

Example 9

Find the Laplace transform of $e^{-t}\int_0^t \frac{\sin t}{t}\mathrm{d}t$.

Solution

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}\mathrm{d}s$$

$$= \int_s^\infty \frac{1}{s^2+1}\,\mathrm{d}s$$

$$= \left|\tan^{-1} s\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$L\left\{\int_0^t \frac{\sin t}{t}\mathrm{d}t\right\} = \frac{1}{s}L\left\{\frac{\sin t}{t}\right\}$$

$$= \frac{1}{s}\cot^{-1} s$$

$$L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\mathrm{d}t\right\} = \frac{1}{s+1}\cot^{-1}(s+1)$$

Example 10

Find the Laplace transform of $\int_0^t e^t \frac{\sin t}{t}\,\mathrm{d}t$. **[Winter 2016]**

Solution

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}\, ds$$

$$= \int_s^\infty \frac{1}{s^2+1}\, ds$$

$$= \left| \tan^{-1} s \right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$L\left\{e^t \frac{\sin t}{t}\right\} = \cot^{-1}(s-1)$$

$$L\left\{\int_0^t e^t \frac{\sin t}{t}\, dt\right\} = \frac{1}{s} L\left\{e^t \frac{\sin t}{t}\right\}$$

$$= \frac{1}{s} \cot^{-1}(s-1)$$

Example 11

Find the Laplace transform of $t \int_0^t e^{-4t} \sin 3t\, dt$.

Solution

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{e^{-4t} \sin 3t\} = \frac{3}{(s+4)^2+9}$$

$$= \frac{3}{s^2+8s+25}$$

$$L\left\{\int_0^t e^{-4t} \sin 3t\, dt\right\} = \frac{1}{s} L\{e^{-4t} \sin 3t\}$$

$$= \frac{3}{s^3+8s^2+25s}$$

$$L\left\{t \int_0^t e^{-4t} \sin 3t\, dt\right\} = -\frac{d}{ds} L\left\{\int_0^t e^{-4t} \sin 3t\, dt\right\}$$

$$= -\frac{d}{ds}\left(\frac{3}{s^3+8s^2+25s}\right)$$

$$=\frac{3(3s^2+16s+25)}{(s^3+8s^2+25s)^2}$$

Example 12

Find the Laplace transform of $\int_0^t t\,e^{-3t}\sin^2 t\,dt$.

Solution

$$L\{\sin^2 t\}=L\left\{\frac{1-\cos 2t}{2}\right\}$$

$$=\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$L\{t\sin^2 t\}=-\frac{d}{ds}L\{\sin^2 t\}$$

$$=-\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$=-\frac{1}{2}\left[-\frac{1}{s^2}-\left\{\frac{s^2+4-s(2s)}{(s^2+4)^2}\right\}\right]$$

$$=\frac{1}{2}\left[\frac{1}{s^2}-\frac{s^2-4}{(s^2+4)^2}\right]$$

$$L\{t\,e^{-3t}\sin^2 t\}=\frac{1}{2}\left[\frac{1}{(s+3)^2}-\frac{(s+3)^2-4}{\{(s+3)^2+4\}^2}\right]$$

$$=\frac{1}{2}\left[\frac{1}{(s+3)^2}-\frac{s^2+6s+5}{(s^2+6s+13)^2}\right]$$

$$L\left\{\int_0^t t\,e^{-3t}\sin^2 t\,dt\right\}=\frac{1}{s}L\{t\,e^{-3t}\sin^2 t\}$$

$$=\frac{1}{2s}\left[\frac{1}{(s+3)^2}-\frac{s^2+6s+5}{(s^2+6s+13)^2}\right]$$

Example 13

Find the Laplace transform of $\int_0^t\int_0^t \sin at\,dt\,dt$. **[Summer 2013]**

Solution

$$L\{\sin at\}=\frac{a}{s^2+a^2}$$

$$L\left\{\int_0^t \sin at\, dt\right\} = \frac{1}{s} L\{\sin at\}$$

$$= \frac{1}{s}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{a}{s(s^2+a^2)}$$

$$L\left\{\int_0^t\int_0^t \sin at\, dt\, dt\right\} = \frac{1}{s} L\left\{\int_0^t \sin at\, dt\right\}$$

$$= \frac{1}{s} \frac{a}{s(s^2+a^2)}$$

$$= \frac{a}{s^2(s^2+a^2)}$$

Example 14

Find the Laplace transform of $\int_0^t\int_0^t\int_0^t t \sin t\, dt\, dt\, dt$.

Solution

$$L\{t \sin t\} = -\frac{d}{ds} L\{\sin t\}$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{2s}{(s^2+1)^2}$$

$$L\left\{\int_0^t t \sin t\, dt\right\} = \frac{1}{s} L\{t \sin t\}$$

$$L\left\{\int_0^t\int_0^t t \sin t\, dt\right\} = \frac{1}{s} L\left\{\int_0^t t \sin t\, dt\right\}$$

$$= \frac{1}{s}\cdot\frac{1}{s} L\{t \sin t\}$$

$$L\left\{\int_0^t\int_0^t\int_0^t t \sin t\, dt\right\} = \frac{1}{s} L\left\{\int_0^t\int_0^t t \sin t\, dt\right\}$$

$$= \frac{1}{s}\cdot\frac{1}{s}\cdot\frac{1}{s} L\{t \sin t\}$$

$$= \frac{1}{s^3}\cdot\frac{2s}{(s^2+1)^2}$$

$$= \frac{2}{s^2(s^2+1)^2}$$

EXERCISE 2.8

Find the Laplace transforms of the following functions:

1. $\int_0^t e^{-t} t^4 \, dt$

$$\left[\text{Ans.}: \frac{4!}{s(s+1)^5}\right]$$

2. $\int_0^t \frac{1+e^{-t}}{t} dt$

$$\left[\text{Ans.}: \frac{1}{s}\log\left[s(s+1)\right]\right]$$

3. $\int_0^t \frac{e^t \sin t}{t} dt$

$$\left[\text{Ans.}: \frac{1}{s}\cot^{-1}(s-1)\right]$$

4. $\int_0^t t e^{-2t} \sin 3t \, dt$

$$\left[\text{Ans.}: \frac{1}{s} \cdot \frac{3(2s+4)}{(s^2+4s+13)^2}\right]$$

5. $e^{-3t} \int_0^t t \sin 3t \, dt$

$$\left[\text{Ans.}: -\frac{6}{(s^2+6s+18)^2}\right]$$

6. $\int_0^t t^2 \sin t \, dt$

$$\left[\text{Ans.}: -\frac{2(1-3s^2)}{s(s^2+1)^3}\right]$$

7. $\int_0^t t \cos^2 t \, dt$

$$\left[\text{Ans.}: \frac{1}{2s^3} + \frac{1}{2} \cdot \frac{s^2-4}{s(s^2+4)^2}\right]$$

8. $\int_0^t t e^{-3t} \cos^2 2t \, dt$

$$\left[\text{Ans.}: \frac{1}{2s(s+3)^2} + \frac{1}{2} \cdot \frac{s^2+6s-7}{s(s^2+6s+25)^2}\right]$$

2.9 UNIT STEP FUNCTION (HEAVISIDE FUNCTION)

Unit step function (Fig. 2.1) is defined as

$$u(t) = 0 \qquad t < 0$$
$$= 1 \qquad t > 0$$

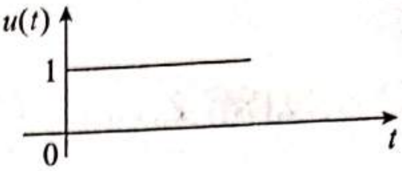


Fig. 2.1 ***Unit step function***

The displaced (delayed) unit step function $u(t-a)$ (Fig. 2.2) represents the function $u(t)$ which is displaced by a distance 'a' to the right.

$$u(t-a) = 0 \qquad t < a$$
$$= 1 \qquad t > a$$

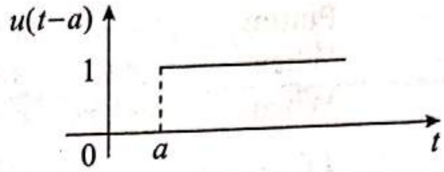


Fig. 2.2 Delayed *unit step function*

Laplace Transforms of Unit Step Functions

(i) Laplace transform of the unit step function $u(t)$

$$u(t) = 0 \qquad t < 0$$
$$= 1 \qquad t > 0$$

$$L\{u(t)\} = \int_0^\infty e^{-st} u(t) \mathrm{d}t$$
$$= \int_0^\infty e^{-st} \mathrm{d}t$$
$$= \left| \frac{e^{-st}}{-s} \right|_0^\infty$$
$$= \frac{1}{s}$$

(ii) Laplace transform of the displaced unit step function $u(t-a)$

$$u(t-a) = 0 \qquad t < a$$
$$= 1 \qquad t > a$$

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a) \mathrm{d}t$$
$$= \int_a^\infty e^{-st} \mathrm{d}t$$
$$= \left| \frac{e^{-st}}{-s} \right|_a^\infty$$
$$= \frac{1}{s} e^{-as}$$

(iii) Laplace transform of the function $f(t)\, u(t-a)$

$$f(t)\,u(t-a) = 0 \qquad t < a$$
$$= f(t) \qquad t > a$$

$$L\{f(t)\,u(t-a)\} = \int_0^\infty e^{-st} f(t)u(t-a)\mathrm{d}t$$
$$= \int_a^\infty e^{-st} f(t)\mathrm{d}t$$

Putting $t-a=x$, $\mathrm{d}t=\mathrm{d}x$

When $t=a$, $x=0$

When $t\to\infty$, $x\to\infty$

$$L\{f(t)\,u(t-a)\} = \int_0^\infty e^{-s(x+a)} f(x+a)\mathrm{d}x$$
$$= e^{-as}\int_0^\infty e^{-sx} f(x+a)\mathrm{d}x$$
$$= e^{-as}\int_0^\infty e^{-st} f(t+a)\mathrm{d}t$$
$$= e^{-as} L\{f(t+a)\}$$
$$= e^{-as} F(s+a)$$

(iv) Laplace transform of the function $f(t-a)\, u(t-a)$

$$f(t-a)\,u(t-a) = 0 \qquad t < a$$
$$= f(t-a) \qquad t > a$$

$$L\{f(t-a)\,u(t-a)\} = \int_0^\infty e^{-st} f(t-a)u(t-a)\mathrm{d}t$$
$$= \int_a^\infty e^{-st} f(t-a)\mathrm{d}t$$

Putting $t-a=x$, $\mathrm{d}t=\mathrm{d}x$

When $t=a$, $x=0$

When $t\to\infty$, $x\to\infty$

$$L\{f(t-a)\,u(t-a)\} = \int_0^\infty e^{-s(a+x)} f(x)\mathrm{d}x$$
$$= e^{-as}\int_0^\infty e^{-sx} f(x)\mathrm{d}x$$
$$= e^{-as} L\{f(x)\}$$
$$= e^{-as} F(s)$$

Example 1

Find the Laplace transform of $e^{-3t}u(t-2)$. **[Winter 2017]**

Solution

$$L\{f(t)u(t-a)\} = e^{-as}L\{f(t+a)\}$$

$$L\{e^{-3t}u(t-2)\} = e^{-2s}L\{e^{-3(t+2)}\}$$

$$= e^{-2s}e^{-6}L\{e^{-3t}\}$$

$$= e^{-(2s+6)}\frac{1}{s+3}$$

Example 2

Find the Laplace transform of $t^2\, u(t-2)$. **[Winter 2014]**

Solution

$$L\{f(t)\,u(t-a)\} = e^{-as}L\{f(t+a)\}$$

$$L\{t^2\,u(t-2)\} = e^{-2s}L\{(t+2)^2\}$$

$$= e^{-2s}L\{t^2+4t+4\}$$

$$= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)$$

Example 3

Find the Laplace transform of $\sin t\, u\left(t-\frac{\pi}{2}\right) - u\left(t-\frac{3\pi}{2}\right)$.

Solution

$$L\{f(t)\,u(t-a)\} = e^{-as}L\{f(t+a)\}$$

$$L\left\{\sin t\, u\left(t-\frac{\pi}{2}\right)-u\left(t-\frac{3\pi}{2}\right)\right\} = L\left\{\sin t\, u\left(t-\frac{\pi}{2}\right)\right\}-L\left\{u\left(t-\frac{3\pi}{2}\right)\right\}$$

$$= e^{-\frac{\pi s}{2}}L\left\{\sin\left(t+\frac{\pi}{2}\right)\right\}-\frac{e^{-\frac{3\pi s}{2}}}{s}$$

$$= e^{\frac{-\pi s}{2}}L\{\cos t\}-\frac{e^{-\frac{3\pi s}{2}}}{s}$$

$$= e^{\frac{-\pi s}{2}}\frac{s}{s^2+1}-e^{-\frac{3\pi s}{2}}\frac{1}{s}$$

Example 4

Find the Laplace transform of $e^{-t} \sin t \, u(t-\pi)$.

Solution

$$L\{f(t)\, u(t-a)\} = e^{-as} L\{ f(t+a)\}$$

$$L\{e^{-t} \sin t \, u(t-\pi)\} = e^{-\pi s} L\{e^{-(t+\pi)} \sin (t+\pi)\}$$

$$= -e^{-\pi s} e^{-\pi} L\{e^{-t} \sin t\}$$

$$= -e^{-\pi(s+1)} \frac{1}{(s+1)^2+1}$$

$$= -e^{-\pi(s+1)} \frac{1}{s^2+2s+2}$$

Example 5

Find the Laplace transform of $(1+2t-3t^2+4t^3)\, u(t-2)$ *and, hence evaluate* $\int_0^\infty e^{-t}(1+2t-3t^2+4t^3)\, u(t-2)dt$.

Solution

$$L\{f(t)\, u(t-a)\} = e^{-as} L\{f(t+a)\}$$

$$L\{(1+2t-3t^2+4t^3)\, u(t-2)\} = e^{-2s} L[1+2(t+2)-3(t+2)^2+4(t+2)^3]$$

$$= e^{-2s} L\{1+2(t+2)-3(t^2+4t+4) + 4(t^3+6t^2+12t+8)\}$$

$$= e^{-2s} L\{25+38t+21t^2+4t^3\}$$

$$= e^{-2s}\left(\frac{25}{s}+38\cdot\frac{1}{s^2}+21\cdot\frac{2!}{s^3}+4\cdot\frac{3!}{s^4}\right)$$

$$= e^{-2s}\left(\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right)$$

Now, $\int_0^\infty e^{-st}(1+2t-3t^2+4t^3)\, u(t-2)\, dt = e^{-2s}\left(\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right)$ ...(1)

Putting $s=1$ in Eq. (1),

$$\int_0^\infty e^{-t}(1+2t-3t^2+4t^3)\, u(t-2)dt = e^{-2}\left(\frac{25}{1}+\frac{38}{1^2}+\frac{42}{1^3}+\frac{24}{1^4}\right)$$

$$= \frac{129}{e^2}$$

Example 6

Find the Laplace transform of $f(t) = t^2 \quad 0 < t < 1$
$= 4t \quad t > 1$

Solution

Expressing $f(t)$ in terms of the unit step function,

$$f(t) = t^2 u(t) - t^2 u(t-1) + 4t\, u(t-1)$$

$$L\{f(t)\} = L\{t^2 u(t) - t^2 u(t-1) + 4t\, u(t-1)\}$$

$$= L\{t^2 u(t)\} - L\{t^2 u(t-1) + 4\, L\{t\, u(t-1)\}$$

$$= \frac{2}{s^3} - e^{-s} L\{(t+1)^2\} + 4e^{-s} L\{(t+1)\}$$

$$= \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) + 4e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

$$= \frac{2}{s^3} + e^{-s}\left(-\frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}\right)$$

Example 7

Find the Laplace transform of $f(t) = \sin 2t \quad 2\pi < t < 4\pi$
$= 0$ *otherwise*

Solution

Expressing $f(t)$ in terms of the unit step function,

$$f(t) = \sin 2t\, u(t-2\pi) - \sin 2t\, u(t-4\pi)$$

$$L\{f(t)\} = L\{\sin 2t\, u(t-2\pi) - \sin 2t\, u(t-4\pi)\}$$

$$= L\{\sin 2t\, u(t-2\pi)\} - L\{\sin 2t\, u(t-4\pi)\}$$

$$= e^{-2\pi s} L\{\sin 2(t+2\pi)\} - e^{-4\pi s} L\{\sin 2(t+4\pi)\}$$

$$= e^{-2\pi s} L\{\sin 2t\} - e^{-4\pi s} L\{\sin 2t\}$$

$$= e^{-2\pi s} \frac{2}{s^2+4} - e^{-4\pi s} \frac{2}{s^2+4} = \frac{2}{s^2+4}\left(e^{-2\pi s} - e^{-4\pi s}\right)$$

Example 8

Find the Laplace transform of $f(t) = \cos t \quad 0 < t < \pi$
$= \sin t \quad t > \pi$

Solution

Expressing $f(t)$ in terms of the unit step function,

$$f(t) = \cos t\, u(t) - \cos t\, u(t-\pi) + \sin t\, u(t-\pi)$$

$$L\{f(t)\} = L\{\cos t\, u(t) - \cos t\, u(t-\pi) + \sin t\, u(t-\pi)\}$$
$$= L\{\cos t\, u(t)\} - L\{\cos t\, u(t-\pi)\} + L\{\sin t\, u(t-\pi)\}$$
$$= \frac{s}{s^2+1} - e^{-\pi s} L\{\cos(t+\pi)\} + e^{-\pi s} L\{\sin(t+\pi)\}$$
$$= \frac{s}{s^2+1} - e^{-\pi s} L\{-\cos t\} + e^{-\pi s} L\{-\sin t\}$$
$$= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos t\} - e^{-\pi s} L\{\sin t\}$$
$$= \frac{s}{s^2+1} + e^{-\pi s} \cdot \frac{s}{s^2+1} - e^{-\pi s} \cdot \frac{1}{s^2+1}$$
$$= \frac{1}{s^2+1}[s + e^{-\pi s}(s-1)]$$

Example 9

Find the Laplace transform of
$$f(t) = \cos t \qquad 0 < t < \pi$$
$$= \cos 2t \qquad \pi < t < 2\pi$$
$$= \cos 3t \qquad t > 2\pi$$

Solution

Expressing $f(t)$ in terms of the unit step function,

$$f(t) = [\cos t\, u(t) - \cos t\, u(t-\pi)] + [\cos 2t\, u(t-\pi) - \cos 2t\, u(t-2\pi)] + \cos 3t\, u(t-2\pi)$$
$$= \cos t\, u(t) + (\cos 2t - \cos t)\, u(t-\pi) + (\cos 3t - \cos 2t)\, u(t-2\pi)$$

$$L\{f(t)\} = L\{\cos t\, u(t)\} + L\{(\cos 2t - \cos t)\, u(t-\pi)\} + L\{(\cos 3t - \cos 2t)\, u(t-2\pi)\}$$
$$= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos 2(t+\pi) - \cos(t+\pi)\} + e^{-2\pi s} L\{\cos 3(t+2\pi) - \cos 2(t+2\pi)\}$$
$$= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos 2t + \cos t\} + e^{-2\pi s} L\{\cos 3t - \cos 2t\}$$
$$= \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

EXERCISE 2.9

(I) Find the Laplace transforms of the following functions:

1. $t^4\, u(t-2)$

$$\left[\textbf{Ans.: } e^{-2s}\left(\frac{16}{s} + \frac{32}{s^2} + \frac{48}{s^3} + \frac{48}{s^4} + \frac{24}{s^5}\right)\right]$$

2. $(1+3t-4t^2+2t^3)\,u(t-3)$

$$\left[\textbf{Ans.: } e^{-3s}\left(\frac{28}{s}+\frac{33}{s^2}+\frac{28}{s^3}+\frac{12}{s^4}\right)\right]$$

3. $t\,e^{-2t}\,u(t-1)$

$$\left[\textbf{Ans.: } e^{-(s+2)}\frac{s+3}{(s+2)^2}\right]$$

4. $\cos t\; u(t-1)$

$$\left[\textbf{Ans.: } e^{-s}\left(\frac{s\cos 1-\sin 1}{s^2+1}\right)\right]$$

(II) Express the following functions in terms of the unit step function and, hence, find the Laplace transform.

1. $f(t)=t \qquad 0<t<2$
   $\quad\; = t^2 \qquad t>2$

$$\left[\textbf{Ans.: } \frac{1}{s^2}+e^{-2s}\left(\frac{2}{s^3}+\frac{3}{s^2}+\frac{2}{s}\right)\right]$$

2. $f(t)=e^t\cos t \qquad 0<t<\pi$
   $\quad\; = e^t\sin t \qquad t>\pi$

$$\left[\textbf{Ans.: } \frac{s-1}{s^2-2s+2}+e^{-\pi(s-1)}\cdot\frac{s-2}{s^2-2s+2}\right]$$

3. $f(t)=\sin t \qquad 0<t<\pi$
   $\quad\; = \sin 2t \qquad \pi<t<2\pi$
   $\quad\; = \sin 3t \qquad t>2\pi$

$$\left[\textbf{Ans.: } \frac{1}{s^2+1}+e^{-\pi s}\left(\frac{2}{s^2+4}+\frac{1}{s^2+1}\right)-e^{-2\pi s}\left(\frac{3}{s^2+9}+\frac{2}{s^2+4}\right)\right]$$

4. $f(t)=t-1 \qquad 1<t<2$
   $\quad\; = 3-t \qquad 2<t<3$
   $\quad\; = 0 \qquad t>3$

$$\left[\textbf{Ans.: } \frac{(1-e^{-s})^2}{s^2}\right]$$

5. $f(t)=\sin t \qquad 0<t<\pi$
   $\quad\; = t \qquad t>\pi$

$$\left[\textbf{Ans.: } \frac{1+e^{-\pi s}}{s^2+1}+e^{-\pi s}\left(\frac{\pi s+1}{s^2}\right)\right]$$

2.10 DIRAC'S DELTA FUNCTION

Consider the function $f(t)$ as shown in Fig. 2.3.

$$f(t) = \frac{1}{T} \qquad -\frac{T}{2} < t < \frac{T}{2}$$
$$= 0 \qquad \text{otherwise}$$

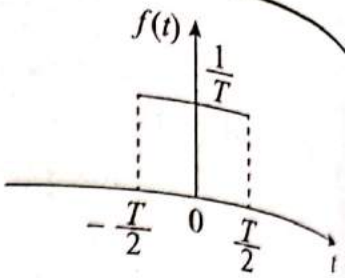


Fig. 2.3 *A function*

The width of this function is T and its amplitude is $\frac{1}{T}$. Hence, the area of this function is one unit. As $T \to 0$, the function becomes a delta function or a unit impulse function.

$$\lim_{T \to 0} f(t) = \delta(t)$$

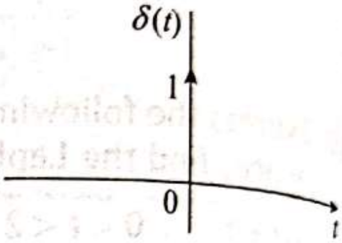


Fig. 2.4 *Unit impulse function*

Dirac's delta, or unit impulse (Fig. 2.4), function has zero amplitude everywhere except at $t = 0$. At $t = 0$, the amplitude of the function is infinitely large such that the area under its curve is equal to one unit. Hence, it is defined as

$$\delta(t) = 0 \qquad t \neq 0$$

and $$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \qquad t = 0$$

The displaced (delayed) delta or unit impulse function $\delta(t-a)$ (Fig. 2.5) represents the function $\delta(t)$ which is displaced by a distance 'a' to the right.

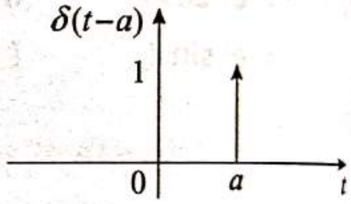


Fig. 2.5 *Delayed unit impulse function*

$$\delta(t-a) = 0 \qquad t \neq a$$

and $$\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1 \qquad t = a$$

Properties of unit impulse functions

(i) $\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)$

(ii) $\int_{0}^{\infty} f(t)\delta(t)\,dt = f(0)$

(iii) $\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a)$

(iv) $\int_{0}^{\infty} f(t)\delta(t-a)\,dt = f(a)$

Laplace Transforms of the Unit Impulse Functions

(i) Laplace transform of $\delta(t)$

$$\delta(t) = 0 \qquad t \neq 0$$
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \qquad t = 0$$
$$L\{\delta(t)\} = \int_{0}^{\infty} e^{-st}\,\delta(t)\,dt$$

$$= [e^{-st}]_{t=0}$$
$$= 1$$

(ii) Laplace transform of $\delta(t-a)$

$$\delta(t-a) = 0 \qquad t \neq a$$

$$\text{and } \int_{-\infty}^{\infty} \delta(t-a)\,dt = 1 \qquad t = a$$

$$L\{\delta(t-a)\} = \int_0^{\infty} e^{-st}\delta(t-a)\,dt$$
$$= [e^{-st}]_{t=a} \qquad \text{[From Property (iii)]}$$
$$= e^{-as}$$

(iii) Laplace transform of $f(t)\,\delta(t-a)$

$$f(t)\,\delta(t-a) = 0 \qquad t \neq a$$

$$\text{and } \int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a) \qquad t = a$$

$$L\{f(t)\,\delta(t-a)\} = \int_0^{\infty} e^{-st} f(t)\delta(t-a)\,dt$$
$$= [e^{-st} f(t)]_{t=a} \qquad \text{[From Property (iii)]}$$
$$= e^{-as} f(a)$$

Example 1

Find the Laplace transform of $\sin 2t\ \delta\left(t-\frac{\pi}{4}\right) - t^2\,\delta(t-2)$.

Solution

$$L\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$$

$$L\left\{\sin 2t\,\delta\left(t-\frac{\pi}{4}\right) - t^2\delta(t-2)\right\} = e^{\frac{-\pi s}{4}} \sin 2\left(\frac{\pi}{4}\right) - e^{-2s}(2)^2$$

$$= e^{\frac{-\pi s}{4}} \sin\frac{\pi}{2} - 4e^{-2s}$$

$$= e^{\frac{-\pi s}{4}} - 4e^{-2s}$$

Example 2

Find the Laplace transform of $t\,u(t-4) + t^2\,\delta(t-4)$.

Solution

$$L\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$$

$$L\{t\,u(t-4)+t^2\,\delta(t-2)\} = e^{-4s}\,L\{f(t+4)\} + L\{t^2\,\delta(t-4)\}$$
$$= e^{-4s}\,L\{t+4\} + e^{-4s}\,(4)^2$$
$$= e^{-4s}\left(\frac{1}{s^2}+\frac{4}{s}\right) + 16\,e^{-4s}$$
$$= e^{-4s}\left(\frac{1}{s^2}+\frac{4}{s}+16\right)$$

Example 3

Find the Laplace transform of $t^2\,u(t-2) - \cosh t\,\delta(t-2)$.

Solution

$$L\{f(t)\,\delta(t-a)\} = e^{-as}\,f(a)$$

and
$$L\{f(t)\,u(t-a)\} = e^{-as}\,L\{f(t+a)\}$$

$$L\{t^2\,u(t-2)-\cosh t\,\delta(t-2)\} = L\{t^2\,u(t-2)\} - L\{\cosh t\,\delta(t-2)\}$$
$$= e^{-2s}\,L\{(t+2)^2\} - e^{-2s}\cosh 2$$
$$= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right) - e^{-2s}\cosh 2$$
$$= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}-\cosh 2\right)$$

Example 4

Evaluate $\int_0^\infty \cos 2t\,\delta\left(t-\frac{\pi}{4}\right)dt$.

Solution

$$\int_0^\infty f(t)\delta(t-a)\,dt = f(a)$$

$$\int_0^\infty \cos 2t\,\delta\left(t-\frac{\pi}{4}\right)dt = \cos\frac{2\pi}{4} = 0$$

Example 5

Evaluate $\int_0^\infty t^2\,e^{-t}\,\sin t\,\delta(t-2)\,dt$.

Solution

$$\int_0^\infty f(t)\delta(t-a)\,dt = f(a)$$

$$\int_0^\infty t^2\,e^{-t}\,\sin t\,\delta(t-2)\,dt = (2)^2 e^{-2}\sin 2 = 4e^{-2}\sin 2$$

Example 6

Evaluate $\int_0^\infty t^m (\log t)^n \, \delta(t-3)\, dt$.

Solution

$$\int_0^\infty f(t)\delta(t-a)\,dt = f(a)$$

$$\int_0^\infty t^m (\log t)^n \, \delta(t-3)\, dt = 3^m (\log 3)^n$$

EXERCISE 2.10

(I) Find the Laplace transforms of the following functions:

1. $t\,u(t-4) - t^2\,\delta(t-2)$

$$\left[\text{Ans.}: e^{-4s}\frac{1}{s^2}(1+4s) - 4e^{-2s}\right]$$

2. $\sin 2t\;\delta(t-2)$

$$[\text{Ans.}: e^{-2s}\sin 4]$$

3. $t^2 u(t-2) - \cosh t\;\delta(t-4)$

$$\left[\text{Ans.}: \frac{2e^{-2s}}{s^3}(2s^2+2s+1) - e^{-4s}\cosh 4\right]$$

4. $t\,e^{-2t}\,\delta(t-2)$

$$[\text{Ans.}: 2e^{-(4+2s)}]$$

5. $\dfrac{e^{-t}\sin t}{t}\delta(t-3)$

$$\left[\text{Ans.}: \frac{1}{3}e^{-(s+3)}\sin 3\right]$$

6. $(e^{-4t} + \log t)\;\delta(t-2)$

$$[\text{Ans.}: (e^{-8} + \log 2)\, e^{-2s}]$$

(II) Evaluate the following integrals:

1. $\int_0^\infty \sin 4t\,\delta\left(t - \frac{\pi}{8}\right)dt$

$$\left[\text{Ans.}: e^{\frac{-\pi s}{8}}\right]$$

2. $\int_0^\infty e^{-t} \sin t\, \delta(t-a)\,dt$ **[Ans. :** $e^{-a}(\sin a - \cos a)$**]**

2.11 LAPLACE TRANSFORMS OF PERIODIC FUNCTIONS

A function $f(t)$ is said to be periodic if there exists a constant $T(T > 0)$ such that $f(t+T) = f(t)$, for all values of t.

$$f(t+2T) = f(t+T+T) = f(t+T) = f(t)$$

In general, $f(t+nT) = f(t)$ for all t, where n is an integer (positive or negative) and T is the period of the function.

If $f(t)$ is a piecewise continuous periodic function with period T then

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\,dt$$

Proof: $$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$

In the second integral, putting $t = x + T$, $dt = dx$

When $t = T$, $x = 0$

When $t \to \infty$, $x \to \infty$

$$\begin{aligned} L\{f(t)\} &= \int_0^T e^{-st} f(t)\,dt + \int_0^\infty e^{-s(x+T)} f(x+T)\,dx \\ &= \int_0^T e^{-st} f(t)\,dt + e^{-Ts} \int_0^\infty e^{-sx} f(x)\,dx \\ &= \int_0^T e^{-st} f(t)\,dt + e^{-Ts} \int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^T e^{-st} f(t)\,dt + e^{-Ts} L\{f(t)\} \end{aligned}$$

$$(1-e^{-Ts}) L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt$$

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\,dt$$

Example 1

Find the Laplace transform of $f(t) = e^t \quad 0 < t < 2\pi$ *if* $f(t) = f(t+2\pi)$.

Solution

The function $f(t)$ is a periodic function with period 2π.

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st}e^{t}\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{(1-s)t}\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-2\pi s}}\left|\frac{e^{(1-s)t}}{1-s}\right|_0^{2\pi}$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{(1-s)2\pi}}{1-s}-\frac{1}{1-s}\right]$$

$$= \frac{e^{(1-s)2\pi}-1}{(1-e^{-2\pi s})(1-s)}$$

Example 2

Find the Laplace transform of $f(t) = t^2 \quad 0 < t < 2$ *if* $f(t) = f(t+2)$.

Solution

The function $f(t)$ is a periodic function with period 2.

$$L\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^{2} e^{-st} f(t)\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-2s}}\int_0^{2} e^{-st}\, t^2\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-2s}}\left|t^2\left(\frac{e^{-st}}{-s}\right)-2t\left(\frac{e^{-st}}{s^2}\right)+2\left(\frac{e^{-st}}{-s^3}\right)\right|_0^{2}$$

$$= \frac{1}{1-e^{-2s}}\left(-4\frac{e^{-2s}}{s}-4\frac{e^{-2s}}{s^2}-2\frac{e^{-2s}}{s^3}+\frac{2}{s^3}\right)$$

$$= \frac{1}{(1-e^{-2s})s^3}(2-2e^{-2s}-4se^{-2s}-4s^2e^{-2s})$$

Example 3

Find the Laplace transform of

$$f(t) = 1 \quad 0 < t < a$$
$$= -1 \quad a < t < 2a$$

and f(t) is periodic with period 2a.

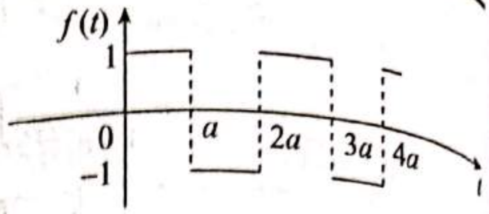


Fig. 2.6

Solution

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\,dt + \int_a^{2a} e^{-st}(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left|\frac{e^{-st}}{-s}\right|_0^a + \left|\frac{e^{-st}}{s}\right|_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left(-\frac{e^{-as}}{s} + \frac{1}{s} + \frac{e^{-2as}}{s} - \frac{e^{-as}}{s}\right)$$

$$= \frac{(1-e^{-as})^2}{s(1+e^{-as})(1-e^{-as})}$$

$$= \frac{1-e^{-as}}{s(1+e^{-as})}$$

$$= \frac{1}{s}\cdot\frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{\left(e^{\frac{as}{2}} + e^{-\frac{as}{2}}\right)}$$

$$= \frac{1}{s}\tanh\left(\frac{as}{2}\right)$$

Example 4

Find the Laplace transform of

$$f(t) = \frac{t}{T} \qquad 0 < t < T$$

if $f(t) = f(t+T)$.

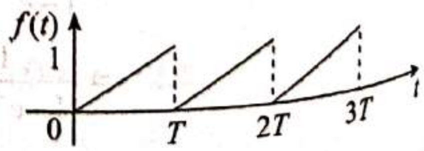


Fig. 2.7

Solution

The function $f(t)$ is a periodic function with period T.

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)\mathrm{d}t$$

$$= \frac{1}{1-e^{-Ts}}\int_0^T e^{-st}\frac{t}{T}\mathrm{d}t$$

$$= \frac{1}{1-e^{-Ts}}\frac{1}{T}\int_0^T e^{-st}\, t\,\mathrm{d}t$$

$$= \frac{1}{T(1-e^{-Ts})}\left| t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_0^T$$

$$= \frac{1}{T(1-e^{-Ts})}\left(-T\frac{e^{-Ts}}{s} - \frac{e^{-Ts}}{s^2} + \frac{1}{s^2}\right)$$

$$= \frac{1}{T(1-e^{-Ts})}\left[-\frac{Te^{-Ts}}{s} + \frac{1}{s^2}(1-e^{-Ts})\right]$$

$$= \frac{1}{Ts^2} - \frac{e^{-Ts}}{s(1-e^{-Ts})}$$

Example 5

Find the Laplace transform of

$$f(t) = \frac{2}{3}t \qquad 0 \le t \le 3$$

if $f(t) = f(t+3)$. **[Winter 2017]**

Solution

The function $f(t)$ is a periodic function with period 3.

$$L\{f(t)\} = \frac{1}{1-e^{-3s}}\int_0^3 e^{-st} f(t)\mathrm{d}t$$

$$= \frac{1}{1-e^{-3s}}\int_0^3 e^{-st}\frac{2}{3}t\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-3s}}\frac{2}{3}\int_0^3 e^{-st}\, t\,\mathrm{d}t$$

$$= \frac{2}{3(1-e^{-3s})}\left| t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_0^3$$

$$= \frac{2}{3(1-e^{-3s})}\left(-3\frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}\right)$$

$$= \frac{2}{3(1-e^{-3s})}\left[-\frac{3e^{-3s}}{s}+\frac{1}{s^2}(1-e^{-3s})\right]$$

$$= \frac{2}{3s^2}-\frac{2e^{-3s}}{s(1-e^{-3s})}$$

Example 6

Find the Laplace transform of

$$f(t) = t \qquad 0<t<1$$
$$= 0 \qquad 1<t<2$$

if $f(t) = f(t+2)$.

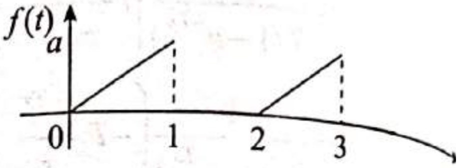


Fig. 2.8

Solution

The function $f(t)$ is a periodic function with period 2.

$$L\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\, t\,dt + \int_1^2 e^{-st}\cdot 0\,dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\left|\frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2}\right|_0^1 + 0\right]$$

$$= \frac{1}{1-e^{-2s}}\left(\frac{e^{-s}}{-s}-\frac{e^{-s}}{s^2}+\frac{1}{s^2}\right)$$

$$= \frac{1}{s^2(1-e^{-2s})}(1-e^{-s}-se^{-s})$$

Example 7

Find the Laplace transform of

$$f(t) = t \qquad 0<t<a$$
$$= 2a-t \qquad a<t<2a$$

if $f(t) = f(t+2a)$.

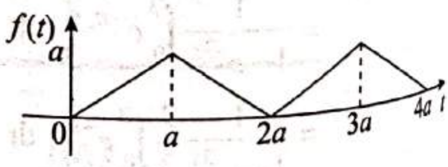


Fig. 2.9

Solution

The function $f(t)$ is a periodic function with period $2a$.

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}t\,\mathrm{d}t + \int_a^{2a} e^{-st}(2a-t)\,\mathrm{d}t\right]$$

$$= \frac{1}{(1-e^{-2as})}\left[\left|\frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2}\right|_0^a + \left|\frac{e^{-st}}{-s}(2a-t) + \frac{e^{-st}}{s^2}\right|_a^{2a}\right]$$

$$= \frac{1}{(1-e^{-2as})}\left(-\frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} - \frac{e^{-as}}{s^2}\right)$$

$$= \frac{-2e^{-as}+1+e^{-2as}}{s^2(1-e^{-2as})}$$

$$= \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

$$= \frac{1-e^{-as}}{s^2(1+e^{-as})}$$

$$= \frac{e^{\frac{as}{2}} - e^{\frac{-as}{2}}}{s^2\left(e^{\frac{as}{2}} + e^{\frac{-as}{2}}\right)}$$

$$= \frac{\tanh\left(\frac{as}{2}\right)}{s^2}$$

Example 8

Find the Laplace transform of

$$f(t) = \sin \omega t \qquad 0 < t < \frac{\pi}{\omega}$$

$$= 0 \qquad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$$

if $f(t) = f\left(t + \frac{2\pi}{\omega}\right)$.

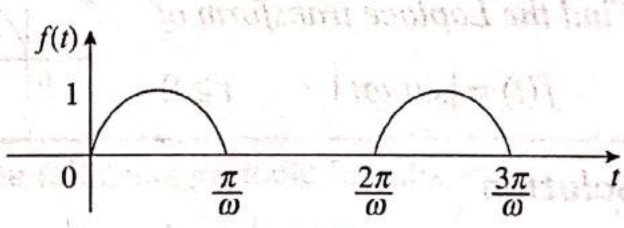


Fig. 2.10

Solution

The function $f(t)$ is a periodic function with period $\frac{2\pi}{\omega}$.

$$L\{f(t)\} = \frac{1}{1-e^{-\left(\frac{2\pi}{\omega}\right)s}}\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left(\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t\,dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st}\cdot 0\,dt\right)$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left|\frac{1}{s^2+\omega^2}\cdot e^{-st}(-s\sin\omega t-\omega\cos\omega t)\right|_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\cdot\frac{1}{s^2+\omega^2}\left[e^{-\frac{\pi s}{\omega}}(\omega)+\omega\right]$$

$$= \frac{\omega\left(1+e^{-\frac{\pi s}{\omega}}\right)}{\left(1+e^{\frac{-\pi s}{\omega}}\right)\left(1-e^{\frac{-\pi s}{\omega}}\right)}\cdot\frac{1}{s^2+\omega^2}$$

$$= \frac{\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)}\cdot\frac{1}{s^2+\omega^2}$$

$$= \frac{\omega}{\left(s^2+\omega^2\right)\left(1-e^{\frac{-\pi s}{\omega}}\right)}$$

Example 9

Find the Laplace transform of

$$f(t) = |\sin\omega t| \qquad t \ge 0$$

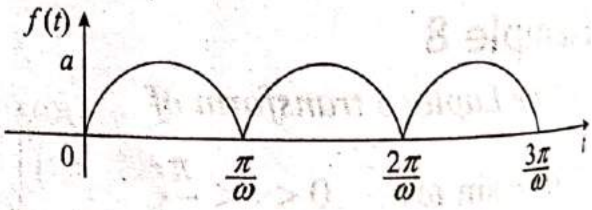


Fig. 2.11

Solution

$$f\left(t+\frac{\pi}{\omega}\right) = \left|\sin\omega\left(t+\frac{\pi}{\omega}\right)\right|$$

$$= |\sin(\omega t+\pi)|$$

$$= |-\sin\omega t|$$

$$= |\sin\omega t|$$

Hence, the function $f(t)$ is periodic with period $\frac{\pi}{\omega}$.

$$L\{f(t)\} = \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st} |\sin \omega t|\,dt$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t\,dt \qquad \left[\because |\sin \omega t| = \sin \omega t \quad 0 < t < \frac{\pi}{\omega}\right]$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\left|\frac{e^{-st}}{s^2+\omega^2}(-s\sin \omega t - \omega\cos \omega t)\right|_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\frac{1}{s^2+\omega^2}\left[e^{-\frac{\pi s}{\omega}}(\omega) - (-\omega)\right]$$

$$= \frac{1}{s^2+\omega^2}\cdot\frac{1}{1-e^{-\frac{\pi s}{\omega}}}\omega\left(1+e^{-\frac{\pi s}{\omega}}\right)$$

$$= \frac{\omega}{s^2+\omega^2}\left(\frac{e^{\frac{\pi s}{2\omega}}+e^{-\frac{\pi s}{2\omega}}}{e^{\frac{\pi s}{2\omega}}-e^{-\frac{\pi s}{2\omega}}}\right)$$

$$= \frac{\omega}{s^2+\omega^2}\cdot\coth\left(\frac{\pi s}{2\omega}\right)$$

EXERCISE 2.11

Find the Laplace transforms of the following periodic functions:

1. $f(t) = 1 \qquad 0 < t < 1$
   $\quad = 0 \qquad 1 < t < 2$
   $\quad = -1 \qquad 2 < t < 3$
   $f(t) = f(t+3)$

   $$\left[\textbf{Ans.: } \frac{1}{s}\left(\frac{3}{1-e^{-3s}} - \frac{1}{1-e^{-s}} - 1\right)\right]$$

2. $f(t) = t \qquad 0 < t < a$
   $\quad = \frac{2a-t}{a} \qquad a < t < 2a$
   $f(t) = f(t+2a)$

   $$\left[\textbf{Ans.: } \frac{1}{as^2}\tanh\frac{as}{2}\right]$$

3. $f(t) = t \quad 0 < t < \pi$

$\quad = \pi - t \quad \pi < t < 2\pi$

$f(t) = f(t + 2\pi)$

$$\left[\textbf{Ans.}: \frac{1-(1+\pi s)e^{-\pi s}}{(1+e^{-\pi s})s^2}\right]$$

4. $f(t) = |\cos \omega t| \quad t > 0$

$$\left[\textbf{Ans.}: \frac{1}{s^2+\omega^2}\left(s + \omega \operatorname{cosech} \frac{\pi s}{2\omega}\right)\right]$$

5. $f(t) = \cos \omega t \quad 0 < t < \frac{\pi}{\omega}$

$\quad = 0 \quad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$

$$\left[\textbf{Ans.}: \frac{s}{\left(1-e^{-\frac{\pi s}{\omega}}\right)(s^2+\omega^2)}\right]$$

6. $f(t) = E \quad 0 < t < \frac{\pi}{2}$

$\quad = -E \quad \frac{\pi}{2} < t < \pi$

$f(t) = f(t + \pi)$

$$\left[\textbf{Ans.}: \frac{E}{s} \tanh\left(\frac{\pi s}{4}\right)\right]$$

7. $f(t) = \left(\frac{\pi - t}{2}\right)^2 \quad 0 < t < 2\pi$

$f(t) = f(t + 2\pi)$

$$\left[\textbf{Ans.}: \frac{1}{s^3}(2\pi s \coth \pi s - \pi^2 s^2 - 2)\right]$$

2.12 INVERSE LAPLACE TRANSFORM

If $L\{f(t)\} = F(s)$ then $f(t)$ is called the inverse Laplace transform of $F(s)$ and is symbolically written as

$$f(t) = L^{-1}\{F(s)\}$$

where L^{-1} is called the *inverse Laplace transform operator.*

Inverse Laplace transforms of simple functions can be found from the properties of Laplace transforms.

Table of Inverse Laplace Transforms

Sr. No.	$F(s)$	$f(t)$
1	$\frac{1}{s}$	1
2	$\frac{1}{s^n}$	$\frac{t^{n-1}}{\lceil n}$
3	$\frac{1}{s-a}$	e^{at}
4	$\frac{1}{(s-a)^n}$	$e^{at}\frac{t^{n-1}}{\lceil n}$
5	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
6	$\frac{s}{s^2+a^2}$	$\cos at$
7	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$
8	$\frac{s}{s^2-a^2}$	$\cosh at$
9	$\frac{1}{(s+b)^2+a^2}$	$\frac{1}{a}e^{-bt}\sin at$
10	$\frac{s+b}{(s+b)^2+a^2}$	$e^{-bt}\cos at$
11	$\frac{1}{(s+b)^2-a^2}$	$\frac{1}{a}e^{-bt}\sinh at$
12	$\frac{s+b}{(s+b)^2-a^2}$	$e^{-bt}\cosh at$

2.12.1 Linearity

If $L^{-1}\{F_1(s)\} = f_1(t)$ and $L^{-1}\{F_2(s)\} = f_2(t)$ then $L^{-1}\{aF_1(s) + bF_2(s)\} = af_1(t) + bf_2(t)$ where a and b are constants.

Example 1

Find the inverse Laplace transform of $\frac{s^2 - 3s + 4}{s^3}$.

Solution

Let
$$F(s) = \frac{s^2 - 3s + 4}{s^3}$$
$$= \frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}$$
$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s}\right\} - 3L^{-1}\left\{\frac{1}{s^2}\right\} + 4L^{-1}\left\{\frac{1}{s^3}\right\}$$
$$= 1 - 3t + 2t^2$$

Example 2

Find the inverse Laplace transform of $\frac{6s}{s^2 - 16}$. **[Winter 2012]**

Solution

Let
$$F(s) = \frac{6s}{s^2 - 16}$$
$$L^{-1}\{F(s)\} = 6L^{-1}\left\{\frac{s}{s^2 - 16}\right\} = 6\cosh 4t$$

Example 3

Find the inverse Laplace transform of $\frac{3(s^2 - 2)^2}{2s^5}$.

Solution

Let
$$F(s) = \frac{3(s^2 - 2)^2}{2s^5}$$

$$= \frac{3}{2}\frac{(s^2-2)^2}{s^5}$$

$$= \frac{3}{2}\frac{s^4-4s^2+4}{s^5}$$

$$= \frac{3}{2}\left(\frac{1}{s}-\frac{4}{s^3}+\frac{4}{s^5}\right)$$

$$L^{-1}\{F(s)\} = \frac{3}{2}\left[L^{-1}\left\{\frac{1}{s}\right\}-4L^{-1}\left\{\frac{1}{s^3}\right\}+4L^{-1}\left\{\frac{1}{s^5}\right\}\right]$$

$$= \frac{3}{2}\left[1-4\left(\frac{t^2}{2!}\right)+4\left(\frac{t^4}{4!}\right)\right]$$

$$= \frac{3}{2}\left[1-2t^2+\frac{t^4}{6}\right]$$

$$= \frac{3}{2}-3t^2+\frac{t^4}{4}$$

$$= \frac{1}{4}(t^4-12t^2+6)$$

Example 4

Find the inverse Laplace transform of $\frac{2s+1}{s(s+1)}$.

Solution

Let $$F(s) = \frac{2s+1}{s(s+1)}$$

$$= \frac{s+(s+1)}{s(s+1)}$$

$$= \frac{1}{s+1}+\frac{1}{s}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\}+L^{-1}\left\{\frac{1}{s}\right\}$$

$$= e^{-t}+1$$

Example 5

Find the inverse Laplace transform of $\frac{3s+4}{s^2+9}$.

Solution

Let
$$F(s) = \frac{3s+4}{s^2+9}$$
$$= \frac{3s}{s^2+9} + \frac{4}{s^2+9}$$
$$L^{-1}\{F(s)\} = 3L^{-1}\left\{\frac{s}{s^2+9}\right\} + 4L^{-1}\left\{\frac{1}{s^2+9}\right\}$$
$$= 3\cos 3t + \frac{4}{3}\sin 3t$$

Example 6

Find the inverse Laplace transform of $\frac{s^2+9s-9}{s^3-9s}$.

Solution

Let
$$F(s) = \frac{s^2+9s-9}{s^3-9s}$$
$$= \frac{(s^2-9)+9s}{s(s^2-9)}$$
$$= \frac{1}{s} + \frac{9}{s^2-9}$$
$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s}\right\} + 9L^{-1}\left\{\frac{1}{s^2-9}\right\}$$
$$= 1 + 3\sinh 3t$$

Example 7

Find the inverse Laplace transform of $\frac{4s+15}{16s^2-25}$.

Solution

Let $F(s) = \frac{4s+15}{16s^2-25}$
$$= \frac{4s+15}{16\left(s^2-\frac{25}{16}\right)}$$
$$= \frac{1}{4}\frac{s}{s^2-\frac{25}{16}} + \frac{15}{16}\frac{1}{s^2-\frac{25}{16}}$$

$$L^{-1}\{F(s)\} = \frac{1}{4}\left[L^{-1}\left\{\frac{s}{s^2-\frac{25}{16}}\right\}+\frac{15}{4}L^{-1}\left\{\frac{1}{s^2-\frac{25}{16}}\right\}\right]$$

$$= \frac{1}{4}\cosh\frac{5}{4}t+\frac{3}{4}\sinh\frac{5}{4}t$$

EXERCISE 2.12

Find the inverse Laplace transforms of the following functions:

1. $\frac{2s-5}{s^2-4}$

$\left[\text{Ans.}: 2\cosh 2t-\frac{5}{2}\sinh 2t\right]$

2. $\frac{3s-8}{4s^2+25}$

[Ans.: $e^{-t}+1$]

3. $\frac{3s-12}{s^2+18}$

$\left[\text{Ans.}: 3\cos 2\sqrt{2}t-3\sqrt{2}\sin 2\sqrt{2}t\right]$

4. $\frac{s+1}{s^{\frac{4}{3}}}$

$\left[\text{Ans.}: \frac{t^{\frac{-2}{3}}+3t^{\frac{1}{3}}}{\left\lceil\frac{1}{3}\right.}\right]$

5. $\left(\frac{\sqrt{s}-1}{s}\right)^2$

$\left[\text{Ans.}: 1+t-\frac{4\sqrt{t}}{\sqrt{\pi}}\right]$

6. $\frac{s^2-1}{s^5}$

$\left[\text{Ans.}: 1-t^2-\frac{t^4}{24}\right]$

2.12.2 First Shifting Theorem

If $L^{-1}\{F(s)\}=f(t)$ then $L^{-1}\{F(s+a)\}=e^{-at}f(t)$.

Example 1

Find the inverse Laplace transform of $\frac{1}{(s+2)^3}$.

Solution

Let
$$F(s)=\frac{1}{(s+2)^3}$$

$$L^{-1}\{F(s)\}=e^{-2t}L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$=e^{-2t}\frac{t^2}{2!}$$

$$=\frac{e^{-2t}}{2}t^2$$

Example 2

Find the inverse Laplace transform of $\frac{10}{(s-2)^4}$. [Winter 2012]

Solution

Let
$$F(s)=\frac{10}{(s-2)^4}$$

$$L^{-1}\{F(s)\}=10e^{2t}\,L^{-1}\left\{\frac{1}{s^4}\right\}$$

$$=10e^{2t}\frac{t^3}{3!}$$

$$=\frac{5}{3}e^{2t}\,t^3$$

Example 3

Find the inverse Laplace transform of $\frac{1}{\sqrt{s+2}}$.

Solution

Let
$$F(s)=\frac{1}{\sqrt{s+2}}$$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{(s+2)^{\frac{1}{2}}}\right\}$$

$$=e^{-2t}L^{-1}\left\{\frac{1}{s^{\frac{1}{2}}}\right\}$$

$$= e^{-2t} \frac{t^{-\frac{1}{2}}}{\boxed{\frac{1}{2}}}$$

$$= \frac{e^{-2t}}{\sqrt{\pi}} \frac{1}{\sqrt{t}} \qquad \left[\because \boxed{\frac{1}{2}} = \sqrt{\pi}\right]$$

Example 4

Find the inverse Laplace transform of $\frac{1}{s^2+4s+4}$.

Solution

Let

$$F(s) = \frac{1}{s^2+4s+4}$$

$$= \frac{1}{(s+2)^2}$$

$$L^{-1}\{F(s)\} = e^{-2t} L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= e^{-2t}\, t$$

Example 5

Find the inverse Laplace transform of $\frac{s}{(2s+1)^2}$.

Solution

Let

$$F(s) = \frac{s}{(2s+1)^2}$$

$$= \frac{1}{4} \frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2}$$

$$= \frac{1}{4}\left[\frac{1}{s+\frac{1}{2}} - \frac{1}{2}\cdot\frac{1}{\left(s+\frac{1}{2}\right)^2}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} - \frac{1}{8}e^{-\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= \frac{1}{4}e^{-\frac{t}{2}} - \frac{1}{8}e^{-\frac{t}{2}}t$$

$$= e^{-\frac{t}{2}}\left(\frac{1}{4} - \frac{1}{8}t\right)$$

Example 6

Find the inverse Laplace transform of $\frac{1}{\sqrt{2s+3}}$.

Solution

Let

$$F(s) = \frac{1}{\sqrt{2s+3}}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\left(s+\frac{3}{2}\right)^{\frac{1}{2}}}$$

$$L^{-1}\{F(s)\} = \frac{1}{\sqrt{2}} e^{-\frac{3t}{2}} L^{-1}\left\{\frac{1}{s^{\frac{1}{2}}}\right\}$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{3t}{2}} \frac{t^{-\frac{1}{2}}}{\left\lceil\frac{1}{2}\right.}$$

$$= \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{-\frac{3t}{2}} \qquad \left[\because \left\lceil\frac{1}{2}\right. = \sqrt{\pi}\right]$$

Example 7

Find the inverse Laplace transform of $\frac{3s+1}{(s+1)^4}$.

Solution

Let

$$F(s) = \frac{3s+1}{(s+1)^4}$$

$$= \frac{3(s-1)-2}{(s+1)^4}$$

$$= \frac{3}{(s+1)^3} - \frac{2}{(s+1)^4}$$

$$L^{-1}\{F(s)\} = 3e^{-t}L\left\{\frac{1}{s^3}\right\} - 2e^{-t}\left\{\frac{1}{s^4}\right\}$$

$$= 3e^{-t}\frac{t^2}{2!} - 2e^{-t}\frac{t^3}{3!}$$

$$= \frac{3}{2}e^{-t}t^2 - \frac{1}{3}e^{-t}t^3$$

$$= e^{-t}\left(\frac{3}{2}t^2 - \frac{1}{3}t^3\right)$$

Example 8

Find the inverse Laplace transform of $\dfrac{s+2}{s^2+4s+8}$.

Solution

Let
$$F(s) = \frac{s+2}{s^2+4s+8}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+2}{s^2+4s+8}\right\}$$

$$= L^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\}$$

$$= e^{-2t}L^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$= e^{-2t}\cos 2t$$

Example 9

Find the inverse Laplace transform of $\dfrac{2s+2}{s^2+2s+10}$.

Solution

Let
$$F(s) = \frac{2s+2}{s^2+2s+10}$$

$$= \frac{2(s+1)}{(s+1)^2+9}$$

$$L^{-1}\{F(s)\} = 2e^{-t}L^{-1}\left\{\frac{s}{s^2+9}\right\}$$

$$= 2\,e^{-t}\cos 3t$$

Example 10

Find the inverse Laplace transform of $\dfrac{s}{(s+2)^2+1}$.

Solution

Let $$F(s) = \frac{s}{(s+2)^2+1}$$

$$= \frac{s+2-2}{(s+2)^2+1}$$

$$= \frac{s+2}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} - 2L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$

$$= e^{-2t}L^{-1}\left\{\frac{s}{s^2+1}\right\} - 2e^{-2t}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-2t}\cos t - 2e^{-2t}\sin t$$

$$= e^{-2t}(\cos t - 2\sin t)$$

Example 11

Find the inverse Laplace transform of $\dfrac{2s+3}{s^2-4s+13}$.

Solution

Let $$F(s) = \frac{2s+3}{s^2-4s+13}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{2s+3}{s^2-4s+13}\right\}$$

$$= L^{-1}\left\{\frac{2s+3}{(s-2)^2+13-4}\right\}$$

$$= L^{-1}\left\{\frac{2s+3}{(s-2)^2+9}\right\}$$

$$= L^{-1}\left\{\frac{2s-4+7}{(s-2)^2+9}\right\}$$

$$= L^{-1}\left\{\frac{2(s-2)}{(s-2)^2+9}\right\} + 7L^{-1}\left\{\frac{1}{(s-2)^2+9}\right\}$$

$$= 2e^{2t}L^{-1}\left\{\frac{s}{s^2+9}\right\} + 7e^{2t}L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= 2e^{2t}\cos 3t + \frac{7}{3}e^{2t}\sin 3t$$

$$= \frac{1}{3}e^{2t}(6\cos 3t + 7\sin 3t)$$

Example 12

Find the inverse Laplace transform of $\dfrac{s+7}{s^2+8s+25}$. **[Summer 2017]**

Solution

Let $F(s) = \dfrac{s+7}{s^2+8s+25}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+7}{s^2+8s+16+9}\right\}$$

$$= L^{-1}\left\{\frac{s+7}{(s+4)^2+9}\right\}$$

$$= L^{-1}\left\{\frac{(s+4)+3}{(s+4)^2+9}\right\}$$

$$= L^{-1}\left\{\frac{s+4}{(s+4)^2+9}\right\} + L^{-1}\left\{\frac{3}{(s+4)^2+9}\right\}$$

$$= e^{-4t}\,L^{-1}\left\{\frac{s}{s^2+9}\right\} + e^{-4t}\,L^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= e^{-4t}\cos 3t + e^{-4t}\sin 3t$$

$$= e^{-4t}(\sin 3t + \cos 3t)$$

Example 13

Find the inverse Laplace transform of $\dfrac{3s+7}{s^2-2s-3}$.

Solution

Let
$$F(s) = \frac{3s+7}{s^2-2s-3}$$
$$= \frac{3(s-1)+10}{(s-1)^2-4}$$
$$= \frac{3(s-1)}{(s-1)^2-4} + 10\frac{1}{(s-1)^2-4}$$
$$L^{-1}\{F(s)\} = 3e^t L^{-1}\left\{\frac{s}{s^2-4}\right\} + 10e^t L^{-1}\left\{\frac{1}{s^2-4}\right\}$$
$$= 3\,e^t \cosh 2t + 5\,e^t \sinh 2t$$
$$= e^t\,(3\cosh 2t + 5\sinh 2t)$$

EXERCISE 2.13

Find the inverse Laplace transforms of the following functions:

1. $\frac{5}{(s+2)^5}$ $\left[\textbf{Ans.: } \frac{5}{24}t^4 e^{-2t}\right]$

2. $\frac{4s+12}{s^2+8s+16}$ $\left[\textbf{Ans.: } 4e^{-4t}(1-t)\right]$

3. $\frac{1}{(s^2+2s+5)^2}$ $\left[\textbf{Ans.: } \frac{e^{-t}}{16}(\sin 2t - 2t\cos 2t)\right]$

4. $\frac{s}{(s-2)^6}$ $\left[\textbf{Ans.: } e^{2t}\left(\frac{t^4}{24}+\frac{t^5}{60}\right)\right]$

5. $\frac{5}{s^2+2s+2}$ $\left[\textbf{Ans.: } e^{-t}(\cos t - \sin t)\right]$

6. $\frac{1}{(s+2)^4}$ $\left[\textbf{Ans.: } \frac{1}{6}e^{-2t}t^3\right]$

2.12.3 Second Shifting Theorem

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{e^{-as}F(s)\} = g(t)$

where $g(t) = f(t-a) \qquad t > a$

$= 0 \qquad t < a$

The above result can also be expressed as

$$L^{-1}\{e^{-as}F(s)\} = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$$

Or $L^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$

Example 1

Find the inverse Laplace transform of $\dfrac{e^{-as}}{s}$.

Solution

Let $\qquad F(s) = \dfrac{1}{s}$

$$L^{-1}\{F(s)\} = 1$$

$$L^{-1}\{e^{-as}F(s)\} = 1 \cdot u(t-a) = u(t-a)$$

Example 2

Find the inverse Laplace transform of $\dfrac{e^{-2s}}{s-3}$.

Solution

Let $\qquad F(s) = \dfrac{1}{s-3}$

$$L^{-1}\{F(s)\} = e^{3t}$$

$$L^{-1}\{e^{-2s}F(s)\} = e^{3(t-2)}u(t-2)$$

Example 3

Find the inverse Laplace transform of $e^{-s}\left(\dfrac{1+\sqrt{s}}{s^3}\right)$.

Solution

Let $\qquad F(s) = \left(\dfrac{1+\sqrt{s}}{s^3}\right)$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^3} + \frac{1}{s^{\frac{5}{2}}}\right\}$$

$$= \frac{t^2}{2!} + \frac{t^{\frac{3}{2}}}{\left\lceil\frac{5}{2}\right.}$$

$$= \frac{t^2}{2} + \frac{t^{\frac{3}{2}}}{\frac{3}{2}\,\frac{1}{2}\left\lceil\frac{1}{2}\right.}$$

$$= \frac{t^2}{2} + \frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}}$$

$$L^{-1}\{e^{-s}F(s)\} = \left[\frac{(t-1)^2}{2} + \frac{4(t-1)^{\frac{3}{2}}}{3\sqrt{\pi}}\right]u(t-1)$$

Example 4

Find the inverse Laplace transform of $\dfrac{e^{-2s}}{(s+4)^3}$.

Solution

Let $$F(s) = \frac{1}{(s+4)^3}$$

$$L^{-1}\{F(s)\} = e^{-4t}\, L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= e^{-4t}\,\frac{t^2}{2}$$

$$L^{-1}\{e^{-2s}F(s)\} = e^{-4(t-2)}\,\frac{(t-2)^2}{2}\,u(t-2)$$

Example 5

Find the inverse Laplace transform of $\dfrac{e^{4-3s}}{(s+4)^{\frac{5}{2}}}$.

Solution

Let $$F(s) = \frac{1}{(s+4)^{\frac{5}{2}}}$$

$$L^{-1}\{F(s)\} = e^{-4t}\, L^{-1}\left\{\frac{1}{s^{\frac{5}{2}}}\right\}$$

$$= e^{-4t}\frac{t^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}$$

$$= \frac{e^{-4t}\, t^{\frac{3}{2}}}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{4e^{-4t}\, t^{\frac{3}{2}}}{3\sqrt{\pi}}$$

$$L^{-1}\{e^{4-3s}F(s)\} = \frac{e^{4}\cdot 4}{3\sqrt{\pi}}\, e^{-4(t-3)}\, (t-3)^{\frac{3}{2}}\, u(t-3)$$

$$= \frac{4}{3\sqrt{\pi}}\, e^{-4(t-4)}\, (t-3)^{\frac{3}{2}}\, u(t-3)$$

Example 6

Find the inverse Laplace transform of $\frac{e^{-3s}}{s^2+4}$.

Solution

Let $$F(s) = \frac{1}{s^2+4}$$

$$L^{-1}\{F(s)\} = \frac{1}{2}\sin 2t$$

$$L^{-1}\{e^{-3s}F(s)\} = \frac{1}{2}\sin 2(t-3)\, u(t-3)$$

Example 7

Find the inverse Laplace transform of $\dfrac{se^{-\left(\frac{\pi}{2}\right)s}}{s^2+4}$.

Solution

Let $$F(s)=\frac{s}{s^2+4}$$

$$L^{-1}\{F(s)\}=\cos 2t$$

$$L^{-1}\left\{e^{-\left(\frac{\pi}{2}\right)s}F(s)\right\}=\cos 2\left(t-\frac{\pi}{2}\right)u\left(t-\frac{\pi}{2}\right)$$

$$=\cos(2t-\pi)u\left(t-\frac{\pi}{2}\right)$$

$$=\cos(\pi-2t)u\left(t-\frac{\pi}{2}\right)$$

$$=-\cos 2t\, u\left(t-\frac{\pi}{2}\right)$$

Example 8

Find the inverse Laplace transform of $\dfrac{e^{-3s}}{s^2+8s+25}$. **[Winter 2016]**

Solution

Let $$F(s)=\frac{1}{s^2+8s+25}$$

$$=\frac{1}{s^2+8s+16+9}$$

$$=\frac{1}{(s+4)^2+9}$$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{(s+4)^2+9}\right\}$$

$$=e^{-4t}\cdot\frac{1}{3}\sin 3t$$

$$L^{-1}\{e^{-3s}F(s)\}=\frac{1}{3}e^{-4(t-3)}\sin 3(t-3)\,u(t-3)$$

$$=\frac{1}{3}e^{12-4t}\sin(3t-9)\,u(t-3)$$

Example 9

Find the inverse Laplace transform of $\dfrac{e^{-2s}}{(s^2+2)(s^2-3)}$. **[Winter 2015]**

Solution

Let
$$F(s) = \frac{1}{(s^2+2)(s^2-3)}$$

$$L^{-1}\{F(s)\} = \frac{1}{5}L^{-1}\left\{\frac{1}{s^2-3} - \frac{1}{s^2+2}\right\}$$

$$= \frac{1}{5}\left[L^{-1}\left\{\frac{1}{s^2-3}\right\} - L^{-1}\left\{\frac{1}{s^2+2}\right\}\right]$$

$$= \frac{1}{5}\left[\frac{1}{\sqrt{3}}\sinh\sqrt{3}t - \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right]$$

$$L^{-1}\{e^{-2s}F(s)\} = \left[\frac{1}{5\sqrt{3}}\sinh\sqrt{3}(t-2) - \frac{1}{5\sqrt{2}}\sin\sqrt{2}(t-2)\right]u(t-2)$$

Example 10

Find the inverse Laplace transform of $\dfrac{e^{-\pi s}}{s^2-2s+2}$.

Solution

Let
$$F(s) = \frac{1}{s^2-2s+2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= e^t\, L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^t \sin t$$

$$L^{-1}\{e^{-\pi s}F(s)\} = e^{(t-\pi)}\sin(t-\pi)u(t-\pi)$$

Example 11

Find the inverse Laplace transform of $\dfrac{(s+1)e^{-2s}}{s^2+2s+2}$.

Solution

Let $$F(s)=\frac{s+1}{s^2+2s+2}$$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{(s+1)}{(s+1)^2+1}\right\}$$

$$=e^{-t}\,L^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$=e^{-t}\cos t$$

$$L^{-1}\{e^{-2s}F(s)\}=e^{-(t-2)}\cos(t-2)u(t-2)$$

Example 12

Find the inverse Laplace transform of $\dfrac{se^{-2s}}{s^2+2s+2}$.

Solution

Let $$F(s)=\frac{s}{s^2+2s+2}$$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{s+1-1}{(s+1)^2+1}\right\}$$

$$=L^{-1}\left\{\frac{s+1}{(s+1)^2+1}-\frac{1}{(s+1)^2+1}\right\}$$

$$=e^{-t}\,L^{-1}\left\{\frac{s}{s^2+1}-\frac{1}{s^2+1}\right\}$$

$$=e^{-t}(\cos t-\sin t)$$

$$L^{-1}\{e^{-2s}F(s)\}=e^{-(t-2)}\left[\cos(t-2)-\sin(t-2)\right]u(t-2)$$

EXERCISE 2.14

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{e^{-as}}{(s+b)^{\frac{5}{2}}}$

$$\left[\text{Ans.}: \frac{4}{3\sqrt{\pi}}e^{-b(t-a)}(t-a)^{\frac{3}{2}}u(t-a)\right]$$

2. $\dfrac{e^{-\pi s}}{s^2+9}$

$$\left[\text{Ans.}: \frac{1}{3}\sin 3(t-\pi)u(t-\pi)\right]$$

3. $\dfrac{e^{-\pi s}}{s^2(s^2+1)}$

$$\left[\text{Ans.}: [(t-\pi)+\sin(t-\pi)]u(t-\pi)\right]$$

4. $\dfrac{e^{-4s}}{\sqrt{2s+7}}$

$$\left[\text{Ans.}: \frac{e^{\frac{-7(t-4)}{2}}}{\sqrt{2\pi(t-4)}}u(t-4)\right]$$

5. $\dfrac{(s+1)e^{-s}}{s^2+s+1}$

$$\left[\text{Ans.}: e^{\frac{-(t-1)}{2}}\left[\cos\left(\sqrt{3}\frac{(t-1)}{2}\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}(t-1)}{2}\right)\right]u(t-1)\right]$$

6. $\dfrac{se^{-3s}}{s^2-1}$

$$\left[\text{Ans.}: \cosh(t-3)u(t-3)\right]$$

7. $\dfrac{se^{-as}}{s^2+b^2}$

$$\left[\text{Ans.}: \cos b(t-a)u(t-a)\right]$$

8. $e^{-s}\left\{\dfrac{1-\sqrt{s}}{s^2}\right\}^2$

$$\left[\text{Ans.}: \left[\frac{(t-1)^3}{6}-\frac{16}{15\sqrt{\pi}}(t-1)^{\frac{5}{2}}+\frac{(t-1)^2}{2}\right]u(t-1)\right]$$

2.12.4 Multiplication by *s*

If $L^{-1}\{F(s)\}=f(t)$ and $f(0)=0$ then $L^{-1}\{sF(s)\}=f'(t)=\dfrac{d}{dt}\left[L^{-1}\{F(s)\}\right]$.

In general, $L^{-1}\{s^n F(s)\}=f^n(t)$, if $f(0)=0=f'(0)\ldots=f^{n-1}(0)$.

Example 1

Find the inverse Laplace transform of $\frac{s}{s^2 - a^2}$.

Solution

Let $$F(s) = \frac{1}{s^2 - a^2}$$

$$L^{-1}\{F(s)\} = \frac{1}{a}\sinh at$$

$$L^{-1}\{sF(s)\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L^{-1}\{F(s)\}\right]$$

$$L^{-1}\left\{\frac{s}{s^2 - a^2}\right\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L^{-1}\left\{\frac{1}{s^2 - a^2}\right\}\right]$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{a}\sinh at\right]$$

$$= \frac{1}{a}\cosh at(a)$$

$$= \cosh at$$

Example 2

Find the inverse Laplace transform of $\frac{s}{2s^2 - 1}$.

Solution

Let $$F(s) = \frac{1}{2s^2 - 1}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{2s^2 - 1}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{s^2 - \frac{1}{2}}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{s^2 - \frac{1}{2}}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{s^2 - \left(\frac{1}{\sqrt{2}}\right)^2}\right\}$$

$$= \frac{1}{2}\frac{1}{\frac{1}{\sqrt{2}}}\sinh\left(\frac{1}{\sqrt{2}}t\right)$$

$$= \frac{1}{\sqrt{2}}\sinh\left(\frac{t}{\sqrt{2}}\right)$$

$$L^{-1}\{sF(s)\} = \frac{d}{dt}\left[L^{-1}\{F(s)\}\right]$$

$$L^{-1}\left[\frac{s}{2s^2-1}\right] = \frac{d}{dt}\left[\frac{1}{\sqrt{2}}\sinh\left(\frac{t}{\sqrt{2}}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\cosh\left(\frac{t}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\cosh\left(\frac{t}{\sqrt{2}}\right)$$

Example 3

Find the inverse Laplace transform of $\frac{s}{(s+2)^4}$.

Solution

Let $F(s) = \frac{1}{(s+2)^4}$

$\frac{1}{s^4}$ e^{-2t} $\frac{t^3}{3!}$

$$L^{-1}\{F(s)\} = e^{-2t}\, L^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= e^{-2t}\frac{1}{6}L^{-1}\left[\frac{6}{s^4}\right]$$

$$= \frac{e^{-2t}}{6}t^3$$

$$L^{-1}\{sF(s)\} = \frac{d}{dt}\left[L^{-1}\{F(s)\}\right]$$

$$L^{-1}\left\{\frac{s}{(s+2)^4}\right\} = \frac{d}{dt}\left[L^{-1}\left\{\frac{1}{(s+2)^4}\right\}\right]$$

$$= \frac{d}{dt}\left[\frac{e^{-2t}}{6}t^3\right]$$

$$= \frac{1}{6}\frac{d}{dt}\left[e^{-2t}t^3\right]$$

$$= \frac{1}{6}\left[e^{-2t}(3t^2) + t^3 e^{-2t}(-2)\right]$$

$$= \frac{1}{6}\left[3t^2 e^{-2t} - 2t^3 e^{-2t}\right]$$

$$= \frac{1}{6} t^2 e^{-2t}(3-2t)$$

Example 4

Find the inverse Laplace transform of $\dfrac{s^2}{(s-3)^2}$.

Solution

Let
$$F(s) = \frac{1}{(s-3)^2}$$

$$L^{-1}\{F(s)\} = e^{3t} L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= e^{3t} t$$

$$L^{-1}\{s^2 F(s)\} = \frac{d^2}{dt^2}\left[L^{-1}\{F(s)\}\right]$$

$$= \frac{d^2}{dt^2}\left[te^{3t}\right]$$

$$= \frac{d}{dt}\left[t(3e^{3t}) + e^{3t}(1)\right]$$

$$= \frac{d}{dt}\left[e^{3t}(3t+1)\right]$$

$$= e^{3t}(3) + (3t+1)3e^{3t}$$

$$= 3e^{3t}(3t+2)$$

EXERCISE 2.15

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{s}{(s+2)^2}$ **[Ans.:** $e^{-2t}(1-2t)$**]**

2. $\dfrac{s^2}{(s^2-a^2)^2}$

$\left[\text{Ans.}: \dfrac{1}{2a}(\sinh at + at\cosh at)\right]$

3. $\dfrac{s^2}{(s-1)^3}$

$\left[\text{Ans.}: \dfrac{e^t}{2}(t^2+4t+2)\right]$

4. $\dfrac{s^2}{(s+4)^3}$

$[\text{Ans.}: e^{-4t}(8t^2-8t+1)]$

5. $\dfrac{s-3}{s^2+4s+13}$

$\left[\text{Ans.}: e^{-2t}\left(\cos 3t - \dfrac{5}{3}\sin 3t\right)\right]$

2.12.5 Division by *s*

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\left\{\dfrac{F(s)}{s}\right\} = \int_0^t f(t)\,dt = \int_0^t L^{-1}\{F(s)\}\,dt.$

Similarly, $L^{-1}\left\{\dfrac{F(s)}{s^2}\right\} = \int_0^t\int_0^t f(t)dt\,dt$

Example 1

Find the inverse Laplace transform of $\dfrac{1}{s(s+2)}$.

Solution

Let

$$F(s) = \frac{1}{s+2}$$

$$L^{-1}\{F(s)\} = e^{-2t}$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\}\,dt$$

$$L^{-1}\left\{\frac{1}{s(s+2)}\right\} = \int_0^t e^{-2t}\,dt$$

$$= \left|\frac{e^{-2t}}{-2}\right|_0^t$$

$$= -\frac{1}{2}(e^{-2t} - 1)$$

$$= \frac{1}{2}(1 - e^{-2t})$$

Example 2

Find the inverse Laplace transform of $\frac{1}{s(s^2+a^2)}$.

Solution

Let

$$F(s) = \frac{1}{s^2+a^2}$$

$$L^{-1}\{F(s)\} = \frac{1}{a}\sin at$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\}dt$$

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \frac{1}{a}\sin at\, dt$$

$$= \frac{1}{a}\int_0^t \sin at\, dt$$

$$= \frac{1}{a}\left|\frac{-\cos at}{a}\right|_0^t$$

$$= -\frac{1}{a^2}\left|\cos at\right|_0^t$$

$$= -\frac{1}{a^2}(\cos at - 1)$$

$$= \frac{1}{a^2}(1 - \cos at)$$

Example 3

Find the inverse Laplace transform of $\frac{1}{s(s^2+2s+2)}$.

Solution

Let

$$F(s) = \frac{1}{s^2+2s+2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+2s+2}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\, L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-t} \sin t$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\}\, dt$$

$$L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\} = \int_0^t e^{-t} \sin t\, dt$$

$$= \left|\frac{e^{-t}}{2}(-\sin t - \cos t)\right|_0^t$$

$$= \frac{-1}{2}\left[\left|e^{-t}(\sin t + \cos t)\right|_0^t\right]$$

$$= -\frac{1}{2}\left[e^{-t}(\sin t + \cos t) - (0+1)\right]$$

$$= \frac{1}{2}\left[1 - e^{-t}(\sin t + \cos t)\right]$$

Example 4

Find the inverse Laplace transform of $\dfrac{1}{s(s^2-3s+3)}$. **[Winter 2015]**

Solution

Let $$F(s) = \frac{1}{s^2-3s+3}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2-3s+\frac{9}{4}+3-\frac{9}{4}}\right\}$$

$$= L^{-1}\left\{\frac{1}{\left(s-\frac{3}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= L^{-1}\left\{\frac{1}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{2}{\sqrt{3}}e^{\frac{3t}{2}}L^{-1}\left\{\frac{\frac{2}{\sqrt{3}}}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{2}{\sqrt{3}}e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\}dt$$

$$L^{-1}\left\{\frac{1}{s(s^2-3s+3)}\right\} = \int_0^t \frac{2}{\sqrt{3}}e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{3}t}{2}\right)dt$$

$$= \frac{2}{\sqrt{3}}\left|\frac{e^{\frac{3t}{2}}}{\frac{9}{4}+\frac{3}{4}}\left\{\frac{3}{2}\sin\left(\frac{\sqrt{3}t}{2}\right)-\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}t}{2}\right)\right\}\right|_0^t$$

$$= \frac{2}{3\sqrt{3}}\left|e^{\frac{3t}{2}}\left\{\frac{3}{2}\sin\left(\frac{\sqrt{3}t}{2}\right)-\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}t}{2}\right)\right\}\right|_0^t$$

$$= e^{\frac{3t}{2}}\left[\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)-\frac{1}{3}\cos\left(\frac{\sqrt{3}t}{2}\right)\right]+\frac{2}{3\sqrt{3}}\cdot\frac{\sqrt{3}}{2}$$

$$= e^{\frac{3t}{2}}\left[\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)-\frac{1}{3}\cos\left(\frac{\sqrt{3}t}{2}\right)+\frac{1}{3}\right]$$

Example 5

Find the inverse Laplace transform of $\frac{1}{s(s^2-1)(s^2+1)}$.

Solution

Let
$$F(s) = \frac{1}{(s^2-1)(s^2+1)}$$

$$= \frac{1}{2}\left[\frac{(s^2+1)-(s^2-1)}{(s^2-1)(s^2+1)}\right]$$

$$= \frac{1}{2}\left(\frac{1}{s^2-1} - \frac{1}{s^2+1}\right)$$

$$L^{-1}\{F(s)\} = \frac{1}{2}\left[L^{-1}\left\{\frac{1}{s^2-1}\right\} - L^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

$$= \frac{1}{2}(\sinh t - \sin t)$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\}\,dt$$

$$L^{-1}\left\{\frac{1}{s(s^2-1)(s^2+1)}\right\} = \int_0^t \frac{1}{2}(\sinh t - \sin t)dt$$

$$= \frac{1}{2}\int_0^t (\sinh t - \sin t)dt$$

$$= \frac{1}{2}\left|\cosh t + \cos t\right|_0^t$$

$$= \frac{1}{2}\left[(\cosh t + \cos t) - (1+1)\right]$$

$$= \frac{1}{2}\left[\cosh t + \cos t - 2\right]$$

Example 6

Find the inverse Laplace transform of $\frac{1}{s^2(1+s^2)}$.

Solution

Let
$$F(s) = \frac{1}{1+s^2}$$

$$L^{-1}\{F(s)\} = \sin t$$

$$L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \int_0^t L^{-1}\{F(s)\}\,dt\,dt$$

$$L^{-1}\left\{\frac{1}{s^2(1+s^2)}\right\} = \int_0^t \int_0^t \sin t\,dt\,dt$$

$$= \int_0^t \left|-\cos t\right|_0^t\,dt$$

$$= \int_0^t (-\cos t + 1)dt$$

$$= \int_0^t (1 - \cos t)\,dt$$

$$= |t - \sin t|_0^t$$
$$= (t - \sin t) - (0 - 0)$$
$$= t - \sin t$$

EXERCISE 2.16

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{1}{(s^2 + a^2)^2}$ $\left[\text{Ans.: } \dfrac{1}{2a^3}(\sin at - at \cos at)\right]$

2. $\dfrac{s^2 + 2}{s(s^2 + 4)}$ $\left[\text{Ans.: } \dfrac{1}{2}(1 + \cos 2t)\right]$

3. $\dfrac{s}{(s^2 + 4)^2}$ $\left[\text{Ans.: } \dfrac{1}{4} t \sin 2t\right]$

4. $\dfrac{1}{s^2(s^2 + a^2)}$ $\left[\text{Ans.: } \dfrac{1}{a^2}\left(t - \dfrac{1}{a}\sin at\right)\right]$

5. $\dfrac{s + 1}{s^2(s^2 + 1)}$ [Ans.: $1 + t - \cos t - \sin t$]

6. $\dfrac{1}{s(s^2 + 4s + 5)}$ $\left[\text{Ans.: } \dfrac{1}{5}\left[1 - e^{-2t}(2 \sin t + \cos t)\right]\right]$

2.12.6 Differentiation of Laplace Transforms

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(s)\} = -\dfrac{1}{t} L^{-1}\{F'(s)\}$.

Example 1

Find the inverse Laplace transform of $\log\left(\dfrac{s + a}{s + b}\right)$. **[Summer 2013]**

Solution

Let
$$F(s) = \log\left(\frac{s+a}{s+b}\right)$$
$$= \log(s+a) - \log(s+b)$$
$$F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$
$$L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\}$$
$$= -\frac{1}{t}(e^{-at} - e^{-bt})$$

Example 2

Find the inverse Laplace transform of $\log\left(1+\frac{\omega^2}{s^2}\right)$. **[Summer 2014]**

Solution

Let
$$F(s) = \log\left(1+\frac{\omega^2}{s^2}\right)$$
$$= \log\left(\frac{s^2+\omega^2}{s^2}\right)$$
$$= \log(s^2+\omega^2) - \log s^2$$
$$F'(s) = \frac{2s}{s^2+\omega^2} - \frac{2s}{s^2}$$
$$= \frac{2s}{s^2+\omega^2} - \frac{2}{s}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$
$$L^{-1}\left\{\log\left(1+\frac{\omega^2}{s^2}\right)\right\} = -\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+\omega^2} - \frac{2}{s}\right\}$$
$$= -\frac{2}{t}L^{-1}\left\{\frac{s}{s^2+\omega^2} - \frac{1}{s}\right\}$$
$$= -\frac{2}{t}(\cos\omega t - 1)$$
$$= \frac{2}{t}(1-\cos\omega t)$$

Example 3

Find the inverse Laplace transform of $\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$.

Solution

Let
$$F(s)=\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$
$$=\log(s^2+b^2)-\log(s^2+a^2)$$
$$F'(s)=\frac{2s}{s^2+b^2}-\frac{2s}{s^2+a^2}$$
$$L^{-1}\{F(s)\}=-\frac{1}{t}L^{-1}\{F'(s)\}$$
$$L^{-1}\left\{\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right\}=-\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+b^2}-\frac{2s}{s^2+a^2}\right\}$$
$$=-\frac{1}{t}(2\cos bt-2\cos at)$$
$$=\frac{2}{t}(\cos at-\cos bt)$$

Example 4

Find the inverse Laplace transform of $\log\frac{s^2+a^2}{(s+b)^2}$.

Solution

Let
$$F(s)=\log\frac{s^2+a^2}{(s+b)^2}$$
$$=\log(s^2+a^2)-\log(s+b)^2$$
$$=\log(s^2+a^2)-2\log(s+b)$$
$$F'(s)=\frac{2s}{s^2+a^2}-\frac{2}{s+b}$$
$$L^{-1}\{F(s)\}=-\frac{1}{t}L^{-1}\{F'(s)\}$$
$$L^{-1}\left\{\log\frac{s^2+a^2}{(s+b)^2}\right\}=-\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+a^2}-\frac{2}{s+b}\right\}$$
$$=-\frac{1}{t}(2\cos at-2e^{-bt})$$

$$= \frac{2}{t}(e^{-bt} - \cos at)$$

Example 5

Find the inverse Laplace transform of $\log\sqrt{\dfrac{s^2-a^2}{s^2}}$.

Solution

Let
$$F(s) = \log\sqrt{\frac{s^2-a^2}{s^2}}$$

$$= \log\sqrt{s^2-a^2} - \log\sqrt{s^2}$$

$$= \frac{1}{2}\log(s^2-a^2) - \log s$$

$$F'(s) = \frac{1}{2}\frac{2s}{s^2-a^2} - \frac{1}{s}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\log\sqrt{\frac{s^2-a^2}{s^2}}\right\} = -\frac{1}{t}L^{-1}\left\{\frac{s}{s^2-a^2} - \frac{1}{s}\right\}$$

$$= -\frac{1}{t}(\cosh at - 1)$$

$$= \frac{1}{t}(1 - \cosh at)$$

Example 6

Find the inverse Laplace transform of $\log\sqrt{\dfrac{s-1}{s+1}}$.

Solution

Let
$$F(s) = \log\sqrt{\frac{s-1}{s+1}}$$

$$= \log\sqrt{s-1} - \log\sqrt{s+1}$$

$$= \frac{1}{2}\log(s-1) - \frac{1}{2}\log(s+1)$$

$$F'(s) = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\log\sqrt{\frac{s-1}{s+1}}\right\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right\}$$

$$= -\frac{1}{t}\left(\frac{1}{2}e^{t} - \frac{1}{2}e^{-t}\right)$$

$$= -\frac{1}{t}\sinh t$$

Example 7

Find the inverse Laplace transform of $\log\sqrt{\dfrac{s^2+1}{s(s+1)}}$.

Solution

Let
$$F(s) = \log\sqrt{\frac{s^2+1}{s(s+1)}}$$

$$= \frac{1}{2}\left[\log(s^2+1) - \log s - \log(s+1)\right]$$

$$F'(s) = \frac{1}{2}\left(\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right)$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\log\sqrt{\frac{s^2+1}{s(s+1)}}\right\} = -\frac{1}{t}L^{-1}\frac{1}{2}\left\{\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= -\frac{1}{2t}(2\cos t - 1 - e^{-t})$$

Example 8

Find the inverse Laplace transform of $s\log\left(\dfrac{s^2+a^2}{s^2+b^2}\right)$.

Solution

Let
$$F(s) = \log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$= \log(s^2+a^2) - \log(s^2+b^2)$$

$$F'(s) = \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\} = -\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right\}$$

$$= -\frac{1}{t}(2\cos at - 2\cos bt)$$

$$= \frac{2}{t}(\cos bt - \cos at)$$

$$L^{-1}\{s\,F(s)\} = \frac{d}{dt}\left[L^{-1}\{F(s)\}\right]$$

$$L^{-1}\left\{s\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\} = \frac{d}{dt}\left[\frac{2}{t}(\cos bt - \cos at)\right]$$

$$= \frac{2b}{t}(-\sin bt)\frac{-2\cos bt}{t^2} + \frac{2a\sin at}{t} + \frac{2\cos at}{t^2}$$

$$= \frac{1}{t}\left[2(a\sin at - b\sin bt) - \frac{2(\cos bt - \cos at)}{t}\right]$$

Example 9

Find the inverse Laplace transform of $\tan^{-1} s$.

Solution

Let
$$F(s) = \tan^{-1} s$$

$$F'(s) = \frac{1}{s^2+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\{\tan^{-1} s\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= -\frac{1}{t}\sin t$$

Example 10

Find the inverse Laplace transform of $\tan^{-1}\left(\frac{s+a}{b}\right)$.

Solution

Let
$$F(s) = \tan^{-1}\left(\frac{s+a}{b}\right)$$

$$F'(s) = \frac{1}{1+\left(\frac{s+a}{b}\right)^2} \cdot \frac{1}{b}$$

$$= \frac{b}{(s+a)^2 + b^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\tan^{-1}\left(\frac{s+a}{b}\right)\right\} = -\frac{1}{t} L^{-1}\left\{\frac{b}{(s+a)^2 + b^2}\right\}$$

$$= -\frac{1}{t} e^{-at} \sin bt$$

Example 11

Find the inverse Laplace transform of $\tan^{-1}\left(\frac{2}{s}\right)$.

Solution

Let

$$F(s) = \tan^{-1}\left(\frac{2}{s}\right)$$

$$F'(s) = \frac{1}{1+\frac{4}{s^2}}\left(-\frac{2}{s^2}\right)$$

$$= -\frac{2}{s^2+4}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\tan^{-1}\frac{2}{s}\right\} = -\frac{1}{t} L^{-1}\left\{-\frac{2}{s^2+4}\right\}$$

$$= \frac{2}{t} L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{2}{t} \cdot \frac{1}{2} \sin 2t$$

$$= \frac{1}{t} \sin 2t$$

Example 12

Find the inverse Laplace transform of $\tan^{-1}\left(\frac{2}{s^2}\right)$.

Solution

Let
$$F(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$$

$$F'(s) = \frac{1}{1+\frac{4}{s^4}}\left(-\frac{4}{s^3}\right)$$

$$= -\frac{4s}{s^4+4}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\tan^{-1}\frac{2}{s^2}\right\} = -\frac{1}{t}L^{-1}\left\{-\frac{4s}{s^4+4}\right\}$$

$$= \frac{4}{t}L^{-1}\left\{\frac{s}{s^4+4}\right\}$$

$$= \frac{4}{t}L^{-1}\left\{\frac{s}{(s^2+2)^2-(2s)^2}\right\}$$

$$= \frac{4}{t}\cdot\frac{1}{4}L^{-1}\left\{\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right\}$$

$$= \frac{1}{t}L^{-1}\left\{\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right\}$$

$$= \frac{1}{t}(e^t \sin t - e^{-t}\sin t)$$

$$= \frac{\sin t}{t}(e^t - e^{-t})$$

$$= \frac{2}{t}\sin t \sinh t$$

Example 13

Find the inverse Laplace transform of $\cot^{-1} s$.

Solution

Let
$$F(s) = \cot^{-1} s$$

$$F'(s) = -\frac{1}{s^2+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\{\cot^{-1} s\} = -\frac{1}{t} L^{-1}\left\{-\frac{1}{s^2+1}\right\}$$

$$= \frac{1}{t} \sin t$$

Example 14

Find the inverse Laplace transform of $\cot^{-1}\left(\frac{k}{s}\right)$.

Solution

Let $$F(s) = \cot^{-1}\left(\frac{k}{s}\right)$$

$$F'(s) = -\frac{1}{1+\frac{k^2}{s^2}}\left(-\frac{k}{s^2}\right) = \frac{k}{s^2+k^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\cot^{-1}\left(\frac{k}{s}\right)\right\} = -\frac{1}{t} L^{-1}\left\{\frac{k}{s^2+k^2}\right\}$$

$$= -\frac{1}{t} \sin kt$$

Example 15

Find the inverse Laplace transform of $\cot^{-1}(s+1)$.

Solution

Let $$F(s) = \cot^{-1}(s+1)$$

$$F'(s) = -\frac{1}{(s+1)^2+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\{F'(s)\}$$

$$L^{-1}\{\cot^{-1}(s+1)\} = -\frac{1}{t} L^{-1}\left\{-\frac{1}{(s+1)^2+1}\right\}$$

$$= \frac{1}{t} e^{-t} \sin t$$

Example 16

Find the inverse Laplace transform of $2\tanh^{-1} s$.

Solution

Let
$$F(s) = 2\tanh^{-1} s$$
$$= 2\cdot\frac{1}{2}\log\frac{1+s}{1-s}$$
$$= \log(1+s) - \log(1-s)$$
$$F'(s) = \frac{1}{1+s} + \frac{1}{1-s}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$
$$L^{-1}\{2\tanh^{-1} s\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{1+s} + \frac{1}{1-s}\right\}$$
$$= -\frac{1}{t}(e^{-t} - e^{t})$$
$$= \frac{2}{t}\sinh t$$

EXERCISE 2.17

Find the inverse Laplace transforms of the following functions:

1. $\log\left(1+\frac{a^2}{s^2}\right)$

$$\left[\text{Ans.}: \frac{2}{t}(1-\cos at)\right]$$

2. $\log\left(1-\frac{1}{s^2}\right)$

$$\left[\text{Ans.}: \frac{2}{t}(1-\cosh t)\right]$$

3. $\log\frac{s^2-4}{(s-3)^2}$

$$\left[\text{Ans.}: \frac{2}{t}(e^{3t}-\cosh 2t)\right]$$

4. $\log\sqrt{\frac{s^2+1}{s^2}}$

$$\left[\text{Ans.}: \frac{1}{t}(1-\cos t)\right]$$

5. $\log\frac{(s-2)^2}{s^2+1}$ $\left[\text{Ans.}: \frac{2}{t}(\cos t - e^{2t})\right]$

6. $\log\left(\frac{s^2-4}{s^2}\right)^{\frac{1}{3}}$ $\left[\text{Ans.}: \frac{2}{3t}(1-\cosh 2t)\right]$

7. $\log\frac{1}{s}\left(1+\frac{1}{s^2}\right)$ $\left[\text{Ans.}: \int_0^t \frac{2(1-\cos t)}{t}dt\right]$

8. $\frac{1}{s}\log\frac{s+1}{s+2}$ $\left[\text{Ans.}: \int_0^t \frac{e^{-2t}-e^{-t}}{t}dt\right]$

9. $\tan^{-1}(s+1)$ $\left[\text{Ans.}: -\frac{1}{t}e^{-t}\sin t\right]$

10. $\tan^{-1}\left(\frac{s}{2}\right)$ $\left[\text{Ans.}: -\frac{1}{t}\sin 2t\right]$

11. $\cot^{-1}(as)$ $\left[\text{Ans.}: \frac{1}{t}\sin\frac{t}{a}\right]$

12. $\cot^{-1}\left(\frac{2}{s^2}\right)$ $\left[\text{Ans.}: -\frac{2}{7}\sin t \sinh t\right]$

2.12.7 Integration of Laplace Transforms

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(s)\} = t\,L^{-1}\left[\int_s^\infty F(s)\,ds\right]$.

Example 1

Find the inverse Laplace transform of $\frac{1}{(s+1)^2}$.

Solution

Let

$$F(s)=\frac{1}{(s+1)^2}$$

$$\int_s^\infty F(s)\,\mathrm{d}s=\int_s^\infty \frac{1}{(s+1)^2}\,\mathrm{d}s$$

$$=\left|-\left(\frac{1}{s+1}\right)\right|_s^\infty$$

$$=-\left(0-\frac{1}{s+1}\right)$$

$$=\frac{1}{s+1}$$

$$L^{-1}\{F(s)\}=t\,L^{-1}\left[\int_s^\infty F(s)\,\mathrm{d}s\right]$$

$$=t\,L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$=t\,e^{-t}$$

Example 2

Find the inverse Laplace transform of $\frac{2}{(s-a)^3}$.

Solution

Let

$$F(s)=\frac{2}{(s-a)^3}$$

$$\int_s^\infty F(s)\,\mathrm{d}s=\int_s^\infty \frac{2}{(s-a)^3}\,\mathrm{d}s$$

$$=2\left|-\frac{1}{2(s-a)^2}\right|_s^\infty$$

$$=\frac{1}{(s-a)^2}$$

$$L^{-1}\{F(s)\} = t\, L^{-1}\left[\int_s^\infty F(s)\, ds\right]$$

$$= t\, L^{-1}\left[\frac{1}{(s-a)^2}\right]$$

$$= t\, e^{at}\, L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= t\, e^{at} . t$$

$$= t^2\, e^{at}$$

Example 3

Find the inverse Laplace transform of $\frac{2s}{(s^2+1)^2}$.

Solution

Let $$F(s) = \frac{2s}{(s^2+1)^2}$$

$$\int_s^\infty F(s)\, ds = \int_s^\infty \frac{2s}{(s^2+1)^2}\, ds$$

$$= \left|\frac{-1}{s^2+1}\right|_s^\infty$$

$$= 0 + \frac{1}{s^2+1}$$

$$= \frac{1}{s^2+1}$$

$$L^{-1}\{F(s)\} = t\, L^{-1}\left[\int_s^\infty F(s)\, ds\right]$$

$$= t\, L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= t \sin t$$

Example 4

Find the inverse Laplace transform of $\frac{s}{(s^2-a^2)^2}$.

Solution

Let $$F(s) = \frac{s}{(s^2-a^2)^2}$$

$$\int_s^\infty F(s)\,ds = \int_s^\infty \frac{s}{(s^2-a^2)^2}\,ds$$

$$= \frac{1}{2}\int_s^\infty \frac{2s}{(s^2-a^2)^2}\,ds$$

$$= \frac{1}{2}\left|-\left(\frac{1}{s^2-a^2}\right)\right|_s^\infty$$

$$= \frac{1}{2}\frac{1}{s^2-a^2}$$

$$L^{-1}\{F(s)\} = t\,L^{-1}\left[\int_s^\infty F(s)\,ds\right]$$

$$= t\,L^{-1}\left\{\frac{1}{2}\frac{1}{s^2-a^2}\right\}$$

$$= \frac{t}{2}\frac{1}{a}\sinh at$$

$$= \frac{t}{2a}\sinh at$$

EXERCISE 2.18

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{2s}{(s^2-4)^2}$

   $\left[\textbf{Ans.:}\ \dfrac{t}{2}\sinh 2t\right]$

2. $\dfrac{s+2}{(s^2+4s+5)^2}$

   $\left[\textbf{Ans.:}\ \dfrac{t}{2}e^{-2t}\sin t\right]$

3. $\dfrac{s}{s^2-a^2}$

   $\left[\textbf{Ans.:}\ \dfrac{t}{2a}\sinh at\right]$

2.12.8 Partial Fraction Expansion

Any function $F(s)$ can be written as $\dfrac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials in s. To perform partial fraction expansion, the degree of $P(s)$ must be less than the degree of $Q(s)$. If not, $P(s)$ must be divided by $Q(s)$, so that the degree of $P(s)$ becomes less than that of $Q(s)$. Assuming that the degree of $P(s)$ is less than that of $Q(s)$, four possible cases arise depending upon the factors of $Q(s)$.

Case I **Factors are linear and distinct**

$$F(s) = \frac{P(s)}{(s+a)(s+b)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B}{s+b}$$

Case II **Factors are linear and repeated**

$$F(s) = \frac{P(s)}{(s+a)(s+b)^n}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B_1}{s+b} + \frac{B_2}{(s+b)^2} + ... + \frac{B_n}{(s+b)^n}$$

Case III **Factors are quadratic and distinct**

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)}$$

By partial fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{Cs+D}{s^2+cs+d}$$

Case IV **Factors are quadratic and repeated**

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)^n}$$

By partial fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{C_1 s+D_1}{s^2+cs+d} + \frac{C_2 s+D_2}{(s^2+cs+d)^2} + ... + \frac{C_n s+D_n}{(s^2+cs+d)^n}$$

Example 1

Find the inverse Laplace transform of $\frac{1}{(s+1)(s+2)}$. **[Summer 2018]**

Solution

Let $F(s) = \frac{1}{(s+1)(s+2)}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$1 = A(s+2) + B(s+1)$$

Putting $s = -1$ in Eq. (1),

$$A = 1$$

Putting $s = -2$ in Eq. (2),

$$1 = B\,(-1)$$

$$B = -1$$

$$F(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= e^{-t} - e^{-2t}$$

Example 2

Find the inverse Laplace transform of $\frac{1}{(s-2)(s+3)}$. **[Summer 2015]**

Solution

Let $F(s) = \frac{1}{(s-2)(s+3)}$

By partial-fraction expansion,

$$F(s) = \frac{A}{s-2} + \frac{B}{s+3}$$

$$1 = A(s+3) + B(s-2) \qquad \text{...(1)}$$

Putting $s = 2$ in Eq. (1),

$$A = \frac{1}{5}$$

Putting $s = -3$ in Eq. (1),

$$B = -\frac{1}{5}$$

$$F(s) = \frac{1}{5} \cdot \frac{1}{s-2} - \frac{1}{5} \cdot \frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{1}{5}L^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{5}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t}$$

Example 3

Find the inverse Laplace transform of $\frac{1}{(s+\sqrt{2})(s-\sqrt{3})}$.

[Summer 2016]

Solution

Let $f(s) = \frac{1}{(s+\sqrt{2})(s-\sqrt{3})}$

By partial fractional expansion,

$$F(s) = \frac{A}{(s+\sqrt{2})} + \frac{B}{(s-\sqrt{3})}$$

$$1 = A(s-\sqrt{3}) + B(s+\sqrt{2}) \quad ...(1)$$

Putting $s = -\sqrt{2}$ in Eq. (1),

$$1 = A(-\sqrt{2}-\sqrt{3})$$

$$A = -\frac{1}{(\sqrt{3}+\sqrt{2})}$$

Putting $s = \sqrt{3}$ in Eq.(1),

$$1 = B(\sqrt{3}+\sqrt{2})$$

$$B = \frac{1}{(\sqrt{3}+\sqrt{2})}$$

$$F(s) = \frac{1}{(\sqrt{3}+\sqrt{2})}\left[-\frac{1}{(s+\sqrt{2})} + \frac{1}{(s-\sqrt{3})}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{\sqrt{3}+\sqrt{2}}\left[L^{-1}\left\{-\frac{1}{(s+\sqrt{2})} + \frac{1}{(s-\sqrt{3})}\right\}\right]$$

$$= \frac{1}{\sqrt{3}+\sqrt{2}}[-e^{-\sqrt{2}t} + e^{\sqrt{3}t}]$$

$$= \frac{e^{\sqrt{3}t} - e^{-\sqrt{2}t}}{\sqrt{3} + \sqrt{2}}$$

Example 4

Find the inverse Laplace transform of $\dfrac{s+2}{s(s+1)(s+3)}$.

Solution

Let $$F(s) = \frac{s+2}{s(s+1)(s+3)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$s + 2 = A(s + 1)(s + 3) + Bs(s + 3) + Cs(s + 1) \qquad \dots (1)$$

Putting $s = 0$ in Eq. (1),

$$2 = 3A$$

$$A = \frac{2}{3}$$

Putting $s = -1$ in Eq. (1),

$$1 = B(-1)(2)$$

$$B = -\frac{1}{2}$$

Putting $s = -3$ in Eq. (1),

$$-1 = C(-3)(-2)$$

$$C = -\frac{1}{6}$$

$$F(s) = \frac{2}{3}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{6}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{2}{3}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{6}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{2}{3} - \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t}$$

Example 5

Find the inverse Laplace transform of $\dfrac{3s^2+2}{(s+1)(s+2)(s+3)}$.

[Winter 2014]

Solution

Let $$F(s) = \frac{3s^2+2}{(s+1)(s+2)(s+3)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$3s^2 + 2 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2) \quad \ldots(1)$$

Putting $s = -1$ in Eq. (1),

$$5 = 2A$$

$$A = \frac{5}{2}$$

Putting $s = -2$ in Eq. (1),

$$14 = -B$$

$$B = -14$$

Putting $s = -3$ in Eq. (1),

$$29 = 2C$$

$$C = \frac{29}{2}$$

$$F(s) = \frac{5}{2}\cdot\frac{1}{s+1} - 14\cdot\frac{1}{s+2} + \frac{29}{2}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{5}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - 14L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{29}{2}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{5}{2}e^{-t} - 14e^{-2t} + \frac{29}{2}e^{-3t}$$

Example 6

Find the inverse Laplace transform of $\dfrac{s+2}{s^2(s+3)}$.

Solution

Let $$F(s) = \frac{s+2}{s^2(s+3)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$$s + 2 = As(s+3) + B(s+3) + Cs^2 \quad \dots (1)$$

Putting $s = 0$ in Eq. (1),

$$2 = 3B$$

$$B = \frac{2}{3}$$

Putting $s = -3$ in Eq. (1),

$$-1 = 9C$$

$$C = -\frac{1}{9}$$

Equating the coefficients of s^2,

$$0 = A + C$$

$$A = \frac{1}{9}$$

$$F(s) = \frac{1}{9}\cdot\frac{1}{s} + \frac{2}{3}\cdot\frac{1}{s^2} - \frac{1}{9}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{1}{9}L^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{9}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{9} + \frac{2}{3}t - \frac{1}{9}e^{-3t}$$

Example 7

Find the inverse Laplace transform of $\dfrac{s}{(s+1)(s-1)^2}$. **[Winter 2014]**

Solution

Let $\quad F(s) = \dfrac{s}{(s+1)(s-1)^2}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$s = A(s-1)^2 + B(s+1)(s-1) + C(s+1) \quad \dots (1)$$

Putting $s = -1$ in Eq. (1),

$$-1 = 4A$$

$$A = -\frac{1}{4}$$

Putting $s = 1$ in Eq. (1),

$$1 = 2C$$

$$C = \frac{1}{2}$$

Putting $s = 0$ in Eq. (1),

$$0 = A - B + C$$

$$B = A + C = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

$$F(s) = -\frac{1}{4} \cdot \frac{1}{s+1} + \frac{1}{4} \cdot \frac{1}{s-1} + \frac{1}{2} \cdot \frac{1}{(s-1)^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{4} L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{4} L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= -\frac{1}{4} e^{-t} + \frac{1}{4} e^{t} + \frac{1}{2} t e^{t}$$

Example 8

Find the inverse Laplace transform of $\frac{5s^2 - 15s - 11}{(s+1)(s-2)^2}$.

Solution

Let $F(s) = \frac{5s^2 - 15s - 11}{(s+1)(s-2)^2}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$5s^2 - 15s - 11 = A(s-2)^2 + B(s+1)(s-2) + C(s+1) \quad \ldots (1)$$

Putting $s = -1$ in Eq. (1),

$$9 = 9A$$

$$A = 1$$

Putting $s = 2$ in Eq. (1),

$$-21 = 3C$$

$$C = -7$$

Equating the coefficients of s^2,

$$5 = A + B$$

$$B = 4$$

$$F(s) = \frac{1}{s+1} + \frac{4}{s-2} - \frac{7}{(s-2)^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} + 4L^{-1}\left\{\frac{1}{s-2}\right\} - 7L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$= e^{-t} + 4e^{2t} - 7\,te^{2t}$$

Example 9

Find the inverse Laplace transform of $\dfrac{s+2}{(s+3)(s+1)^3}$.

Solution

Let $$F(s) = \frac{s+2}{(s+3)(s+1)^3}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+3} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

$$s + 2 = A(s+1)^3 + B(s+3)(s+1)^2 + C(s+3)(s+1) + D(s+3) \qquad \text{... (1)}$$

Putting $s = -3$ in Eq. (1),

$$-1 = -8A$$

$$A = \frac{1}{8}$$

Putting $s = -1$ in Eq. (1),

$$1 = 2D$$

$$D = \frac{1}{2}$$

Equating the coefficients of s^3,

$$0 = A + B$$

$$B = -\frac{1}{8}$$

Equating the coefficients of s^2,

$$0 = 3A + 5B + C$$

$$C = -\frac{3}{8} + \frac{5}{8} = \frac{1}{4}$$

$$F(s) = \frac{1}{8}\cdot\frac{1}{s+3} - \frac{1}{8}\cdot\frac{1}{s+1} + \frac{1}{4}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{(s+1)^3}$$

$$L^{-1}\{F(s)\} = \frac{1}{8}L^{-1}\left\{\frac{1}{s+3}\right\} - \frac{1}{8}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$= \frac{1}{8}e^{-3t} - \frac{1}{8}e^{-t} + \frac{1}{4}te^{-t} + \frac{1}{2}\cdot\frac{t^2}{2}\cdot e^{-t}$$

$$= \frac{1}{8}\left[e^{-3t} + (2t^2 + 2t - 1)e^{-t}\right]$$

Example 10

Find the inverse Laplace transform of $\frac{s^3+6s^2+14s}{(s+2)^4}$.

Solution

Let $$F(s) = \frac{s^3+6s^2+14s}{(s+2)^4}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} + \frac{D}{(s+2)^4}$$

$$s^3 + 6s^2 + 14s = A(s+2)^3 + B(s+2)^2 + C(s+2) + D$$

$$= As^3 + (6A + B)s^2 + (12A + 4B + C)s + (8A + 4B + 2C + D) \quad \text{...(1)}$$

Equating the coefficients of s^3,

$$A = 1$$

Equating the coefficients of s^2,

$$6 = 6A + B$$

$$B = 0$$

Equating the coefficients of s,

$$14 = 12A + 4B + C$$

$$C = 14 - 12 - 0 = 2$$

Equating the coefficients of s^0,

$$0 = 8A + 4B + 2C + D$$

$$D = -8 - 0 - 4 = -12$$

$$F(s) = \frac{1}{s+2} + \frac{2}{(s+2)^3} - \frac{12}{(s+2)^4}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} + 2L^{-1}\left\{\frac{1}{(s+2)^3}\right\} - 12L^{-1}\left\{\frac{1}{(s+2)^4}\right\}$$

$$= e^{-2t} + 2\cdot\frac{t^2}{2}\cdot e^{-2t} - 12\cdot\frac{t^3}{6}\cdot e^{-2t}$$

$$= e^{-2t}(1 + t^2 - 2t^3)$$

Example 11

Find the inverse Laplace transform of $\frac{s^2+1}{(s+1)(s-2)^2}$.

Solution

Let $$F(s)=\frac{s^2+1}{(s+1)(s-2)^2}$$

By partial fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$$

$$s^2+1=A(s-2)^2+B(s+1)(s-2)+C(s+1) \quad \ldots(1)$$

Putting $s=-1$ in Eq. (1),

$$2=9A$$

$$A=\frac{2}{9}$$

Putting $s=2$ in Eq. (1),

$$5=3C$$

$$C=\frac{5}{3}$$

Equating the coefficients of s^2,

$$1=A+B$$

$$B=\frac{7}{9}$$

$$F(s)=\frac{2}{9}\cdot\frac{1}{s+1}+\frac{7}{9}\cdot\frac{1}{s-2}+\frac{5}{3}\cdot\frac{1}{(s-2)^2}$$

$$L^{-1}\{F(s)\}=\frac{2}{9}L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{7}{9}L^{-1}\left\{\frac{1}{s-2}\right\}+\frac{5}{3}L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$=\frac{2}{9}e^{-t}+\frac{7}{9}e^{2t}+\frac{5}{3}e^{2t}\,L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$=\frac{2}{9}e^{-t}+\frac{7}{9}e^{2t}+\frac{5}{3}te^{2t}$$

Example 12

Find the inverse Laplace transform of $\dfrac{1}{(s+1)(s^2+1)}$.

Solution

Let $$F(s)=\frac{1}{(s+1)(s^2+1)}$$

By partial fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{Bs+C}{s^2+1}$$

$$1=A(s^2+1)+(Bs+C)(s+1) \quad \ldots(1)$$

Putting $s=-1$, in Eq. (1),

$$1=2A$$

$$A=\frac{1}{2}$$

Equating the coefficients of s^2,

$$0=A+B$$

$$B=-\frac{1}{2}$$

Equating the coefficients of s,

$$0=B+C$$

$$C=\frac{1}{2}$$

$$F(s)=\frac{1}{2}\cdot\frac{1}{s+1}-\frac{1}{2}\cdot\frac{s}{s^2+1}+\frac{1}{2}\cdot\frac{1}{s^2+1}$$

$$L^{-1}\{F(s)\}=\frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{2}L^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$=\frac{1}{2}e^{-t}-\frac{1}{2}\cos t+\frac{1}{2}\sin t$$

$$=\frac{1}{2}(e^{-t}-\cos t+\sin t)$$

Example 13

Find the inverse Laplace transform of $\dfrac{3s+1}{(s+1)(s^2+2)}$.

Solution

Let $$F(s) = \frac{3s+1}{(s+1)(s^2+2)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+2}$$

$$3s + 1 = A(s^2 + 2) + (Bs + C)(s + 1) \quad \text{... (1)}$$

Putting $s = -1$ in Eq. (1),

$$-2 = 3A$$

$$A = -\frac{2}{3}$$

Equating the coefficients of s^2,

$$0 = A + B$$

$$B = -\frac{2}{3}$$

Equating the coefficients of s^0,

$$1 = 2A + C$$

$$C = 1 + \frac{4}{3} = \frac{7}{3}$$

$$F(s) = -\frac{2}{3} \cdot \frac{1}{s+1} + \frac{2}{3} \cdot \frac{s}{s^2+2} + \frac{7}{3} \cdot \frac{1}{s^2+2}$$

$$L^{-1}\{F(s)\} = -\frac{2}{3} L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{2}{3} L^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{7}{3} L^{-1}\left\{\frac{1}{s^2+2}\right\}$$

$$= -\frac{2}{3} e^{-t} + \frac{2}{3} \cos\sqrt{2}t + \frac{7}{3\sqrt{2}} \sin\sqrt{2}t$$

Example 14

Find the inverse Laplace transform of $\frac{s+4}{s(s-1)(s^2+4)}$.

Solution

Let $$F(s) = \frac{s+4}{s(s-1)(s^2+4)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+4}$$

$$s+4 = A(s-1)(s^2+4) + Bs(s^2+4) + (Cs+D)\,s(s-1) \quad \ldots(1)$$

Putting $s = 0$ in Eq. (1),

$$4 = -4A$$
$$A = -1$$

Putting $s = 1$ in Eq. (1),

$$5 = 5B$$
$$B = 1$$

Equating the coefficients of s^3,

$$0 = A + B + C$$
$$C = 1 - 1 = 0$$

Equating the coefficients of s,

$$1 = 4A + 4B - D$$
$$D = -4 + 4 - 1 = -1$$
$$F(s) = -\frac{1}{s} + \frac{1}{s+1} - \frac{1}{s^2+4}$$
$$L^{-1}\{F(s)\} = -L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s^2+4}\right\}$$
$$= -1 + e^t - \frac{1}{2}\sin 2t$$

Example 15

Find the inverse Laplace transform of $\frac{1}{s^4-81}$. **[Summer 2016]**

Solution

Let
$$F(s) = \frac{1}{s^4-81} = \frac{1}{(s^2-9)(s^2+9)}$$
$$= \frac{1}{(s-3)(s+3)(s^2+9)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s-3} + \frac{B}{s+3} + \frac{Cs+D}{s^2+9}$$
$$1 = A(s+3)(s^2+9) + B(s-3)(s^2+9) + (Cs+D)(s^2-9) \quad \ldots(1)$$

Putting $s = 3$ in Eq. (1),

$$1 = 6A\,(18)$$

$$A = \frac{1}{108}$$

Putting $s = -3$ in Eq. (1),

$$1 = (-6)\,B\,(18)$$

$$B = -\frac{1}{108}$$

Putting $s = 0$ in Eq. (1),

$$1 = 27A - 27B - 9D$$

$$9D = 27A - 27B - 1 = \frac{1}{4} + \frac{1}{4} - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$D = -\frac{1}{18}$$

Equating the coefficient of s^3 from Eq. (1),

$$A + B + C = 0$$

$$\frac{1}{108} - \frac{1}{108} + C = 0$$

$$C = 0$$

$$F(s) = \frac{1}{108}\frac{1}{s-3} + \left(-\frac{1}{108}\right)\frac{1}{s+3} + \left(-\frac{1}{18}\right)\frac{1}{s^2+9}$$

$$L^{-1}\{F(s)\} = \frac{1}{108}L^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{108}L^{-1}\left\{\frac{1}{s+3}\right\} - \frac{1}{18}L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \frac{1}{108}e^{3t} - \frac{1}{108}e^{-3t} - \frac{1}{54}\sin 3t$$

Example 16

Find the inverse Laplace transform of $\dfrac{s}{(s^2+1)(s^2+4)}$.

Solution

Let $$F(s) = \frac{s}{(s^2+1)(s^2+4)}$$

$$= \frac{s}{3}\left[\frac{s^2+4-s^2-1}{(s^2+1)(s^2+4)}\right]$$

$$= \frac{1}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{3}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{s}{s^2+4}\right\}\right]$$

$$= \frac{1}{3}(\cos t - \cos 2t)$$

Example 17

Find the inverse Laplace transform of $\frac{s^3}{s^4 - a^4}$. [Summer 2018]

Solution

Let $F(s) = \frac{s^3}{s^4 - a^4}$

$$= \frac{s^3}{(s^2+a^2)(s^2-a^2)}$$

$$= \frac{s}{2}\left[\frac{(s^2+a^2)+(s^2-a^2)}{(s^2+a^2)(s^2-a^2)}\right]$$

$$= \frac{s}{2}\left[\frac{1}{s^2-a^2} + \frac{1}{s^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2-a^2} + \frac{s}{s^2+a^2}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{2}\left[L^{-1}\left\{\frac{s}{s^2-a^2}\right\} + L^{-1}\left\{\frac{s}{s^2+a^2}\right\}\right]$$

$$= \frac{1}{2}\left[\cosh at + \cos at\right]$$

Example 18

Find the inverse Laplace transform of $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$.

Solution

Let $F(s) = \frac{s^2}{(s^2+a^2)(s^2+b^2)}$

$$s^2 = x$$

Let

$$G(x) = \frac{x}{(x+a^2)(x+b^2)}$$

By partial fraction expansion,

$$G(x) = \frac{A}{x+a^2} + \frac{B}{x+b^2}$$

$$x = A(x+b^2) + B(x+a^2) \quad \dots (1)$$

Putting $x = -a^2$ in Eq. (1),

$$-a^2 = A(-a^2 + b^2)$$

$$A = \frac{a^2}{a^2 - b^2}$$

Putting $x = -b^2$ in Eq. (1),

$$-b^2 = B(-b^2 + a^2)$$

$$B = -\frac{b^2}{a^2 - b^2}$$

$$G(x) = \frac{a^2}{a^2-b^2}\frac{1}{x+a^2} - \frac{b^2}{a^2-b^2}\frac{1}{x+b^2}$$

$$F(s) = \frac{a^2}{a^2-b^2}\frac{1}{s^2+a^2} - \frac{b^2}{a^2-b^2}\frac{1}{s^2+b^2}$$

$$L^{-1}\{F(s)\} = \frac{a^2}{a^2-b^2}L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - \frac{b^2}{a^2-b^2}L^{-1}\left\{\frac{1}{s^2+b^2}\right\}$$

$$= \frac{a^2}{a^2-b^2}\frac{1}{a}\sin at - \frac{b^2}{a^2-b^2}\frac{1}{b}\sin bt$$

$$= \frac{1}{a^2-b^2}(a \sin at - b \sin bt)$$

Example 19

Find the inverse Laplace transform of $\dfrac{2s^2-1}{(s^2+1)(s^2+4)}$.

[Winter 2016]

Solution

Let $$F(s)=\frac{2s^2-1}{(s^2+1)(s^2+4)}$$

Let $$s^2=x$$

$$G(x)=\frac{2x-1}{(x+1)(x+4)}$$

By partial fraction expansion,

$$G(x)=\frac{A}{x+1}+\frac{B}{x+4}$$

$$2x-1=A(x+4)+B(x+1) \quad \text{...(1)}$$

Putting $x=-1$ in Eq. (1),

$$-3=3A$$
$$A=-1$$

Putting $x=-4$ in Eq. (1),

$$-9=B(-3)$$
$$B=3$$

$$G(x)=-\frac{1}{x+1}+\frac{3}{x+4}$$

$$F(s)=-\frac{1}{s^2+1}+\frac{3}{s^2+4}$$

$$L^{-1}\{F(s)\}=-L^{-1}\left\{\frac{1}{s^2+1}\right\}+3L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$=-\sin t+\frac{3}{2}\sin 2t$$

Example 20

Find the inverse Laplace transform of $\frac{5s+3}{(s-1)(s^2+2s+5)}$.

[Summer 2014]

Solution

Let $$F(s)=\frac{5s+3}{(s-1)(s^2+2s+5)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$$

$$5s + 3 = A(s^2 + 2s + 5) + (Bs + C)(s - 1)$$

$$= s^2(A+B) + s(2A - B + C) + (5A - C)$$

Equating the coefficients of s^2, s and s^0,

$$A + B = 0$$

$$2A - B + C = 5$$

$$5A - C = 3$$

Solving these equations,

$$A = 1, \quad B = -1, \quad C = 2$$

$$F(s) = \frac{1}{s-1} + \frac{-s+2}{s^2+2s+5}$$

$$= \frac{1}{s-1} - \frac{s+1-3}{(s+1)^2+2^2}$$

$$= \frac{1}{s-1} - \frac{s+1}{(s+1)^2+2^2} + \frac{3}{(s+1)^2+2^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + 3L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= e^t - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t$$

Example 21

Find the inverse Laplace transform of $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$.

Solution

Let $$F(s) = \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$$

Let $$s^2 + 2s = x$$

$$G(x) = \frac{x+3}{(x+5)(x+2)}$$

By partial fraction expansion,

$$G(x) = \frac{A}{x+5} + \frac{B}{x+2}$$

$$x + 3 = A(x + 2) + B(x + 5) \qquad \ldots (1)$$

Putting $x = -5$ in Eq. (1),

$$-2 = -3A$$

$$A = \frac{2}{3}$$

Putting $x = -2$ in Eq. (1),

$$1 = 3B$$

$$B = \frac{1}{3}$$

$$G(x) = \frac{2}{3} \cdot \frac{1}{x+5} + \frac{1}{3} \cdot \frac{1}{x+2}$$

$$F(s) = \frac{2}{3} \cdot \frac{1}{(s^2+2s+5)} + \frac{1}{3} \cdot \frac{1}{(s^2+2s+2)}$$

$$= \frac{2}{3} \cdot \frac{1}{(s+1)^2+4} + \frac{1}{3} \cdot \frac{1}{(s+1)^2+1}$$

$$L^{-1}\{F(s)\} = \frac{2}{3} L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} + \frac{1}{3} L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= \frac{2}{3} e^{-t} \cdot \frac{1}{2} \sin 2t + \frac{1}{3} e^{-t} \sin t$$

$$= \frac{1}{3} e^{-t} (\sin 2t + \sin t)$$

Example 22

Find the inverse Laplace transform of $\dfrac{s+2}{(s^2+4s+8)(s^2+4s+13)}$.

Solution

Let

$$F(s) = \frac{s+2}{(s^2+4s+8)(s^2+4s+13)}$$

$$= \frac{s+2}{5}\left[\frac{s^2+4s+13-s^2-4s-8}{(s^2+4s+8)(s^2+4s+13)}\right]$$

$$= \frac{1}{5}\left[\frac{s+2}{s^2+4s+8} - \frac{s+2}{s^2+4s+13}\right]$$

$$= \frac{1}{5}\left[\frac{s+2}{(s+2)^2+4} - \frac{s+2}{(s+2)^2+9}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{5}\left[L^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\} - L^{-1}\left\{\frac{s+2}{(s+2)^2+9}\right\}\right]$$

$$= \frac{1}{5}\left[e^{-2t}\, L^{-1}\left\{\frac{s}{s^2+4}\right\} - e^{-2t}\, L^{-1}\left\{\frac{s}{s^2+9}\right\}\right]$$

$$= \frac{1}{5}\left[e^{-2t}\cos 2t - e^{-2t}\cos 3t\right]$$

$$= \frac{e^{-2t}}{5}(\cos 2t - \cos 3t)$$

Example 23

Find the inverse Laplace transform of $\frac{s}{s^4+4a^4}$. **[Winter 2013]**

Solution

Let $$F(s) = \frac{s}{s^4+4a^4}$$

$$= \frac{s}{(s^4+4a^2s^2+4a^4)-4a^2s^2}$$

$$= \frac{s}{(s^2+2a^2)^2-(2as)^2}$$

$$= \frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)}$$

$$= \frac{1}{4a}\left[\frac{s^2+2as+2a^2-s^2+2as-2a^2}{(s^2+2as+2a^2)(s^2-2as+2a^2)}\right]$$

$$= \frac{1}{4a}\left[\frac{1}{s^2-2as+2a^2} - \frac{1}{s^2+2as+2a^2}\right]$$

$$= \frac{1}{4a}\left[\frac{1}{(s-a)^2+a^2} - \frac{1}{(s+a)^2+a^2}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{4a}\left[L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\} - L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[e^{at}L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - e^{-at}L^{-1}\left\{\frac{1}{s^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[e^{at} \cdot \frac{1}{a}\sin at - e^{-at} \cdot \frac{1}{a}\sin at\right]$$

$$= \frac{1}{2a^2}\sin at\left(\frac{e^{at} - e^{-at}}{2}\right)$$

$$= \frac{1}{2a^2}\sin at \sinh at$$

Example 24

Find the inverse Laplace transform of $\frac{s}{s^4 + s^2 + 1}$.

Solution

Let $$F(s) = \frac{s}{s^4 + s^2 + 1}$$

$$= \frac{s}{s^4 + 2s^2 + 1 - s^2}$$

$$= \frac{s}{(s^2+1)^2 - s^2}$$

$$= \frac{s}{(s^2+1+s)(s^2+1-s)}$$

$$= \frac{1}{2}\left[\frac{s^2+1+s-s^2-1+s}{(s^2+1+s)(s^2+1-s)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2+1-s} - \frac{1}{s^2+1+s}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{1}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{2}\left[L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2 + \frac{3}{4}}\right\} - L^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}\right\}\right]$$

$$= \frac{1}{2}\left[e^{\frac{t}{2}} L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\} - e^{-\frac{t}{2}} L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\}\right]$$

$$= \frac{1}{2}\left[e^{\frac{t}{2}} \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t - e^{-\frac{t}{2}} \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t\right]$$

$$= \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t \left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{2}\right)$$

$$= \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t \sinh\frac{t}{2}$$

Example 25

Find the inverse Laplace transform of $\frac{1}{s^3+1}$.

Solution

Let $F(s) = \frac{1}{s^3+1}$

$$= \frac{1}{(s+1)(s^2-s+1)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2-s+1}$$

$$1 = A(s^2-s+1) + (Bs+C)(s+1) \qquad \ldots (1)$$

Putting $s = -1$ in Eq. (1),

$$1 = 3A$$

$$A = \frac{1}{3}$$

Equating the coefficients of s^2,

$$0 = A + B$$

$$B = -\frac{1}{3}$$

Equating the coefficients of s,

$$0 = -A + B + C$$

$$C = \frac{2}{3}$$

$$F(s) = \frac{1}{3}\cdot\frac{1}{s+1} - \frac{1}{3}\cdot\frac{s}{s^2-s+1} + \frac{2}{3}\cdot\frac{1}{s^2-s+1}$$

$$= \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\left(\frac{s-2}{s^2-s+1}\right)$$

$$= \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\left[\frac{s-\frac{1}{2}-\frac{3}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$= \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} + \frac{1}{3}\cdot\frac{\frac{3}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$L^{-1}\{F(s)\} = \frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{3}L^{-1}\left\{\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= \frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{3}e^{\frac{t}{2}}L^{-1}\left\{\frac{s}{s^2+\frac{3}{4}}\right\} + \frac{1}{2}e^{\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\}$$

$$= \frac{1}{3}e^{-t} - \frac{1}{3}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}e^{\frac{t}{2}}\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t$$

$$= \frac{1}{3}e^{-t} - \frac{1}{3}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t$$

Example 26

Find the inverse Laplace transform of $\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$.

Solution

Let $\quad F(s) = \dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$

By partial fraction expansion,

$$F(s) = \frac{As+B}{(s^2-2s+2)} + \frac{Cs+D}{(s^2-2s+2)^2}$$

$$s^3-3s^2+6s-4 = (As+B)(s^2-2s+2)+Cs+D$$

$$= As^3+s^2(B-2A)+s(2A-2B+C)+2B+D$$

Equating the coefficients of s^3,

$$A=1$$

Equating the coefficients of s^2,

$$-3 = B-2A$$

$$B = -3+2 = -1$$

Equating the coefficients of s,

$$6 = 2A-2B+C$$

$$C = 6-2-2 = 2$$

Equating the coefficients of s^0,

$$-4 = 2B+D$$

$$D = -4+2 = -2$$

$$F(s) = \frac{s-1}{(s^2-2s+2)} + \frac{2s-2}{(s^2-2s+2)^2}$$

$$= \frac{s-1}{(s-1)^2+1} + \frac{2(s-1)}{\left[(s-1)^2+1\right]^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} + 2\,L^{-1}\left\{\frac{s-1}{\left[(s-1)^2+1\right]^2}\right\}$$

$$= e^t\,L^{-1}\left\{\frac{s}{s^2+1}\right\} + 2e^t\,L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$= e^t\cos t + 2e^t\,\frac{t}{2}\sin t$$

$$= e^t(\cos t + t\sin t)$$

EXERCISE 2.19

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}$

$$\left[\textbf{Ans.: } -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}\right]$$

2. $\frac{s+2}{s^2(s+3)}$ $\left[\text{Ans.: } \frac{1}{9}(1+6t-e^{-3t})\right]$

3. $\frac{1}{s(s+1)^2}$ $\left[\text{Ans.: } 1-e^{-t}-te^{-t}\right]$

4. $\frac{1}{s^2(s+3)^2}$ $\left[\text{Ans.: } \frac{1}{27}(-2+3t+2e^{-3t}+3t^2e^{-3t})\right]$

5. $\frac{s^2}{(s+4)^3}$ $\left[\text{Ans.: } e^{-4t}(1-8t+8t^2)\right]$

6. $\frac{1}{(s-2)^4(s+3)}$

$$\left[\text{Ans.: } \frac{e^{2t}}{6}\left(\frac{t^3}{5}-\frac{3}{25}t^2+\frac{6}{125}t-\frac{6}{625}\right)+\frac{1}{625}e^{-3t}\right]$$

7. $\frac{5s^2-7s+17}{(s-1)(s^2+4)}$

$$\left[\text{Ans.: } 3e^t+2\cos 2t-\frac{5}{2}\sin 2t\right]$$

8. $\frac{2s^3-s^2-1}{(s+1)^2(s^2+1)^2}$

$$\left[\text{Ans.: } \frac{1}{2}\sin t+\frac{1}{2}t\cos t-te^{-t}\right]$$

9. $\frac{1}{s^3(s-1)}$

$$\left[\text{Ans.: } 1-t+\frac{t^2}{2}-e^{-t}\right]$$

10. $\frac{s}{(s+1)^2(s^2+1)}$

$$\left[\text{Ans.: } \frac{1}{2}(\sin t-te^{-t})\right]$$

11. $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

$$\left[\text{Ans.: } e^t - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t\right]$$

12. $\dfrac{s}{(s^2-2s+2)(s^2+2s+2)}$

$$\left[\text{Ans.: } \frac{1}{2}\sin t \sinh t\right]$$

13. $\dfrac{10}{s(s^2-2s+5)}$

$$\left[\text{Ans.: } 2 - e^t(2\cos 2t - \sin 2t)\right]$$

14. $\dfrac{s^2+8s+27}{(s+1)(s^2+4s+13)}$

$$\left[\text{Ans.: } 2e^{-t} + e^{-2t}(\sin 3t - \cos 3t)\right]$$

15. $\dfrac{2s-1}{s^4+s^2+1}$

$$\left[\text{Ans.: } \frac{1}{2}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}e^{\frac{1}{2}}\sin\frac{\sqrt{3}}{2}t - \frac{1}{2}e^{-\frac{1}{2}}\cos\frac{\sqrt{3}}{2}t - \frac{5}{2\sqrt{3}}e^{-\frac{1}{2}}\sin\frac{\sqrt{3}}{2}t\right]$$

16. $\dfrac{s}{s^4+4a^4}$

$$\left[\text{Ans.: } \frac{1}{2a^2}\sin at \sinh at\right]$$

17. $\dfrac{s^2}{s^4+4a^4}$

$$\left[\text{Ans.: } \frac{1}{2a}[\sinh at \cos at + \cosh at \sin at]\right]$$

2.13 CONVOLUTION THEOREM

If $L^{-1}\{F_1(s)\} = f_1(t)$ and $L^{-1}\{F_2(s)\} = f_2(t)$ then

$$L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(u) f_2(t-u)\,du$$

where $\int_0^t f_1(u) f_2(t-u)\,du = f_1(t) * f_2(t)$

$f_1(t) * f_2(t)$ is called the convolution of $f_1(t)$ and $f_2(t)$.

Proof: $F_1(s)\cdot F_2(s) = L\{f_1(t)\}\cdot L\{f_2(t)\}$

$$= \int_0^\infty e^{-su} f_1(u)\mathrm{d}u \cdot \int_0^\infty e^{-sv} f_2(v)\mathrm{d}v$$

$$= \int_0^\infty \int_0^\infty e^{-s(u+v)} f_1(u) f_2(v)\mathrm{d}u\,\mathrm{d}v$$

$$= \int_0^\infty f_1(u)\left[\int_0^\infty e^{-s(u+v)} f_2(v)\mathrm{d}v\right]\mathrm{d}u$$

Putting $u+v=t,\ \mathrm{d}v=\mathrm{d}t$

When $v=0, \quad t=u$

When $v\to\infty, \quad t\to\infty$

$$F_1(s)\cdot F_2(s) = \int_0^\infty f_1(u)\left[\int_u^\infty e^{-st} f_2(t-u)\mathrm{d}t\right]\mathrm{d}u$$

$$= \int_0^\infty \int_u^\infty e^{-st} f_1(u) f_2(t-u)\mathrm{d}t\,\mathrm{d}u$$

The region of integration is bounded by the lines $u=0$ and $u=t$ (Fig. 2.12). To change the order of integration, draw a vertical strip which starts from the line $u=0$ and terminates on the line $u=t$. Hence, u varies from 0 to t and t varies from 0 to ∞.

$$F_1(s)\cdot F_2(s) = \int_0^\infty e^{-st} \int_0^t f_1(u) f_2(t-u)\mathrm{d}u\,\mathrm{d}t$$

$$= L\left\{\int_0^t f_1(u) f_2(t-u)\mathrm{d}u\right\}$$

Hence, $L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(u) f_2(t-u)\mathrm{d}u$

Note: The convolution operation is commutative, i.e.,

$$L\left\{\int_0^t f_1(u) f_2(t-u)\mathrm{d}u\right\} = L\left\{\int_0^t f_1(t-u) f_2(u)\mathrm{d}u\right\}$$

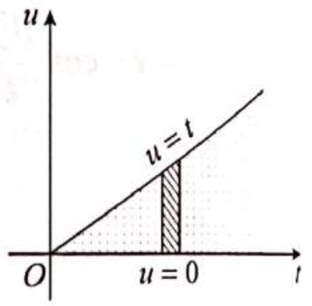


Fig. 2.12 *Region of integration bounded by line*

Example 1

Evaluate $t * e^t$. **[Summer 2015]**

Solution

Let $f(t)=t \quad g(t)=e^t$

$$f(t)*g(t) = \int_0^t f(u)g(t-u)\mathrm{d}u$$

$$t*e^t = \int_0^t u\cdot e^{(t-u)}\,\mathrm{d}u$$

$$= e^t \int_0^t u\,e^{-u}\,\mathrm{d}u$$

$$= e^t \left| u\left(\frac{e^{-u}}{-1}\right) - (1)\left(\frac{e^{-u}}{1}\right)\right|_0^t$$

$$= e^t \left| -u\, e^{-u} - e^{-u} \right|_0^t$$

$$= e^t \left[-te^{-t} - e^{-t} + e^0 \right]$$

$$= -t - 1 + e^t$$

Example 2

State the convolution theorem and verify it for f(t) = t and g(t) = e^{2t}. **[Winter 2015]**

Solution

If $L\{f(t)\} = F(s)$ and $L\{g(t)\} = G(s)$ and $L^{-1}\{F(s)\} = f(t)$ and $L^{-1}\{G(s)\} = g(t)$ then

$$L^{-1}\{F(s)G(s)\} = \int_0^t f(u)\, g(t-u)\, du = f(t) * g(t)$$

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u) du$$

$$= \int_0^t u\, e^{2(t-u)} du$$

$$= e^{2t} \int_0^t u\, e^{-2u} du$$

$$= e^{2t} \left| u\left(\frac{e^{-2u}}{-2}\right) - (1)\left(\frac{e^{-2u}}{4}\right) \right|_0^t$$

$$= e^{2t} \left| -\frac{u}{2} e^{-2u} - \frac{e^{-2u}}{4} \right|_0^t$$

$$= e^{2t} \left[-\frac{t}{2} e^{-2t} - \frac{e^{-2t}}{4} + \frac{1}{4} \right]$$

$$= -\frac{t}{2} - \frac{1}{4} + \frac{1}{4} e^{2t}$$

$$= \frac{1}{4}(e^{2t} - 2t - 1)$$

$$L\{f(t) * g(t)\} = L\left\{ \frac{e^{2t} - 2t - 1}{4} \right\}$$

$$= \frac{1}{4}\left[L\{e^{2t}\} - 2L\{t\} - L\{1\}\right]$$

$$= \frac{1}{4}\left(\frac{1}{s-2} - \frac{2}{s^2} - \frac{1}{s}\right)$$

$$= \frac{1}{4}\left[\frac{s^2 - 2(s-2) - s(s-2)}{s^2(s-2)}\right]$$

$$= \frac{1}{4}\left[\frac{s^2 - 2s + 4 - s^2 + 2s}{s^2(s-2)}\right]$$

$$= \frac{1}{4}\left[\frac{4}{s^2(s-2)}\right]$$

$$= \frac{1}{s^2(s-2)}$$

$$L\{f(t)\} \cdot L\{g(t)\} = L\{t\} \cdot L\{e^{2t}\}$$

$$= \frac{1}{s^2} \cdot \frac{1}{s-2}$$

$$= \frac{1}{s^2(s-2)}$$

Hence, convolution theorem is verified.

Example 3

Find the inverse Laplace transform of $\frac{1}{(s+2)(s-1)}$.

Solution

Let $F(s) = \frac{1}{(s+2)(s-1)}$

Let $F_1(s) = \frac{1}{s+2}$ $\qquad F_2(s) = \frac{1}{s-1}$

$f_1(t) = e^{-2t}$ $\qquad f_2(t) = e^t$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-2u}\, e^{t-u} \mathrm{d}u$$

$$= e^t \int_0^t e^{-3u} \mathrm{d}u$$

$$= e^{t}\left|\frac{e^{-3u}}{-3}\right|_{0}^{t}$$

$$= \frac{e^{t}}{3}(1 - e^{-3t})$$

Example 4

Find the inverse Laplace transform of $\frac{1}{s^2(s+5)}$.

Solution

Let $F(s) = \frac{1}{s^2(s+5)}$

Let $F_1(s) = \frac{1}{s^2}$ $\qquad F_2(s) = \frac{1}{s+5}$

$f_1(t) = t$ $\qquad f_2(t) = e^{-5t}$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t u\, e^{-5(t-u)}\mathrm{d}u$$

$$= \int_0^t u\, e^{-5t+5u}\mathrm{d}u$$

$$= e^{-5t}\int_0^t u\, e^{5u}\mathrm{d}u$$

$$= e^{-5t}\left|u\frac{e^{5u}}{5} - (1)\frac{e^{5u}}{25}\right|_0^t$$

$$= e^{-5t}\left[\left(t\frac{e^{5t}}{5} - \frac{e^{5t}}{25}\right) - \left(0 - \frac{1}{25}\right)\right]$$

$$= e^{-5t}\left[t\frac{e^{5t}}{5} - \frac{e^{5t}}{25} + \frac{1}{25}\right]$$

$$= \frac{t}{5} - \frac{1}{25} + \frac{e^{-5t}}{25}$$

$$= \frac{1}{25}(e^{-5t} + 5t - 1)$$

Example 5

Find the inverse Laplace transform of $\frac{1}{s^2(s+1)^2}$.

Solution

Let $F(s)=\dfrac{1}{s^2(s+1)^2}$

Let $F_1(s)=\dfrac{1}{(s+1)^2}$ $\qquad F_2(s)=\dfrac{1}{s^2}$

$f_1(t)=te^{-t}$ $\qquad f_2(t)=t$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t ue^{-u}(t-u)\,\mathrm{d}u$$

$$=\int_0^t (ut-u^2)e^{-u}\mathrm{d}u$$

$$=\left|(ut-u^2)(-e^{-u})-(t-2u)(e^{-u})+(-2)(-e^{-u})\right|_0^t$$

$$=te^{-t}+2e^{-t}+t-2$$

Example 6

Find the inverse Laplace transform of $\dfrac{1}{(s-2)(s+2)^2}$.

Solution

Let $F(s)=\dfrac{1}{(s-2)(s+2)^2}$

Let $F_1(s)=\dfrac{1}{(s+2)^2}$ $\qquad F_2(s)=\dfrac{1}{s-2}$

$f_1(t)=te^{-2t}$ $\qquad f_2(t)=e^{2t}$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t ue^{-2u}\,e^{2(t-u)}\mathrm{d}u$$

$$=e^{2t}\int_0^t ue^{-4u}\mathrm{d}u$$

$$=e^{2t}\left|\frac{ue^{-4u}}{-4}-\frac{e^{-4u}}{16}\right|_0^t$$

$$=e^{2t}\left[\frac{-te^{-4t}}{4}-\frac{e^{-4t}}{16}+\frac{1}{16}\right]$$

$$=\frac{e^{2t}}{16}-\frac{te^{-2t}}{4}-\frac{e^{-2t}}{16}$$

$$= \frac{1}{16}(e^{2t} - e^{-2t} - 4t\,e^{-2t})$$

Example 7

Find the inverse Laplace transform of $\dfrac{1}{(s-2)^4(s+3)}$.

Solution

Let $F(s) = \dfrac{1}{(s-2)^4(s+3)}$

Let $F_1(s) = \dfrac{1}{(s-2)^4}$ $\qquad F_2(s) = \dfrac{1}{s+3}$

$f_1(t) = e^{2t}\dfrac{t^3}{6}$ $\qquad f_2(t) = e^{-3t}$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{2u}\frac{u^3}{6}e^{-3(t-u)}du$$

$$= \frac{e^{-3t}}{6}\int_0^t u^3 e^{5u}du$$

$$= \frac{e^{-3t}}{6}\left| u^3\frac{e^{5u}}{5} - 3u^2\frac{e^{5u}}{25} + 6u\frac{e^{5u}}{125} - 6\frac{e^{5u}}{625}\right|_0^t$$

$$= \frac{e^{-3t}}{6}\left[t^3\frac{e^{5t}}{5} - 3t^2\frac{e^{5t}}{25} + 6t\frac{e^{5t}}{125} - 6\frac{e^{5t}}{625} + \frac{6}{625}\right]$$

$$= \frac{e^{-3t}}{625} + \frac{e^{2t}}{6}\left[\frac{t^3}{5} - \frac{3t^2}{25} + \frac{6t}{125} - \frac{6}{625}\right]$$

Example 8

Find the inverse Laplace transform of $\dfrac{1}{s(s^2+4)}$.

[Winter 2014; Summer 2015]

Solution

Let $F(s) = \dfrac{1}{s(s^2+4)}$

Let $F_1(s) = \dfrac{1}{s^2+4}$ $\qquad F_2(s) = \dfrac{1}{s}$

$f_1(t) = \dfrac{1}{2}\sin 2t$ $\qquad f_2(t) = 1$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \frac{1}{2}\sin 2u\, du$$

$$= \frac{1}{2}\left| -\frac{\cos 2u}{2} \right|_0^t$$

$$= \frac{1}{4}(1-\cos 2t)$$

Example 9

Find the inverse Laplace transform of $\frac{1}{s^2(s^2+1)}$.

Solution

Let $F(s) = \frac{1}{s^2(s^2+1)}$

Let $F_1(s) = \frac{1}{s^2+1}$ $\qquad F_2(s) = \frac{1}{s^2}$

$f_1(t) = \sin t$ $\qquad f_2(t) = t$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \sin u\,(t-u)du$$

$$= \left|(t-u)(-\cos u) - \sin u\right|_0^t$$

$$= t - \sin t$$

Example 10

Find the inverse Laplace transform of $\frac{1}{(s+1)(s^2+1)}$.

Solution

Let $F(s) = \frac{1}{(s+1)(s^2+1)}$

Let $F_1(s) = \frac{1}{s^2+1}$ $\qquad F_2(s) = \frac{1}{s+1}$

$f_1(t) = \sin t$ $\qquad f_2(t) = e^{-t}$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \sin u\, e^{-(t-u)}du$$

$$= \int_0^t e^{u-t} \sin u \, du$$

$$= e^{-t} \left| \frac{e^u}{2}(\sin u - \cos u) \right|_0^t$$

$$= \frac{e^{-t}}{2}\left[e^t(\sin t - \cos t) + 1 \right]$$

$$= \frac{1}{2}(\sin t - \cos t) + \frac{1}{2}e^{-t}$$

Example 11

Find the inverse Laplace transform of $\dfrac{s}{(s^2+1)(s^2+4)}$.

Solution

Let $F(s) = \dfrac{s}{(s^2+1)(s^2+4)}$

Let $F_1(s) = \dfrac{1}{s^2+1}$ $\qquad F_2(s) = \dfrac{s}{s^2+4}$

$f_1(t) = \sin t$ $\qquad f_2(t) = \cos 2t$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \sin u \cos 2(t-u) du$$

$$= \frac{1}{2}\int_0^t [\sin(2t-u) + \sin(3u-2t)] du$$

$$= \frac{1}{2}\left| \frac{-\cos(2t-u)}{-1} - \frac{\cos(3u-2t)}{3} \right|_0^t$$

$$= \frac{1}{2}\left| \cos(2t-u) - \frac{1}{3}\cos(3u-2t) \right|_0^t$$

$$= \frac{1}{2}\left[\left(\cos t - \frac{1}{3}\cos t \right) - \left(\cos 2t - \frac{1}{3}\cos 2t \right) \right]$$

$$= \frac{1}{2}\left[\frac{2}{3}\cos t - \frac{2}{3}\cos 2t \right]$$

$$= \frac{1}{3}(\cos t - \cos 2t)$$

Example 12

Find the inverse Laplace transform of $\dfrac{1}{(s^2+a^2)^2}$. [Summer 2016]

Solution

Let $$F(s)=\frac{1}{(s^2+a^2)^2}$$

Let $$F_1(s)=F_2(s)=\frac{1}{s^2+a^2}$$

$$f_1(t)=f_2(t)=\frac{1}{a}\sin at$$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t \frac{1}{a}\sin au\cdot\frac{1}{a}\sin a(t-u)\mathrm{d}u$$

$$=\frac{1}{a^2}\int_0^t \sin au\sin a(t-u)\mathrm{d}u$$

$$=\frac{1}{2a^2}\int_0^t 2\sin au\sin a(t-u)\mathrm{d}u$$

$$=\frac{1}{2a^2}\int_0^t [\cos(2au-at)-\cos at]\,\mathrm{d}u$$

$$=\frac{1}{2a^2}\left|\frac{\sin(2au-at)}{2a}-u\cos at\right|_0^t$$

$$=\frac{1}{2a^2}\left[\left(\frac{1}{2a}\sin at-t\cos at\right)-\left(-\frac{\sin at}{2a}\right)\right]$$

$$=\frac{1}{2a^2}\left[\frac{1}{2a}\sin at-t\cos at+\frac{1}{2a}\sin at\right]$$

$$=\frac{1}{2a^2}\left[\frac{1}{a}\sin at-t\cos at\right]$$

$$=\frac{1}{2a^3}[\sin at-at\cos at]$$

Example 13

Find the inverse Laplace transform of $\dfrac{1}{(s^2+4)^2}$. [Summer 2018, 2017]

Solution

Let $$F(s)=\frac{1}{(s^2+4)^2}$$

Let $$F_1(s)=F_2(s)=\frac{1}{s^2+4}$$

$$f_1(t)=f_2(t)=\frac{1}{2}\sin 2t$$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t \frac{1}{2}\sin 2u.\frac{1}{2}\sin 2(t-u)\mathrm{d}u$$

$$=\frac{1}{4}\int_0^t \sin 2u \sin 2(t-u)\mathrm{d}u$$

$$=\frac{1}{8}\int_0^t [\cos(4u-2t)-\cos 2t]\mathrm{d}u$$

$$=\frac{1}{8}\left|\frac{\sin(4u-2t)}{4}-(\cos 2t)u\right|_0^t$$

$$=\frac{1}{8}\left[\left(\frac{\sin 2t}{4}-t\cos 2t\right)-\left(\frac{-\sin 2t}{4}-0\right)\right]$$

$$=\frac{1}{8}\left[\frac{\sin 2t}{4}-t\cos 2t+\frac{\sin 2t}{4}\right]$$

$$=\frac{1}{8}\left[\frac{2\sin 2t}{4}-t\cos 2t\right]$$

$$=\frac{1}{16}(\sin 2t-2t\cos 2t)$$

Example 14

Find the inverse Laplace transform of $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$.

[Winter 2017]

Solution

Let $F(s)=\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

Let $F_1(s)=\frac{s}{s^2+a^2}$ $\quad F_2(s)=\frac{s}{s^2+b^2}$

$f_1(t) = \cos at$ $\qquad f_2(t) = \cos bt$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \cos au \cos b(t-u) du$$
$$= \frac{1}{2}\int_0^t [\cos(au+bt-bu) + \cos(au-bt+bu)du]$$
$$= \frac{1}{2}\int_0^t [\cos\{(a-b)u+bt\} + \cos\{(a+b)u-bt\}]du$$
$$= \frac{1}{2}\left| \frac{\sin\{bt+(a-b)u\}}{a-b} + \frac{\sin\{(a+b)u-bt\}}{a+b} \right|_0^t$$
$$= \frac{1}{2}\left[\left\{\frac{\sin(bt+at-bt)}{a-b} + \frac{\sin(at+bt-bt)}{a+b}\right\} - \left(\frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right)\right]$$
$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$
$$= \frac{1}{2}\left[\frac{2a \sin at}{a^2-b^2} - \frac{2b \sin bt}{a^2-b^2}\right]$$
$$= \frac{1}{2}\left[\frac{2a \sin at - 2b \sin bt}{a^2-b^2}\right]$$
$$= \frac{a \sin at - b \sin bt}{a^2-b^2}$$

Example 15

Find the inverse Laplace transform of $\frac{s^2}{(s^2+a^2)^2}$.

Solution

Let $F(s) = \frac{s^2}{(s^2+a^2)^2}$

Let $F_1(s) = \frac{s}{s^2+a^2}$ $\qquad F_2(s) = \frac{s}{s^2+a^2}$

$f_1(t) = \cos at$ $\qquad f_2(t) = \cos at$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \cos au \cos a(t-u) du$$
$$= \frac{1}{2}\int_0^t [\cos at + \cos(2au - at)] du$$
$$= \frac{1}{2}\left| u \cos at + \frac{1}{2a}\sin(2au - at)\right|_0^t$$

$$= \frac{1}{2}\left(t \cos at + \frac{1}{a} \sin at\right)$$

$$= \frac{1}{2a}(\sin at + at \cos at)$$

Example 16

Find the inverse Laplace transform of $\frac{s}{(s^2 + a^2)^2}$.

[Winter 2016, 2014; Summer 2014]

Solution

Let $F(s) = \frac{s}{(s^2 + a^2)^2}$

Let $F_1(s) = \frac{s}{s^2 + a^2}$ $\quad F_2(s) = \frac{1}{s^2 + a^2}$

$f_1(t) = \cos at$ $\quad f_2(t) = \frac{1}{a} \sin at$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \cos au \cdot \frac{1}{a} \sin a(t-u) du$$

$$= \frac{1}{2a} \int_0^t [\sin at + \sin a(t-2u)] du$$

$$= \frac{1}{2a} \left| u \sin at + \frac{1}{2a} \cos a(t-2u) \right|_0^t$$

$$= \frac{1}{2a} t \sin at$$

Example 17

Find the inverse Laplace transform of $\frac{1}{(s^2 + a^2)(s^2 + b^2)}$.

Solution

Let $F(s) = \frac{1}{(s^2 + a^2)(s^2 + b^2)}$

Let $F_1(s) = \frac{1}{s^2 + a^2}$ $\quad F_2(s) = \frac{1}{s^2 + b^2}$

$f_1(t) = \frac{1}{a} \sin at$ $\quad f_2(t) = \frac{1}{b} \sin bt$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t \frac{1}{a}\sin au\cdot\frac{1}{b}\sin b(t-u)\mathrm{d}u$$

$$=\frac{1}{ab}\int_0^t \sin au\sin b(t-u)\mathrm{d}u$$

$$=-\frac{1}{2ab}\int_0^t\left[\cos\{(a-b)u+bt\}-\cos\{(a+b)u-bt\}\right]\mathrm{d}u$$

$$=-\frac{1}{2ab}\left|\frac{\sin\{(a-b)u+bt\}}{a-b}-\frac{\sin\{(a+b)u-bt\}}{a+b}\right|_0^t$$

$$=-\frac{1}{2ab}\left[\frac{\sin at}{a-b}-\frac{\sin at}{a+b}-\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right]$$

$$=-\frac{1}{2ab}\left[2b\frac{\sin at}{a^2-b^2}-2a\frac{\sin bt}{a^2-b^2}\right]$$

$$=\frac{a\sin bt-b\sin at}{ab(a^2-b^2)}$$

Example 18

Find the inverse Laplace transform of $\dfrac{s(s+1)}{(s^2+1)(s^2+2s+2)}$.

Solution

Let $F(s)=\dfrac{s(s+1)}{(s^2+1)(s^2+2s+2)}$

Let $F_1(s)=\dfrac{s+1}{s^2+2s+2}$ $\qquad F_2(s)=\dfrac{s}{s^2+1}$

$=\dfrac{s+1}{(s+1)^2+1}$ $\qquad f_2(t)=\cos t$

$f_1(t)=e^{-t}\cos t$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t e^{-u}\cos u\cos(t-u)\mathrm{d}u$$

$$=\frac{1}{2}\int_0^t e^{-u}\left[\cos t+\cos(2u-t)\right]\mathrm{d}u$$

$$=\frac{1}{2}\left|-e^{-u}\cos t+\frac{e^{-u}}{5}\{-\cos(2u-t)+2\sin(2u-t)\}\right|_0^t$$

$$=\frac{1}{2}\left[-e^{-t}\cos t+\frac{e^{-t}}{5}(-\cos t+2\sin t)-\frac{1}{5}(-\cos t-2\sin t)+\cos t\right]$$

$$=\frac{1}{10}\left[e^{-t}(2\sin t-6\cos t)+(2\sin t+6\cos t)\right]$$

Example 19

Find the inverse Laplace transform of $\frac{1}{(s^2+4s+13)^2}$.

Solution

Let $F(s)=\frac{1}{(s^2+4s+13)^2}$

Let $F_1(s)=F_2(s)=\frac{1}{s^2+4s+13}$

$$=\frac{1}{(s+2)^2+9}$$

$$f_1(t)=f_2(t)=\frac{e^{-2t}}{3}\sin 3t$$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t\frac{e^{-2u}}{3}\sin 3u\cdot\frac{e^{-2(t-u)}}{3}\sin 3(t-u)\,du$$

$$=\frac{e^{-2t}}{9}\int_0^t\sin 3u\sin 3(t-u)\,du$$

$$=-\frac{e^{-2t}}{18}\int_0^t\left[\cos 3t-\cos(6u-3t)\right]du$$

$$=-\frac{e^{-2t}}{18}\left|u\cos 3t-\frac{\sin(6u-3t)}{6}\right|_0^t$$

$$=-\frac{e^{-2t}}{18}\left[t\cos 3t-\frac{\sin 3t}{6}-\frac{\sin 3t}{6}\right]$$

$$=\frac{e^{-2t}}{18}\left[\frac{\sin 3t}{3}-t\cos 3t\right]$$

Example 20

Find the inverse Laplace transform of $\frac{s+2}{(s^2+4s+5)^2}$. **[Summer 2015]**

Solution

Let $F(s)=\dfrac{s+2}{(s^2+4s+5)^2}$

Let $F_1(s)=\dfrac{s+2}{s^2+4s+5}$ $\qquad F_2(s)=\dfrac{1}{s^2+4s+5}$

$=\dfrac{s+2}{(s+2)^2+1}$ $\qquad =\dfrac{1}{(s+2)^2+1}$

$f_1(t)=e^{-2t}\cos t$ $\qquad f_2(t)=e^{-2t}\sin t$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t e^{-2u}\cos u\cdot e^{-2(t-u)}\sin(t-u)\,du$$

$$=e^{-2t}\int_0^t \cos u\sin(t-u)\,du$$

$$=e^{-2t}\int_0^t \frac{1}{2}\left[\sin t-\sin(-t+2u)\right]du$$

$$=\frac{e^{-2t}}{2}\int_0^t\left[\sin t-\sin(2u-t)\right]du$$

$$=\frac{e^{-2t}}{2}\left|\sin t\,u-\left\{\frac{-\cos(2u-t)}{2}\right\}\right|_0^t$$

$$=\frac{e^{-2t}}{2}\left[t\sin t+\frac{1}{2}(\cos t-\cos t)\right]$$

$$=\frac{e^{-2t}}{2}t\sin t$$

Example 21

Find the inverse Laplace transform of $\dfrac{s+2}{(s^2+4s+13)^2}$.

Solution

Let $F(s)=\dfrac{s+2}{(s^2+4s+13)^2}$

Let $F_1(s)=\dfrac{s+2}{s^2+4s+13}$ $\qquad F_2(s)=\dfrac{1}{s^2+4s+13}$

$=\dfrac{s+2}{(s+2)^2+9}$ $\qquad =\dfrac{1}{(s+2)^2+9}$

$f_1(t)=e^{-2t}\cos 3t$ $\qquad f_2(t)=\dfrac{1}{3}e^{-2t}\sin 3t$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-2u}\cos 3u \cdot \frac{1}{3}e^{-2(t-u)}\sin 3(t-u)du$$

$$= \frac{e^{-2t}}{3}\int_0^t \cos 3u \sin 3(t-u)du$$

$$= \frac{e^{-2t}}{3}\int_0^t \frac{1}{2}[\sin 3t - \sin(-3t+6u)]du$$

$$= \frac{e^{-2t}}{6}\int_0^t [\sin 3t - \sin(6u-3t)]du$$

$$= \frac{e^{-2t}}{6}\left|\sin 3t\, u - \left\{\frac{-\cos(6u-3t)}{6}\right\}\right|_0^t$$

$$= \frac{e^{-2t}}{6}\left[t\sin 3t + \frac{1}{6}(\cos 3t - \cos 3t)\right]$$

$$= \frac{e^{-2t}}{6}t\sin 3t$$

Example 22

Find the inverse Laplace transform of $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$.

Solution

Let $F(s) = \dfrac{(s+2)^2}{(s^2+4s+8)^2}$

Let $F_1(s) = F_2(s) = \dfrac{s+2}{s^2+4s+8}$

$$= \frac{s+2}{(s+2)^2+4}$$

$$f_1(t) = f_2(t) = e^{-2t}\cos 2t$$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-2u}\cos 2u\, e^{-2(t-u)}\cos 2(t-u)\, du$$

$$= e^{-2t}\int_0^t \cos 2u \cos 2(t-u)\, du$$

$$= \frac{e^{-2t}}{2}\int_0^t [\cos 2t + \cos(4u-2t)]du$$

$$= \frac{e^{-2t}}{2}\left| u\cos 2t + \frac{\sin(4u-2t)}{4} \right|_0^t$$

$$= \frac{e^{-2t}}{2}\left[t\cos 2t + \frac{\sin 2t}{4} + \frac{\sin 2t}{4} \right]$$

$$= \frac{e^{-2t}}{4}\left[\sin 2t + 2t\cos 2t\right]$$

Example 23

Find the inverse Laplace transform of $\frac{1}{(s+3)(s^2+2s+2)}$.

Solution

Let $F(s) = \frac{1}{(s+3)(s^2+2s+2)}$

Let $F_1(s) = \frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}$ $\qquad F_2(s) = \frac{1}{s+3}$

$f_2(t) = e^{-3t}$

$f_1(t) = e^{-t}\sin t$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-u}\sin u\, e^{-3(t-u)}du$$

$$= e^{-3t}\int_0^t e^{2u}\sin u\,du$$

$$= e^{-3t}\left| \frac{e^{2u}}{5}(2\sin u - \cos u) \right|_0^t$$

$$= \frac{e^{-3t}}{5}\left[e^{2t}(2\sin t - \cos t) + 1\right]$$

$$= \frac{1}{5}\left[e^{-t}(2\sin t - \cos t) + e^{-3t}\right]$$

Example 24

Find the inverse Laplace transform of $\frac{1}{s(s+a)^3}$. **[Winter 2013]**

Solution

Let $F(s)=\dfrac{1}{s(s+a)^3}$

Let $F_1(s)=\dfrac{1}{(s+a)^3}$ $\qquad F_2(s)=\dfrac{1}{s}$

$f_1(t)=e^{-at}\dfrac{t^2}{2}$ $\qquad f_2(t)=1$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t e^{-au}\frac{u^2}{2}\,du$$

$$=\frac{1}{2}\left|\frac{u^2e^{-au}}{-a}-\frac{2u\,e^{-au}}{a^2}+\frac{2e^{-au}}{-a^3}\right|_0^t$$

$$=\frac{1}{2}\left[-\frac{t^2e^{-at}}{a}-\frac{2te^{-at}}{a^2}-\frac{2e^{-at}}{a^3}+\frac{2}{a^3}\right]$$

$$=-\frac{1}{2a}t^2e^{-at}-\frac{1}{a^2}te^{-at}-\frac{1}{a^3}e^{-at}+\frac{1}{a^3}$$

Example 25

Find the inverse Laplace transform of $\dfrac{1}{(s^2+4)(s+1)^2}$.

Solution

Let $F(s)=\dfrac{1}{(s^2+4)\,(s+1)^2}$

Considering $F(s)$ as a product of three functions,

$$F(s)=\frac{1}{(s^2+4)}\cdot\frac{1}{s+1}\cdot\frac{1}{s+1}$$

Let $F_1(s)=\dfrac{1}{s^2+4}$ $\qquad F_2(s)=\dfrac{1}{s+1}$ $\qquad F_3(s)=\dfrac{1}{s+1}$

$f_1(t)=\dfrac{1}{2}\sin 2t$ $\qquad f_2(t)=e^{-t}$ $\qquad f_3(t)=e^{-t}$

By the convolution theorem,

$$L^{-1}\{F_1(s)\cdot F_2(s)\}=\int_0^t\frac{1}{2}\sin 2u\,e^{-(t-u)}du$$

$$= \frac{e^{-t}}{2}\left|\frac{e^{u}}{5}(\sin 2u - 2\cos 2u)\right|_0^t$$

$$= \frac{e^{-t}}{10}\left[e^{t}(\sin 2t - 2\cos 2t) + 2\right]$$

$$= \frac{\sin 2t - 2\cos 2t}{10} + \frac{e^{-t}}{5}$$

$$L^{-1}\{F_1(s)F_2(s)F_3(s)\} = \int_0^t \left[\frac{\sin 2u - 2\cos 2u}{10} + \frac{e^{-u}}{5}\right] e^{-(t-u)} du$$

$$= \frac{e^{-t}}{10}\int_0^t \left[e^{u}(\sin 2u - 2\cos 2u) + 2\right] du$$

$$= \frac{e^{-t}}{10}\left|\frac{e^{u}}{5}\{(\sin 2u - 2\cos 2u) - 2(\cos 2u + 2\sin 2u)\} + 2u\right|_0^t$$

$$= \frac{e^{-t}}{10}\left[\frac{e^{t}}{5}(-3\sin 2t - 4\cos 2t) + 2t + \frac{4}{5}\right]$$

$$= \frac{2}{25}e^{-t} + \frac{te^{-t}}{5} - \frac{1}{50}(3\sin 2t + 4\cos 2t)$$

EXERCISE 2.20

Find the inverse Laplace transforms of the following functions:

1. $\frac{1}{(s+3)(s-1)}$

$$\left[\text{Ans.: } \frac{e^{t}}{4}(1 - e^{-4t})\right]$$

2. $\frac{1}{s(s^2+4)}$

$$\left[\text{Ans.: } \frac{1}{4}(1 - \cos 2t)\right]$$

3. $\frac{1}{(s-3)(s+3)^2}$

$$\left[\text{Ans.: } \frac{1}{36}(e^{3t} - e^{-3t} - 6te^{-3t})\right]$$

4. $\frac{s}{(s^2+4)^2}$

$$\left[\text{Ans.: } \frac{1}{4}t \sin 2t\right]$$

5. $\dfrac{s^2}{(s^2-a^2)^2}$

$\left[\text{Ans.: } \dfrac{1}{2}(\sinh at + at\cosh at)\right]$

6. $\dfrac{1}{s(s^2-a^2)}$

$\left[\text{Ans.: } \dfrac{1}{a^2}(\cosh at - 1)\right]$

7. $\dfrac{1}{s^3(s^2+1)}$

$\left[\text{Ans.: } \dfrac{t^2}{2} + \cos t - 1\right]$

8. $\dfrac{s^2}{(s^2+4)^2}$

$\left[\text{Ans.: } \dfrac{1}{4}(\sin 2t + 2t\cos 2t)\right]$

9. $\dfrac{s^2}{(s^2+1)(s^2+4)}$

$\left[\text{Ans.: } \dfrac{1}{3}(2\sin 2t - \sin t)\right]$

10. $\dfrac{s}{(s^2-a^2)^2}$

$\left[\text{Ans.: } \dfrac{1}{2a}(at\cosh at + \sinh at)\right]$

11. $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$

$\left[\text{Ans.: } \dfrac{1}{b^2-a^2}(\sin at - \sin bt)\right]$

12. $\dfrac{s}{(s^2+a^2)^3}$

$\left[\text{Ans.: } \dfrac{t}{8a^3}(\sin at - at\cos at)\right]$

13. $\dfrac{s+3}{(s^2+6s+13)^2}$

$\left[\text{Ans.: } \dfrac{1}{4}e^{-3t}\, t\sin 2t\right]$

14. $\dfrac{s}{s^4+8s^2+16}$

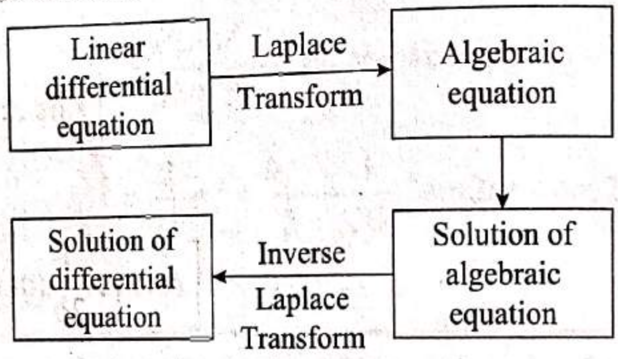
$\left[\text{Ans.: } \dfrac{1}{4}t\sin 2t\right]$

15. $\dfrac{(s+3)^2}{(s^4+6s+5)^2}$ $\left[\text{Ans.}: \dfrac{1}{4}(2t\cosh 2t + \sinh 2t)\right]$

16. $\dfrac{1}{s(s+1)(s+2)}$ $\left[\text{Ans.}: \dfrac{1}{2} + \dfrac{1}{2}e^{-2t} - e^{-t}\right]$

2.14 SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

The Laplace transform is useful in solving linear differential equations with given initial conditions by using algebraic methods. Initial conditions are included from the very beginning of the solution.



Example 1

Solve $\dfrac{dy}{dt} = 1,\ y(0) = 0.$

Solution

Taking Laplace transform of both the sides,

$$sY(s) - y(0) = \frac{1}{s}$$

$$sY(s) - 0 = \frac{1}{s} \qquad [\because y(0) = 0]$$

$$Y(s) = \frac{1}{s^2}$$

Taking inverse Laplace transform of both the sides, $y(t) = t$

Example 2

Solve $y' - 3y = 1$, $y(0) = 2$.

Solution

Taking Laplace transform of both the sides,

$$sY(s) - y(0) - 3Y(s) = \frac{1}{s}$$

$$sY(s) - 2 - 3Y(s) = \frac{1}{s} \qquad [\because y(0) = 2]$$

$$(s-3)Y(s) = \frac{1}{s} + 2 = \frac{2s+1}{s}$$

$$Y(s) = \frac{2s+1}{s(s-3)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s} + \frac{B}{s-3}$$

$$2s+1 = A(s-3) + Bs \qquad ...(1)$$

Putting $s = 0$ in Eq.(1),

$$1 = -3A$$

$$A = -\frac{1}{3}$$

Putting $s = 3$ in Eq. (1),

$$7 = 3B$$

$$B = \frac{7}{3}$$

$$Y(s) = -\frac{1}{3}\frac{1}{s} + \frac{7}{3}\frac{1}{s-3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = -\frac{1}{3} + \frac{7}{3}e^{3t}$$

Example 3

Solve $\frac{dy}{dt} + 2y = e^{-3t}$, $y(0) = 1$.

Solution

Taking Laplace transform of both the sides,

$$sY(s)-y(0)+2Y(s)=\frac{1}{s+3}$$

$$sY(s)-1+2Y(s)=\frac{1}{s+3} \quad [\because y=1]$$

$$(s+2)\,Y(s)=\frac{1}{s+3}+1=\frac{s+4}{s+3}$$

$$Y(s)=\frac{s+4}{(s+2)(s+3)}$$

By partial fraction expansion,

$$Y(s)=\frac{A}{s+2}+\frac{B}{s+3}$$

$$s+4=A(s+3)+B(s+2) \quad \ldots(1)$$

Putting $s=-2$ in Eq. (1),

$A=2$

Putting $s=-3$ in Eq. (1),

$B=-1$

$$Y(s)=\frac{2}{s+2}-\frac{1}{s+3}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=2e^{-2t}-e^{-3t}$$

Example 4

Solve $\dfrac{dy}{dt}+y=\cos 2t,\ y(0)=1.$

Solution

Taking Laplace transform of both the sides,

$$sY(s)-y(0)+Y(s)=\frac{s}{s^2+4}$$

$$sY(s)-1+Y(s)=\frac{s}{s^2+4} \quad [\because y(0)=1]$$

$$(s+1)\,Y(s)=\frac{s}{s^2+4}+1=\frac{s^2+s+4}{(s^2+4)}$$

$$Y(s)=\frac{s^2+s+4}{(s+1)(s^2+4)}$$

By partial fraction expansion,

$$Y(s)=\frac{A}{s+1}+\frac{Bs+C}{s^2+4}$$

$$s^2+s+4=A(s^2+4)+(Bs+C)(s+1) \quad \ldots (1)$$

Putting $s=-1$ in Eq. (1),

$$4=5A$$

$$A=\frac{4}{5}$$

Equating the coefficients of s^2,

$$1=A+B$$

$$B=1-\frac{4}{5}=\frac{1}{5}$$

Equating the coefficients of s^0,

$$4=4A+C$$

$$C=4-4A=4-\frac{16}{5}=\frac{4}{5}$$

$$Y(s)=\frac{4}{5}\cdot\frac{1}{s+1}+\frac{1}{5}\cdot\frac{s}{s^2+4}+\frac{4}{5}\cdot\frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\frac{4}{5}e^{-t}+\frac{1}{5}\cos 2t+\frac{2}{5}\sin 2t$$

Example 5

Solve $y''+6y'=1,\ y(0)=2,\ y'(0)=0.$ **[Winter 2012]**

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s)-sy(0)-y'(0)\right]+6\left[sY(s)-y(0)\right]=\frac{1}{s}$$

$$\left[s^2Y(s)-2s\right]+6\left[sY(s)-2\right]=\frac{1}{s}$$

$$(s^2+6s)Y(s)=2s+12+\frac{1}{s}$$

$$Y(s)=\frac{2s+12}{s^2+6s}+\frac{1}{s(s^2+6s)}$$

$$=\frac{2(s+6)}{s(s+6)}+\frac{1}{s^2(s+6)}$$

$$= \frac{2}{s} + \frac{1}{s^2(s+6)}$$

By partial fraction expansion,

$$\frac{1}{s^2(s+6)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+6}$$

$$1 = As(s+6) + B(s+6) + Cs^2 \qquad \ldots(1)$$

Putting $s = 0$ in Eq. (1),

$$1 = 6B$$

$$B = \frac{1}{6}$$

Putting $s = -6$ in Eq. (1),

$$1 = 36C$$

$$C = \frac{1}{36}$$

Putting $s = 1$ in Eq. (1),

$$1 = 7A + 7B + C$$

$$7A = 1 - \frac{7}{6} - \frac{1}{36}$$

$$= -\frac{7}{36}$$

$$A = -\frac{1}{36}$$

$$Y(s) = \frac{2}{s} - \frac{1}{36}\frac{1}{s} + \frac{1}{6}\frac{1}{s^2} + \frac{1}{36}\frac{1}{(s+6)}$$

$$= \frac{71}{36}\cdot\frac{1}{s} + \frac{1}{6}\cdot\frac{1}{s^2} + \frac{1}{36}\cdot\frac{1}{(s+6)}$$

Taking inverse Laplace transforms of both the sides,

$$y(t) = \frac{71}{36} + \frac{t}{6} + \frac{1}{36}e^{-6t}$$

Example 6

Solve $y'' + 4y' + 8y = 1$, $y(0) = 0$, $y'(0) = 1$.

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 4\left[sY(s) - y(0)\right] + 8Y(s) = \frac{1}{s}$$

$$\left[s^2Y(s) - 1\right] + 4sY(s) + 8Y(s) = \frac{1}{s} \qquad [\because y(0) = 0, y'(0) = 1]$$

$$(s^2+4s+8)Y(s)=\frac{1}{s}+1=\frac{s+1}{s}$$

$$Y(s)=\frac{s+1}{s(s^2+4s+8)}$$

By partial fraction expansion,

$$Y(s)=\frac{A}{s}+\frac{Bs+C}{s^2+4s+8}$$

$$s+1=A(s^2+4s+8)+(Bs+C)s \quad \dots (1)$$

Putting $s=0$ in Eq. (1),

$$1=8A$$

$$A=\frac{1}{8}$$

Equating the coefficients of s^2,

$$0=A+B$$

$$B=-\frac{1}{8}$$

Equating the coefficients of s,

$$1=4A+C$$

$$C=1-4A=1-\frac{1}{2}=\frac{1}{2}$$

$$Y(s)=\frac{1}{8}\cdot\frac{1}{s}-\frac{1}{8}\cdot\frac{s}{s^2+4s+8}+\frac{1}{2}\cdot\frac{1}{s^2+4s+8}$$

$$=\frac{1}{8}\cdot\frac{1}{s}-\frac{1}{8}\cdot\frac{(s+2)-2}{(s+2)^2+4}+\frac{1}{2}\cdot\frac{1}{(s+2)^2+4}$$

$$=\frac{1}{8}\cdot\frac{1}{s}-\frac{1}{8}\frac{s+2}{(s+2)^2+4}+\frac{3}{4}\frac{1}{(s+2)^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\frac{1}{8}-\frac{1}{8}e^{-2t}\cos 2t+\frac{3}{8}e^{-2t}\sin 2t$$

Example 7

Solve $y''+y=t,\ \ y(0)=1,\ \ y'(0)=0.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s)-sy(0)-y'(0)\right]+Y(s)=\frac{1}{s^2}$$

$$s^2Y(s)-s+Y(s)=\frac{1}{s^2} \qquad [\because y(0)=1, y'(0)=0]$$

$$(s^2+1)\,Y(s)=\frac{1}{s^2}+s=\frac{s^3+1}{s^2}$$

$$Y(s)=\frac{s^3+1}{s^2(s^2+1)}$$

$$=\frac{s}{s^2+1}+\frac{1}{s^2(s^2+1)}$$

$$=\frac{s}{s^2+1}+\frac{s^2+1-s^2}{s^2(s^2+1)}$$

$$=\frac{s}{s^2+1}+\frac{1}{s^2}-\frac{1}{s^2+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\cos t+t-\sin t$$

Example 8

Solve $y''-3y'+2y=4t,\ y(0)=1,\ y'(0)=-1.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s)-sy(0)-y'(0)\right]-3\left[sY(s)-y(0)\right]+2Y(s)=\frac{4}{s^2}$$

$$s^2Y(s)-s+1-3sY(s)+3+2Y(s)=\frac{4}{s^2} \qquad [\because y(0)=1, y'(0)=-1]$$

$$(s^2-3s+2)Y(s)-s+4=\frac{4}{s^2}$$

$$(s^2-3s+2)Y(s)=\frac{4}{s^2}+s-4=\frac{4+s^3-4s^2}{s^2}$$

$$Y(s)=\frac{4+s^3-4s^2}{s^2\left(s^2-3s+2\right)}$$

$$=\frac{s^3-4s^2+4}{s^2(s-1)(s-2)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-2}$$

$$s^3 - 4s^2 + 4 = As(s-1)(s-2) + B(s-1)(s-2) + C(s^2)(s-2) + D(s^2)(s-1) \qquad \cdots(1)$$

Putting $s = 0$ in Eq. (1),

$$4 = 2B$$

$$B = 2$$

Putting $s = 1$ in Eq. (1),

$$1 - 4 + 4 = -C$$

$$C = -1$$

Putting $s = 2$ in Eq. (1),

$$8 - 16 + 4 = D(4)$$

$$D = -1$$

Equating the coefficients of s^3,

$$1 = A + C + D$$

$$A = 3$$

$$Y(s) = \frac{3}{s} + \frac{2}{s^2} - \frac{1}{s-1} - \frac{1}{s-2}$$

Taking inverse Laplace transform of both the sides.

$$y(t) = 3 + 2t - e^t - e^{2t}$$

Example 9

Solve $(D^2 + 9)y = 18t,\ y(0) = 0,\ y\left(\frac{\pi}{2}\right) = 1.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2\, Y(s) - sy(0) - y'(0)\right] + 9Y(s) = \frac{18}{s^2}$$

Let $y'(0) = A$

$$s^2\, Y(s) - A + 9Y(s) = \frac{18}{s^2} \qquad [\because y(0) = 0]$$

$$(s^2 + 9)\, Y(s) = \frac{18}{s^2} + A$$

$$Y(s) = \frac{18}{s^2(s^2+9)} + \frac{A}{s^2+9}$$

$$= \frac{18}{9}\left(\frac{1}{s^2} - \frac{1}{s^2+9}\right) + \frac{A}{s^2+9}$$

$$= \frac{2}{s^2} + \frac{A-2}{s^2+9}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2t + \frac{A-2}{3}\sin 3t$$

Putting $t = \frac{\pi}{2}$ and $y\left(\frac{\pi}{2}\right) = 1$,

$$1 = 2\cdot\frac{\pi}{2} + \frac{A-2}{3}\sin\frac{3\pi}{2}$$

$$= \pi - \frac{A-2}{3}$$

$$3 = 3\pi - A + 2$$

$$A = 3\pi - 1$$

Hence,
$$y(t) = 2t + \frac{3\pi - 1 - 2}{3}\sin 3t$$

$$= 2t + (\pi - 1)\sin 3t$$

Example 10

Solve $y'' + y' = t^2 + 2t,\ y(0) = 4,\ y'(0) = -2.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + \left[sY(s) - y(0)\right] = \frac{2}{s^3} + \frac{2}{s^2}$$

$$s^2Y(s) - 4s + 2 + sY(s) - 4 = \frac{2}{s^3} + \frac{2}{s^2} \qquad [\because y(0) = 4,\ y'(0) = -2]$$

$$(s^2+s)\,Y(s) = \frac{2}{s^3} + \frac{2}{s^2} + 4s + 2 = \frac{2(1+s)}{s^3} + 4s + 2$$

$$Y(s) = \frac{2(1+s)}{s^3(s^2+s)} + \frac{4s}{s^2+s} + \frac{2}{s^2+s}$$

$$= \frac{2}{s^4} + \frac{4}{s+1} + \frac{2}{s} - \frac{2}{s+1}$$

$$= \frac{2}{s^4} + \frac{2}{s} - \frac{2}{s+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{t^3}{3} + 2 + 2e^{-t}$$

Example 11

Solve $(D^2 - 2D + 1)y = e^t$, $y = 2$ *and* $Dy = -1$ *at* $t = 0$.

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] - 2\left[sY(s) - y(0)\right] + Y(s) = \frac{1}{s-1}$$

$$\left[s^2Y(s) - 2s + 1\right] - 2\left[sY(s) - 2\right] + Y(s) = \frac{1}{s-1} \quad [\because y(0) = 2, y'(0) = -1]$$

$$(s^2 - 2s + 1)\, Y(s) = \frac{1}{s-1} + 2s - 5$$

$$(s-1)^2\, Y(s) = \frac{1 + 2s(s-1) - 5(s-1)}{s-1}$$

$$Y(s) = \frac{2s^2 - 7s + 6}{(s-1)^3}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}$$

$$2s^2 - 7s + 6 = A(s-1)^2 + B(s-1) + C \qquad \dots (1)$$

Putting $s = 1$ in Eq. (1),

$$C = 1$$

Equating the coefficients of s^2,

$$A = 2$$

Equating the coefficients of s,

$$-7 = -2A + B$$

$$B = -7 + 4 = -3$$

$$Y(s) = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2e^t - 3t\, e^t + \frac{t^2}{2} e^t$$

Example 12

Solve the initial-value problem using Laplace transform

$$y'' + 3y' + 2y = e^t,\ y(0) = 1,\ y'(0) = 0$$ **[Summer 2015]**

Solution

Taking Laplace transform of both the sides,

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] + 3\left[sY(s) - y(0)\right] + 2Y(s) = \frac{1}{s-1}$$

$$s^2 Y(s) - s + 3sY(s) - 3 + 2Y(s) = \frac{1}{s-1}$$

$$(s^2 + 3s + 2)Y(s) = (s+3) + \frac{1}{(s-1)}$$

$$Y(s) = \frac{s+3}{s^2+3s+2} + \frac{1}{(s-1)(s^2+3s+2)}$$

$$= \frac{s+3}{(s+1)(s+2)} + \frac{1}{(s-1)(s+1)(s+2)}$$

$$= \frac{s^2 + 2s - 2}{(s-1)(s+1)(s+2)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$s^2 + 2s - 2 = A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1) \quad \text{...(1)}$$

Putting $s = 1$ in Eq. (1),

$$1 = 6A$$

$$A = \frac{1}{6}$$

Putting $s = -1$ in Eq. (1),

$$-3 = -2B$$

$$B = \frac{3}{2}$$

Putting $s = -2$ in Eq. (1),

$$-2 = 3C$$

$$C = -\frac{2}{3}$$

$$Y(s) = \frac{1}{6}\cdot\frac{1}{s-1} + \frac{3}{2}\cdot\frac{1}{s+1} - \frac{2}{3}\cdot\frac{1}{s+2}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{6}e^{t} + \frac{3}{2}e^{-t} - \frac{2}{3}e^{-2t}$$

Example 13

Solve $y'' + 4y' + 3y = e^{-t}$, $y(0) = y'(0) = 1$. **[Winter 2013]**

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 4\left[sY(s) - y(0)\right] + 3Y(s) = \frac{1}{s+1}$$

$$\left[s^2Y(s) - s - 1\right] + 4\left[sY(s) - 1\right] + 3Y(s) = \frac{1}{s+1} \qquad [\because y(0) = 1, y'(0) = 1]$$

$$(s^2 + 4s + 3)Y(s) - s - 5 = \frac{1}{s+1}$$

$$(s^2 + 4s + 3)Y(s) = s + 5 + \frac{1}{s+1}$$

$$Y(s) = \frac{s+5}{s^2+4s+3} + \frac{1}{(s+1)(s^2+4s+3)}$$

$$= \frac{s+5}{(s+1)(s+3)} + \frac{1}{(s+1)^2(s+3)}$$

$$= \frac{s^2+6s+6}{(s+1)^2(s+3)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3}$$

$$s^2 + 6s + 6 = A(s+1)(s+3) + B(s+3) + C(s+1)^2 \qquad \text{...(1)}$$

Putting $s = -1$ in Eq. (1),

$$1 = 2B$$

$$B = \frac{1}{2}$$

Putting $s = -3$ in Eq. (1),

$$-3 = 4C$$

$$C = -\frac{3}{4}$$

Putting $s = 0$ in Eq. (1),

$$6 = 3A + 3B + C$$

$$A = \frac{7}{4}$$

$$Y(s) = \frac{7}{4}\frac{1}{(s+1)} + \frac{1}{2}\frac{1}{(s+1)^2} - \frac{3}{4}\frac{1}{(s+3)}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{7}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{3}{4}e^{-3t}$$

Example 14

Use Laplace transform to solve the following initial value problem

$$y'' - 3y' + 2y = 12e^{-2t}, y(0) = 2, y'(0) = 6$$ **[Summer 2017]**

Solution

Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{12}{s+2}$$

$$s^2Y(s) - 2s - 6 - 3sY(s) + 6 + 2Y(s) = \frac{12}{s+2}$$

$$(s^2 - 3s + 2)\,Y(s) = \frac{12}{s+2} + 2s$$

$$(s-1)(s-2)\,Y(s) = \frac{12 + 2s^2 + 4s}{s+2}$$

$$Y(s) = \frac{2s^2 + 4s + 12}{(s-1)(s-2)(s+2)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+2}$$

$$2s^2 + 4s + 12 = A(s-2)(s+2) + B(s-1)(s+2) + C(s-1)(s-2) \quad (1)$$

Putting $s = 1$ in Eq. (1),

$$2 + 4 + 12 = A(-1)(3)$$

$$18 = -3A$$

$$A = -6$$

Putting $s = 2$ in Eq. (1),

$$8 + 8 + 12 = B(1)(4)$$

$$28 = 4B$$

$$B = 7$$

Putting $s = -2$ in Eq. (1),

$$8 - 8 + 12 = C(-3)(-4)$$
$$12 = 12C$$
$$C = 1$$

$$Y(s) = -\frac{6}{s-1} + \frac{7}{s-2} + \frac{1}{s+2}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = -6e^{t} + 7e^{2t} + e^{-2t}$$

Example 15

Solve the equation $y'' - 3y' + 2y = 4t + e^{3t}$, *when* $y(0) = 1$ *and* $y'(0) = -1$.

[Winter 2016; Summer 2016]

Solution

Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{4}{s^2} + \frac{1}{s-3}$$

$$s^2Y(s) - s + 1 - 3sY(s) + 3 + 2Y(s) = \frac{4}{s^2} + \frac{1}{s-3}$$

$$(s^2 - 3s + 2)\,Y(s) - s + 4 = \frac{4}{s^2} + \frac{1}{s-3}$$

$$(s^2 - 3s + 2)\,Y(s) = \frac{4}{s^2} + \frac{1}{s-3} + s - 4$$

$$(s^2 - 3s + 2)\,Y(s) = \frac{4(s-3) + s^2 + s^2\,(s-3)(s-4)}{s^2\,(s-3)}$$

$$(s^2 - 3s + 2)\,Y(s) = \frac{4s - 12 + s^2 + s^2\,(s^2 - 7s + 12)}{s^2\,(s-3)}$$

$$(s-1)\,(s-2)\,Y(s) = \frac{4s - 12 + s^2 + s^4 - 7s^3 + 12s^2}{s^2\,(s-3)}$$

$$Y(s) = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2\,(s-1)\,(s-2)\,(s-3)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-2} + \frac{E}{s-3}$$

$$s^4 - 7s^3 + 13s^2 + 4s - 12$$
$$= As(s-1)(s-2)(s-3) + B(s-1)(s-2)(s-3) + Cs^2(s-2)(s-3) + Ds^2(s-1)(s-3) + E(s-1)(s-2)s^2 \quad ...(1)$$

Putting $s = 0$ in Eq. (1),

$$-12 = B(-1)(-2)(-3)$$
$$-12 = -6B$$
$$B = 2$$

Putting $s = 1$ in Eq. (2),

$$1 - 7 + 13 + 4 - 12 = C(-1)(-2)$$
$$-1 = 2C$$
$$C = -\frac{1}{2}$$

Putting $s = 2$ in Eq. (1),

$$16 - 56 + 52 + 8 - 12 = D(4)(1)(-1)$$
$$8 = -4D$$
$$D = -2$$

Putting $s = 3$ in Eq. (1),

$$81 - 189 + 117 + 12 - 12 = 9E(2)(1)$$
$$198 - 189 = 18E$$
$$9 = 18E$$
$$E = \frac{1}{2}$$

Equating the coefficient of s^4,

$$1 = A + C + D + E$$
$$A = 1 - C - D - E$$
$$= 1 + \frac{1}{2} + 2 - \frac{1}{2}$$
$$= 3$$

$$Y(s) = \frac{3}{s} + \frac{2}{s^2} - \frac{1}{2}\cdot\frac{1}{s-1} - 2\cdot\frac{1}{s-2} + \frac{1}{2}\cdot\frac{1}{s-3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 3 + 2t + \frac{1}{2}(e^{3t} - e^{t}) - 2e^{2t}$$

Example 16

Solve $y'' + 9y = \cos 2t,\ y(0) = 1,\ y\left(\frac{\pi}{2}\right) = -1.$

Solution

Taking Laplace transform of both the sides,

$$[s^2Y(s) - s\,y(0) - y'(0)] + 9Y(s) = \frac{s}{s^2+4}$$

Let $y'(0) = A$

$$s^2Y(s) - s - A + 9Y(s) = \frac{s}{s^2+4} \qquad [\because y(0) = 1]$$

$$(s^2+9)\,Y(s) = \frac{s}{s^2+4} + s + A$$

$$Y(s) = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{A}{s^2+9}$$

$$= \frac{s}{5}\left[\frac{(s^2+9)-(s^2+4)}{(s^2+4)(s^2+9)}\right] + \frac{s}{s^2+9} + \frac{A}{s^2+9}$$

$$= \frac{1}{5}\cdot\frac{s}{s^2+4} - \frac{1}{5}\cdot\frac{s}{s^2+9} + \frac{s}{s^2+9} + \frac{A}{s^2+9}$$

$$= \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{4}{5}\cdot\frac{s}{s^2+9} + \frac{A}{s^2+9}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t$$

Putting $t = \frac{\pi}{2}$ and $y\left(\frac{\pi}{2}\right) = -1$,

$$-1 = -\frac{1}{5} - \frac{A}{3}$$

$$A = \frac{12}{5}$$

Hence, $y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t$

Example 17

Solve $\frac{d^2y}{dt^2}+y=\sin 2t,\ y(0)=0,\ y'(0)=0.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s)-sy(0)-y'(0)\right]+Y(s)=\frac{2}{s^2+4}$$

$$s^2Y(s)+Y(s)=\frac{2}{s^2+4} \qquad [\because y(0)=0,\ y'(0)=0]$$

$$(s^2+1)Y(s)=\frac{2}{s^2+4}$$

$$Y(s)=\frac{2}{(s^2+4)(s^2+1)}$$

$$=\frac{2}{3}\left[\frac{(s^2+4)-(s^2+1)}{(s^2+4)(s^2+1)}\right]$$

$$=\frac{2}{3}\left[\frac{1}{s^2+1}-\frac{1}{s^2+4}\right]$$

$$=\frac{2}{3}\frac{1}{s^2+1}-\frac{1}{3}\frac{2}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\frac{2}{3}\sin t-\frac{1}{3}\sin 2t$$

Example 18

Solve $\frac{d^2y}{dt^2}+y=\sin t,\ y(0)=1,\ y'(0)=0.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s)-sy(0)-y'(0)\right]+Y(s)=\frac{1}{s^2+1}$$

$$s^2Y(s)-s-0+Y(s)=\frac{1}{s^2+1} \qquad [\because y(0)=1,\ y'(0)=0]$$

$$(s^2+1)\,Y(s)-s=\frac{1}{s^2+1}$$

$$(s^2+1)\,Y(s)=\frac{1}{s^2+1}+s$$

$$Y(s)=\frac{1}{(s^2+1)^2}+\frac{s}{s^2+1}$$

$$L^{-1}\{Y(s)\}=L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}+L^{-1}\left\{\frac{s}{s^2+1}\right\}$$

Let

$$F(s)=\frac{1}{(s^2+1)^2}$$

$$F_1(s)=F_2(s)=\frac{1}{s^2+1}$$

$$f_1(t)=f_2(t)=\sin t$$

$$L^{-1}\{F(s)\}=\int_0^t \sin u\sin(t-u)\,du$$

$$=\int_0^t \frac{1}{2}[\cos(2u-t)-\cos t]\,du$$

$$=\frac{1}{2}\left|\frac{\sin(2u-t)}{2}-(\cos t)u\right|_0^t$$

$$=\frac{1}{2}\left[\frac{\sin t}{2}-t\cos t-\frac{\sin(-t)}{2}\right]$$

$$=\frac{1}{2}[\sin t-t\cos t]$$

$$L^{-1}\{Y(s)\}=\frac{1}{2}(\sin t-t\cos t)+\cos t$$

Example 19

Solve $y''+y=\sin 2t,\quad y(0)=2,\quad y'(0)=1.$

[Winter 2014; Summer 2018]

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s)-sy(0)-y'(0)\right]+Y(s)=\frac{2}{s^2+4}$$

$$\left[s^2Y(s)-2s-1\right]+Y(s)=\frac{2}{s^2+4} \qquad [\because\ y(0)=2,\ y'(0)=1]$$

$$(s^2+1)Y(s)=2s+1+\frac{2}{s^2+4}$$

$$Y(s) = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}$$

$$= \frac{2s}{s^2+1} + \frac{1}{s^2+1} + \frac{2}{3}\left[\frac{(s^2+4)-(s^2+1)}{(s^2+1)(s^2+4)}\right]$$

$$= \frac{2s}{s^2+1} + \frac{1}{s^2+1} + \frac{2}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right]$$

$$= \frac{2s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

Example 20

Solve $y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2, y'(0) = 6.$

Solution

Taking Laplace transform of both the sides,

$$[s^2Y(s) - s\, y(0) - y'(0)] - 6\,[sY(s) - y(0)] + 9Y(s) = \frac{2}{(s-3)^3}$$

$$[s^2Y(s) - 2s - 6] - 6\,[sY(s) - 2] + 9Y(s) = \frac{2}{(s-3)^3} \quad [\because y(0) = 2, y'(0) = 6]$$

$$(s^2 - 6s + 9)\,Y(s) = \frac{2}{(s-3)^3} + 2s - 6$$

$$(s-3)^2\,Y(s) = \frac{2}{(s-3)^3} + 2(s-3)$$

$$Y(s) = \frac{2}{(s-3)^5} + \frac{2}{s-3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2e^{3t}\frac{t^4}{4!} + 2e^{3t}$$

$$= \frac{1}{12}t^4 e^{3t} + 2e^{3t}$$

Example 21

Solve the initial value problem **[Winter 2015]**

$y'' - 2y' = e^t \sin t, \quad y(0) = y'(0) = 0$, *using Laplace transform.*

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] - 2[sY(s) + y(0)] = \frac{1}{(s-1)^2+1}$$

$$s^2Y(s) - 2sY(s) = \frac{1}{s^2 - 2s + 2}$$

$$Y(s)\{s(s-2)\} = \frac{1}{s^2 - 2s + 2}$$

$$Y(s) = \frac{1}{s(s-2)(s^2 - 2s + 2)}$$

By partial fraction expansion,

$$Y(s) = \;= \frac{A}{s} + \frac{B}{(s-2)} + \frac{Cs+D}{s^2-2s+2}$$

$$1 = A(s-2)(s^2 - 2s + 2) + Bs(s^2 - 2s + 2) + (Cs + D)\,s(s-2) \quad ...(1)$$

Putting $s = 0$ in Eq. (1),

$$A = -\frac{1}{4}$$

Putting $s = 2$ in Eq. (1),

$$B = \frac{1}{4}$$

Equating the coefficients of s^3,

$$A + B + C = 0$$

$$-\frac{1}{4} + \frac{1}{4} + C = 0$$

$$C = 0$$

Equating the coefficients of s,

$$6A + 2B - 2D = 0$$

$$-\frac{3}{2} + \frac{1}{2} = 2D$$

$$-1 = 2D$$

$$D = -\frac{1}{2}$$

$$Y(s) = -\frac{1}{4}\cdot\frac{1}{s}+\frac{1}{4}\cdot\frac{1}{s-2}-\frac{1}{2}\cdot\frac{1}{s^2-2s+2}$$

Taking inverse Laplace transform both the sides,

$$y(t) = -\frac{1}{4}+\frac{1}{4}e^{2t}-\frac{1}{2}e^{t}\sin t$$

Example 22

Solve $(D^2+2D+5)y = e^{-t}\sin t,\ \ y(0)=0,\ \ y'(0)=1.$

Solution

Taking Laplace transform of both the sides,

$$[s^2Y(s)-sy(0)-y'(0)]+2[sY(s)-y(0)]+5Y(s) = \frac{1}{(s+1)^2+1}$$

$$s^2Y(s)-1+2sY(s)+5Y(s) = \frac{1}{s^2+2s+2} \qquad [\because y(0)=0,\ y'(0)=1]$$

$$(s^2+2s+5)\ Y(s) = \frac{1}{s^2+2s+2}+1$$

$$= \frac{s^2+2s+3}{s^2+2s+2}$$

$$Y(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

By partial fraction expansion,

$$Y(s) = \frac{As+B}{s^2+2s+2}+\frac{Cs+D}{(s^2+2s+5)}$$

$$s^2+2s+3 = (As+B)(s^2+2s+5)+(Cs+D)(s^2+2s+2)$$

$$= (A+C)\ s^3+(2A+B+2C+D)s^2$$

$$+(5A+2B+2C+2D)s+(5B+2D)$$

Equating the coefficients of s^3, s^2, s, and s^0,

$$A+C=0$$

$$2A+B+2C+D=1$$

$$5A+2B+2C+2D=2$$

$$5B+2D=3$$

Solving these equations,

$$A=0,\ B=\frac{1}{3},\ C=0,\ D=\frac{2}{3}$$

$$Y(s) = \frac{1}{3}\cdot\frac{1}{s^2+2s+2} + \frac{2}{3}\cdot\frac{1}{s^2+2s+5}$$

$$= \frac{1}{3}\cdot\frac{1}{(s+1)^2+1} + \frac{2}{3}\cdot\frac{1}{(s+1)^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t$$

$$= \frac{e^{-t}}{3}(\sin t + \sin 2t)$$

Example 23

Solve the following initial value problem using Laplace transform

$y''' + 2y'' - y' - 2y = 0,\ y(0) = 1,\ y'(0) = 2,\ y''(0) = 2$ **[Winter 2017]**

Solution

Taking Laplace transform of both the sides,

$$\left[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\right] + 2\left[s^2Y(s) - sy(0) - y'(0)\right] - [sY(s) - y(0)] - 2Y(s) = 0$$

$$\left[s^3Y(s) - s^2 - 2s - 2\right] + 2\left[s^2Y(s) - s - 2\right] - [sY(s) - 1] - 2Y(s) = 0$$

$$(s^3 + 2s^2 - s - 2)Y(s) = s^2 + 4s + 5$$

$$Y(s) = \frac{s^2+4s+5}{s^3+2s^2-s-2}$$

$$= \frac{s^2+4s+5}{(s-1)(s+1)(s+2)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$s^2 + 4s + 5 = A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1) \qquad \text{...(1)}$$

Putting $s = 1$ in Eq. (1),

$$10 = A(2)(3)$$

$$10 = 6A$$

$$A = \frac{5}{3}$$

Putting $s=-1$ in Eq. (1),

$$2 = B(-2)(1)$$
$$2 = -2B$$
$$B = -1$$

Putting $s=-2$ in Eq. (1),

$$1 = C(-3)(-1)$$
$$1 = 3C$$
$$C = \frac{1}{3}$$

$$Y(s) = \frac{5}{3}\cdot\frac{1}{s-1} - \frac{1}{s+1} + \frac{1}{3}\cdot\frac{1}{s+2}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{5}{3}e^{t} - e^{-t} + \frac{1}{3}e^{-2t}$$

Example 24

Solve $\dfrac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t,\ y(0) = 1.$

Solution

Taking Laplace transform of both the sides,

$$sY(s) - y(0) + 2Y(s) + \frac{1}{s}Y(s) = \frac{1}{s^2+1}$$

$$sY(s) - 1 + 2Y(s) + \frac{1}{s}Y(s) = \frac{1}{s^2+1} \qquad [\because y(0)=1]$$

$$\left(s + 2 + \frac{1}{s}\right)Y(s) = \frac{1}{s^2+1} + 1$$

$$= \frac{s^2+2}{s^2+1}$$

$$\frac{s^2+2s+1}{s}Y(s) = \frac{s^2+2}{s^2+1}$$

$$Y(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+2s+1)}$$

$$= \frac{s(s^2+2)}{(s^2+1)(s+1)^2}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1}$$

$$s(s^2+2) = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2 \quad \text{... (1)}$$

Putting $s = -1$ in Eq. (1),

$$-3 = 2B$$

$$B = -\frac{3}{2} \quad \text{... (2)}$$

Equating the coefficients of s^0,

$$0 = A + B + D \quad \text{... (3)}$$

Equating the coefficients of s^3,

$$1 = A + C \quad \text{... (4)}$$

Equating the coefficients of s^2,

$$0 = A + B + 2C + D \quad \text{... (5)}$$

Solving Eqs (2), (3), (4), and (5),

$$A = 1, C = 0, D = \frac{1}{2}$$

$$Y(s) = \frac{1}{s+1} - \frac{3}{2}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = e^{-t} - \frac{3}{2}e^{-t}t + \frac{1}{2}\sin t$$

EXERCISE 2.21

Using Laplace transform, solve the following differential equations:

1. $y' + 4y = 1, y(0) = -3$

$$\left[\textbf{Ans.: } y(t) = \frac{1}{4} - \frac{13}{4}e^{-4t}\right]$$

2. $y' + 6y = e^{4t}, y(0) = 2$

$$\left[\textbf{Ans.: } y(t) = \frac{1}{10}e^{4t} + \frac{19}{10}e^{-6t}\right]$$

3. $y' + 4y = \cos t, y(0) = 0$

$$\left[\textbf{Ans.: } y(t) = -\frac{4}{17}e^{-4t} + \frac{4}{17}\cos t + \frac{1}{17}\sin t\right]$$

4. $y' + 3y = 10\sin t, y(0) = 0$

$$\left[\textbf{Ans.: } y(t) = e^{-3t} - \cos t + 3\sin t\right]$$

5. $y' + 0.2y = 0.01t,\ y(0) = -0.25$ [Ans.: $y(t) = 0.05\,t - 0.25$]

6. $y' - 2y = 1 - t,\ y(0) = 1$ $\left[\text{Ans.: } y(t) = -\frac{1}{4} + \frac{1}{2}t + \frac{5}{4}e^{2t}\right]$

7. $y'' + 5y' + 4y = 0,\ y(0) = 1, y'(0) = -1$

$\left[\text{Ans.: } y(t) = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}\right]$

8. $y'' + 2y' - 3y = 6e^{-2t},\ y(0) = 2, y'(0) = -14$

$\left[\text{Ans.: } y(t) = -2e^{-2t} + \frac{11}{2}e^{-3t} - \frac{3}{2}e^{t}\right]$

9. $y'' - 4y' + 4y = 1,\ y(0) = 1, y'(0) = 4$

$\left[\text{Ans.: } y(t) = \frac{1}{4} + \frac{3}{4}e^{2t} + \frac{5}{2}te^{2t}\right]$

10. $y'' - 4y' + 3y = 6t - 8,\ y(0) = 0,\ y'(0) = 0$

$\left[\text{Ans.: } y(t) = 2t + e^{t} - e^{3t}\right]$

11. $y'' + 2y' + y = 3te^{-t},\ y(0) = 4,\ y'(0) = 2$

$\left[\text{Ans.: } y(t) = 4e^{-t} + 6t\,e^{-t} + \frac{t^3}{2}e^{-t}\right]$

12. $y'' + y = \sin t \cdot \sin 2t,\ y(0) = 1,\ y'(0) = 0$

$\left[\text{Ans.: } y(t) = \frac{15}{16}\cos t + \frac{t}{4}\sin t + \frac{1}{16}\cos 3t\right]$

13. $y'' + y = e^{-2t}\sin t,\ y(0) = 0, y'(0) = 0$

$\left[\text{Ans.: } y(t) = \frac{1}{8}\sin t - \frac{1}{8}\cos t + \frac{1}{8}e^{-2t}\sin t + \frac{1}{8}e^{-2t}\cos t\right]$

14. $y'' + y = t\cos 2t,\ y(0) = 0, y'(0) = 0$

$\left[\text{Ans.: } y(t) = \frac{4}{9}\sin 2t - \frac{5}{9}\sin t - \frac{1}{3}t\cos 2t\right]$

15. $y' + y - 2\int_0^t y\,dt = \frac{t^2}{2}$, $y(0) = 1$, $y'(0) = -2$

$$\left[\textbf{Ans.: } y(t) = \frac{1}{3}e^t + \frac{11}{12}e^{-2t} - \frac{1}{2}t - \frac{1}{4}\right]$$

2.15 SOLUTION OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

The Laplace transform can also be used to solve two or more simultaneous differential equations. The Laplace transform method transforms the differential equations into algebraic equations.

Example 1

Solve $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$

where $x(0) = 0$ *and* $y(0) = 2$.

Solution

Taking Laplace transform of both the equations,

$$sX(s) - x(0) + Y(s) = \frac{1}{s^2+1}$$

$$sX(s) + Y(s) = \frac{1}{s^2+1} \quad \dots (1)$$

and

$$sY(s) - y(0) + X(s) = \frac{s}{s^2+1}$$

$$sY(s) + X(s) = \frac{s}{s^2+1} + 2$$

$$sY(s) + X(s) = \frac{2s^2+s+2}{s^2+1} \quad \dots (2)$$

Multiplying Eq. (1) by s,

$$s^2 X(s) + s\,Y(s) = \frac{s}{s^2+1} \quad \dots (3)$$

Subtracting Eq. (3) from Eq. (2),

$$(s^2 - 1)\,X(s) = -2$$

$$X(s) = -\frac{2}{s^2-1} \quad \cdots(4)$$

Substituting $X(s)$ in Eq. (1),

$$Y(s) = \frac{1}{s^2+1} + 2\frac{s}{s^2-1} \quad \cdots(5)$$

Taking inverse Laplace transform of Eqs (4) and (5),

$$x(t) = -2 \sinh t$$

and

$$y(t) = \sin t + 2 \cosh t$$

Example 2

Solve $\frac{dx}{dt} - y = e^t, \ \frac{dy}{dt} + x = \sin t$

where $x(0) = 1$ *and* $y(0) = 0$.

Solution

Taking Laplace transform of both the equations,

$$s\,X(s) - x(0) - Y(s) = \frac{1}{s-1}$$

$$s\,X(s) - Y(s) = \frac{1}{s-1} + 1 = \frac{s}{s-1} \quad \ldots(1)$$

and

$$s\,Y(s) - y(0) + X(s) = \frac{1}{s^2+1}$$

$$s\,Y(s) + X(s) = \frac{1}{s^2+1} \quad \ldots(2)$$

Multiplying Eq. (1) by s,

$$s^2 X(s) - s\,Y(s) = \frac{s^2}{s-1} \quad \ldots(3)$$

Adding Eqs (2) and (3),

$$(s^2+1)\,X(s) = \frac{1}{s^2+1} + \frac{s^2}{s-1}$$

$$X(s) = \frac{1}{(s^2+1)^2} + \frac{s^2}{(s-1)(s^2+1)}$$

$$= \frac{1}{(s^2+1)^2} + \frac{1}{2}\left(\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right) \quad \ldots(4)$$

Substituting $X(s)$ in Eq. (1),

$$Y(s) = s\,X(s) - \frac{s}{s-1}$$

$$= \frac{s}{(s^2+1)^2} + \frac{s^3}{(s-1)(s^2+1)} - \frac{s}{s-1}$$

$$Y(s) = \frac{s}{(s^2+1)^2} - \frac{s}{(s-1)\,(s^2+1)}$$

$$= \frac{s}{(s^2+1)^2} - \frac{1}{2}\left(\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right) \quad \dots (5)$$

Taking the inverse Laplace transform of Eqs (4) and (5),

$$x(t) = \frac{1}{2}(\sin t - t\cos t) + \frac{1}{2}(e^t + \cos t + \sin t)$$

$$= \frac{1}{2}(e^t + \cos t + 2\sin t - t\cos t)$$

and
$$y(t) = \frac{1}{2}t\sin t - \frac{1}{2}(e^t - \cos t + \sin t)$$

$$= \frac{1}{2}(t\sin t - e^t + \cos t - \sin t)$$

Example 3

Solve $\frac{dx}{dt} + 5x - 2y = t,\ \frac{dy}{dt} + 2x + y = 0$

where $x(0) = 0$ *and* $y(0) = 0$.

Solution

Taking Laplace transform of both the equations,

$$s\,X(s) - x(0) + 5X(s) - 2Y(s) = \frac{1}{s^2}$$

$$(s+5)\,X(s) - 2Y(s) = \frac{1}{s^2} \quad \dots (1)$$

and
$$s\,Y(s) - y(0) + 2X(s) + Y(s) = 0$$

$$2X(s) + (s+1)\,Y(s) = 0 \quad \dots (2)$$

Multiplying Eq. (1) by $\frac{1}{2}(s+1)$,

$$\frac{1}{2}(s+5)(s+1)X(s) - (s+1)Y(s) = \frac{s+1}{2s^2} \quad \dots (3)$$

Adding Eqs (2) and (3),

$$X(s) = \frac{s+1}{s^2(s+3)^2} \quad \ldots(4)$$

Substituting $X(s)$ in Eq. (2),

$$Y(s) = -\frac{2}{s^2(s+3)^2} \quad \ldots(5)$$

Now,

$$X(s) = \frac{s+1}{s^2(s+3)^2}$$

By partial fraction expansion,

$$X(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{(s+3)^2}$$

$$s + 1 = A\, s(s+3)^2 + B(s+3)^2 + C(s+3)s^2 + D\, s^2 \quad \ldots(6)$$

Putting $s = 0$ in Eq. (6),

$$1 = 9B$$

$$B = \frac{1}{9}$$

Putting $s = -3$ in Eq. (6),

$$-2 = 9D$$

$$D = -\frac{2}{9}$$

Equating the coefficients of s^3,

$$A + C = 0$$

$$A = -C$$

Equating the coefficients of s^2,

$$6A + B + 3C + D = 0$$

$$-3C = \frac{1}{9}$$

$$C = -\frac{1}{27}$$

$$A = \frac{1}{27}$$

$$X(s) = \frac{1}{27}\cdot\frac{1}{s} + \frac{1}{9}\cdot\frac{1}{s^2} - \frac{1}{27}\cdot\frac{1}{s+3} - \frac{2}{9}\cdot\frac{1}{(s+3)^2}$$

Taking inverse Laplace transform of both the sides,

$$x(t) = \frac{1}{27} + \frac{1}{9}t - \frac{1}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$

Similarly,

$$Y(s) = \frac{-2}{s^2(s+3)^2}$$

$$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{(s+3)^2}$$

$$= \frac{4}{27}\cdot\frac{1}{s} - \frac{2}{9}\cdot\frac{1}{s^2} - \frac{4}{27}\cdot\frac{1}{s+3} - \frac{2}{9}\cdot\frac{1}{(s+3)^2}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{4}{27} - \frac{2}{9}t - \frac{4}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$

Example 4

Solve $\frac{dx}{dt} + \frac{dy}{dt} + x - y = e^{-t}$, $\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = e^{t}$

where $x(0) = 1$ *and* $y(0) = 0$.

Solution

Taking Laplace transform of both the equations,

$$sX(s) - x(0) + sY(s) - y(0) + X(s) - Y(s) = \frac{1}{s+1}$$

$$(s+1)X(s) + (s-1)Y(s) = \frac{1}{s+1} + 1 = \frac{s+2}{s+1} \quad \dots (1)$$

and $$sX(s) - x(0) + sY(s) - y(0) + 2X(s) + Y(s) = \frac{1}{s-1}$$

$$(s+2)X(s) + (s+1)Y(s) = \frac{1}{s-1} + 1 = \frac{s}{s-1} \quad \dots (2)$$

Multiplying Eq. (1) by $(s+1)$ and Eq. (2) by $(s-1)$,

$$(s+1)^2 X(s) + (s-1)(s+1)Y(s) = s+2 \quad \dots (3)$$

$$(s+2)(s-1)X(s) + (s-1)(s+1)Y(s) = s \quad \dots (4)$$

Subtracting Eq. (4) from Eq. (3),

$$(s+3)X(s) = 2$$

$$X(s) = \frac{2}{s+3} \quad \dots (5)$$

Substituting $X(s)$ in Eq. (1),

$$\frac{2(s+1)}{s+3} + (s-1)Y(s) = \frac{s+2}{s+1}$$

$$Y(s) = \frac{s+2}{(s+1)(s-1)} - \frac{2(s+1)}{(s+3)(s-1)}$$

$$= \frac{(s+2)(s+3)-2(s+1)^2}{(s-1)(s+1)(s+3)}$$

$$= \frac{-s^2+s+4}{(s-1)(s+1)(s+3)}$$

$$= \frac{-s^2+s+4}{(s^2-1)(s+3)}$$

By partial fraction expansion,

$$Y(s) = \frac{As+B}{s^2-1} + \frac{C}{s+3}$$

$$-s^2+s+4 = (As+B)(s+3)+C(s^2-1) \qquad \dots (6)$$

Putting $s = -3$ in Eq. (6),

$-8 = 8C$

$C = -1$

Equating the coefficient of s^2,

$-1 = A + C$

$A = 0$

Equating the coefficient of s^0,

$4 = 3B - C$

$B = 1$

$$Y(s) = \frac{1}{s^2-1} - \frac{1}{s+3} \qquad \dots (7)$$

Taking inverse Laplace transform of Eqs (5) and (7),

$$x(t) = 2e^{-3t}$$

and
$$y(t) = \sinh t - e^{-3t}$$

Example 5

Solve $\frac{d^2x}{dt^2} - \frac{dy}{dt} = te^{-t} - 2e^{-t} - 3$, $\frac{dx}{dt} - 2y - x = -2te^{-t} + e^{-t} - 6t$

where $x(0) = 0$, $x'(0) = 1$ *and* $y(0) = 0$.

Solution

Taking Laplace transform of both the equations,

$$[s^2X(s) - sx(0) - x'(0)] - [sY(s) - y(0)] = \frac{1}{(s+1)^2} - \frac{2}{s+1} - \frac{3}{s}$$

$$s^2X(s) - s\,Y(s) = 1 + \frac{1}{(s+1)^2} - \frac{2}{s+1} - \frac{3}{s}$$

$$s^2 X(s) - s\,Y(s) = \frac{s^2}{(s+1)^2} - \frac{3}{s} \quad \ldots (1)$$

and $$s\,X(s) - x(0) - 2\,Y(s) - X(s) = -\frac{2}{(s+1)^2} + \frac{1}{s+1} - \frac{6}{s^2}$$

$$(s-1)\,X(s) - 2\,Y(s) = \frac{s-1}{(s+1)^2} - \frac{6}{s^2} \quad \ldots (2)$$

Multiplying Eq. (2) by $\frac{s}{2}$,

$$\frac{s(s-1)}{2} X(s) - sY(s) = \frac{s(s-1)}{2(s+1)^2} - \frac{3}{s} \quad \ldots (3)$$

Subtracting Eq. (3) from Eq. (1),

$$\frac{(s^2+s)}{2} X(s) = \frac{s^2+s}{2(s+1)^2}$$

$$X(s) = \frac{1}{(s+1)^2} \quad \ldots (4)$$

Substituting $X(s)$ in Eq. (1),

$$\frac{s^2}{(s+1)^2} - sY(s) = \frac{s^2}{(s+1)^2} - \frac{3}{s}$$

$$Y(s) = \frac{3}{s^2} \quad \ldots (5)$$

Taking inverse Laplace transform of Eqs (4) and (5),

$$x(t) = t\,e^{-t}$$

and $$y(t) = 3t$$

Example 6

Solve $\frac{d^2x}{dt^2} - x - 3y = 0,\ \frac{d^2y}{dt^2} - 4x = -4e^t$

where $x(0) = 2,\ x'(0) = 3,\ y(0) = 1,\ y'(0) = 2.$

Solution

Taking Laplace transform of both the equations,

$$[s^2 X(s) - sx(0) - x'(0)] - X(s) - 3Y(s) = 0$$

$$s^2 X(s) - 2s - 3 - X(s) - 3Y(s) = 0$$

$$(s^2 - 1)\,X(s) - 3\,Y(s) = 2s + 3 \quad \ldots (1)$$

and

$$[s^2Y(s) - sy(0) - y'(0)] - 4X(s) = -\frac{4}{s-1}$$

$$s^2Y(s) - s - 2 - 4\ X(s) = -\frac{4}{s-1}$$

$$s^2Y(s) - 4\ X(s) = -\frac{4}{s-1} + s + 2 \quad \ldots(2)$$

Multiplying Eq. (1) by $\frac{s^2}{3}$,

$$\frac{s^2(s^2-1)}{3}X(s) - s^2Y(s) = \frac{s^2(2s+3)}{3} \quad \ldots(3)$$

Adding Eqs (2) and (3),

$$\left[\frac{s^2(s^2-1)}{3} - 4\right]X(s) = \frac{s^2(2s+3)}{3} + \left[-\frac{4}{s-1} + s + 2\right]$$

$$(s^4 - s^2 - 12)\ X(s) = s^2(2s+3) + \frac{3(s+3)(s-2)}{s-1}$$

$$X(s) = \frac{s^2(2s+3)(s-1) + 3(s+3)(s-2)}{(s-1)(s^2+3)(s^2-4)}$$

$$= \frac{2s^4 + s^3 + 3s - 18}{(s-1)(s^2+3)(s^2-4)}$$

$$= \frac{(s+2)(2s-3)(s^2+3)}{(s-1)(s^2+3)(s^2-4)}$$

$$= \frac{2s-3}{(s-1)(s-2)}$$

$$= \frac{1}{s-1} + \frac{1}{s-2} \quad \ldots(4)$$

Substituting $X(s)$ in Eq. (1),

$$(s^2-1)\frac{(2s-3)}{(s-1)(s-2)} - 3Y(s) = 2s + 3$$

$$Y(s) = \frac{1}{3}\left[\frac{(s+1)(2s-3) - (2s+3)(s-2)}{s-2}\right]$$

$$= \frac{1}{3}\left(\frac{3}{s-2}\right)$$

$$= \frac{1}{s-2} \quad \ldots(5)$$

Taking inverse Laplace transform of Eqs (5) and (6),

$$x(t) = e^t + e^{2t}$$

and $$y(t) = e^{2t}$$

EXERCISE 2.22

Solve the following simultaneous equations:

1. $\frac{dx}{dt} + \frac{dy}{dt} + x = e^{-t}$, $\frac{dx}{dt} + 2\frac{dy}{dt} + 2x + 2y = 0$

   where $x(0) = -1$, $y(0) = 1$

   [**Ans.:** $x(t) = -e^{-t}(\cos t + \sin t)$, $y(t) = e^{-t}(1 + \sin t)$]

2. $\frac{dx}{dt} = 2x - 3y$, $\frac{dy}{dt} = y - 2x$

   where $x(0) = 8$, $y(0) = 3$

   [**Ans.:** $x(t) = 5e^{-t} + 8e^{4t}$, $y(t) = 5e^{-t} - 2e^{4t}$]

3. $\frac{dx}{dt} - \frac{dy}{dt} + 2y = \cos 2t$, $\frac{dx}{dt} + \frac{dy}{dt} - 2x = \sin 2t$

   where $x(0) = 0$, $y(0) = -1$

   [**Ans.:** $x(t) = \frac{1}{2}e^t(\cos t + \sin t) - \frac{1}{2}\cos 2t$, $y(t) = -e^t(\cos t - \sin t) - \sin 2t$]

4. $2\frac{dx}{dt} + \frac{dy}{dt} - x - y = e^{-t}$, $\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = e^{-t}$

   where $x(0) = 2$, $y(0) = 1$

   [**Ans.:** $x(t) = 2\cos t + 8\sin t$, $y(t) = \cos t - 13\sin t + \sinh t$]

5. $\frac{d^2x}{dt^2} + y = -5\cos 2t$, $\frac{d^2y}{dt^2} + x = 5\cos 2t$

   where $x(0) = 1$, $x'(0) = 1$,
   $y'(0) = 1$, $y(0) = -1$

   [**Ans.:** $x(t) = \sin t + \cos 2t$, $y(t) = \sin t - \cos 2t$]

6. $2\frac{d^2x}{dt^2}+3\frac{dy}{dt}=4,\ 2\frac{d^2y}{dt^2}-3\frac{dx}{dt}=0$

where $x(0)=x'(0)=y(0)$
$=y'(0)=0$

$$\left[\text{Ans.: } x(t)=\frac{8}{9}\left(1-\cos\frac{3}{2}t\right),\ y(t)=\frac{8}{9}\left(\frac{3}{2}t-\sin\frac{3}{2}t\right)\right]$$

Points to Remember

Laplace Transform

If $f(t)$ is a function of t defined for all $t \geq 0$ then $\int_0^\infty e^{-st} f(t)\, dt$ is defined as the Laplace transform of $f(t)$, provided the integral exists and is denoted by $L\{f(t)\}$.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Sufficient Conditions for Existence of Laplace Transform

The Laplace transform of the function $f(t)$ exists when the following sufficient conditions are satisfied:

(i) $f(t)$ is piecewise continuous, i.e., $f(t)$ is continuous in every subinterval and $f(t)$ has finite limits at the end points of each subinterval.

(ii) $f(t)$ is of exponential order of α, i.e., there exists M, α such that $|f(t)| \leq Me^{\alpha t}$, for all $t \geq 0$. In other words,

$$\lim_{t\to\infty}\left\{e^{-\alpha t} f(t)\right\} = \text{finite quantity}$$

Properties of Laplace Transform

(i) Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$ then

$L\{a f_1(t) + b f_2(t)\} = a F_1(s) + b F_2(s)$

where a and b are constants.

(ii) First Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\left\{e^{-at} f(t)\right\} = F(s+a)$.

(iii) Second Shifting Theorem

If $L\{f(t)\} = F(s)$

and $g(t) = f(t-a) \quad t > a$

$\quad\quad = 0 \quad\quad t < a$

then $L\{g(t)\} = e^{-as} F(s)$

(iv) Differentiation of Laplace transform (Multiplication by t)

If $L\{f(t)\} = F(s)$ then $L\left\{ t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n} F(s)$.

(v) Integration of Laplace Transform (Division by t)

If $L\{f(t)\} = F(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds$.

(vi) Laplace Transforms of Derivatives

If $L\{f(t)\} = F(s)$ then $L\{f'(t)\} = sF(s) - f(0)$

$L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$

In general,

$L\left\{f^n(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0)$

(vii) Laplace Transforms of Integrals

If $L\{f(t)\} = F(s)$ then $L\left\{ \int_0^t f(t)dt\right\} = \frac{F(s)}{s}$.

Laplace Transform of Periodic Functions

If $f(t)$ is a piecewise continuous periodic function with period T then

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)dt$$

Convolution Theorem

If $L^{-1}\{F_1(s)\} = f_1(t)$ and $L^{-1}\{F_2(s)\} = f_2(t)$ then

$$L^{-1}\{F_1(s) \cdot F_2(s)\} = \int_0^t f_1(u) f_2(t-u)du$$

where $\int_0^t f_1(u) f_2(t-u)du = f_1(t) * f_2(t)$

$f_1(t)*f_2(t)$ is called the convolution of $f_1(t)$ and $f_2(t)$.

Table of Laplace Transforms

Sr. No.	$f(t)$	$F(s)$
1	k	$\frac{k}{s}$
2	t	$\frac{1}{s^2}$
3	t^n	$\frac{\lceil n+1}{s^{n+1}}$
4	e^{at}	$\frac{1}{s-a}$
5	$\sin at$	$\frac{a}{s^2+a^2}$
6	$\cos at$	$\frac{s}{s^2+a^2}$
7	$\sinh at$	$\frac{a}{s^2-a^2}$
8	$\cosh at$	$\frac{s}{s^2-a^2}$
9	$e^{-bt}\sin at$	$\frac{a}{(s+b)^2+a^2}$
10	$e^{-bt}\cos at$	$\frac{s+b}{(s+b)^2+a^2}$
11	$e^{-bt}\sinh at$	$\frac{a}{(s+b)^2-a^2}$
12	$e^{-bt}\cosh at$	$\frac{s+b}{(s+b)^2-a^2}$
13	$u(t)$	$\frac{1}{s}$
14	$u(t-a)$	$\frac{e^{-as}}{s}$
15	$\delta(t)$	1
16	$\delta(t-a)$	e^{-as}

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. For a periodic function $f(t)$ with fundamental period P, its Laplace transform is **[Winter 2015]**

(a) $\frac{1}{1-e^{-Ps}}\int_0^P e^{-st} f(t)\,dt$ (b) $\frac{1}{1+e^{-Ps}}\int_0^P e^{-st} f(t)\,dt$

(c) $\frac{1}{1-e^{Ps}}\int_0^P e^{-st} f(t)dt$ (d) $\frac{1}{1+e^{Ps}}\int_0^P e^{-st} f(t)dt$

2. If $L[f(t)] = \frac{s}{(s-3)^2}$, then $L\{e^{-3t} f(t)\}$ is **[Winter 2015]**

(a) $\frac{s-3}{s^2}$ (b) $\frac{s+3}{s}$ (c) $\frac{s+3}{s^2}$ (d) $\frac{s-3}{s}$

3. $L\{(2t-1)^2\} =$ **[Winter 2015]**

(a) $\frac{8}{s^3}+\frac{4}{s^2}-\frac{1}{s}$ (b) $\frac{8}{s^3}-\frac{4}{s^2}-\frac{1}{3}$

(c) $\frac{8}{s^3}+\frac{4}{s^2}+\frac{1}{s}$ (d) $\frac{8}{s^3}-\frac{4}{s^2}+\frac{1}{s}$

4. $L^{-1}\left\{\frac{1}{(s+a)^2}\right\} =$ **[Summer 2016]**

(a) e^{-at} (b) te^{-at} (c) t^2e^{-at} (d) te^{at}

5. If $f(t)$ is a periodic function with period t, then $L\{f(t)\}$ is **[Winter 2016; Summer 2016]**

(a) $\int_0^\infty e^{st} f(t)\,dt$ (b) $\int_0^\infty e^{-st} f(t)\,dt$

(c) $\int_0^\infty e^{-2st} f(t)\,dt$ (d) $\int_0^\infty e^{2st} f(t)dt$

6. Laplace transform of $t^{-\frac{1}{2}}$ is **[Winter 2016]**

(a) $\frac{\pi}{5}$ (b) $\sqrt{\frac{\pi}{s}}$ (c) $\frac{\pi}{\sqrt{s}}$ (d) $\frac{\sqrt{\pi}}{s}$

7. If $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right)$, then $L\left\{\frac{\sin at}{t}\right\}$ is [Winter 2016]

(a) $\tan^{-1}(s)$ (b) $\tan^{-1}\left(\frac{s}{a}\right)$ (c) $\tan^{-1}\left(\frac{a}{s}\right)$ (d) $\tan^{-1}\left(\frac{1}{s}\right)$

8. If $u(t)$ is a unit step function, $L\{u(t-a)\} =$

(a) $\frac{e^{as}}{s^2}$ (b) $\frac{e^{-as}}{s^2}$ (c) $\frac{e^{-as}}{s}$ (d) $\frac{e^{as}}{s}$

9. Laplace transform of the unit impulse function $s(t-a)$ is

(a) e^{as} (b) e^{-as} (c) e^{s} (d) e^{-s}

10. $L\{f'(t)\} =$

(a) $s\,F(s) - f(0)$ (b) $s\,F(s) + f(0)$
(c) $F(s) - F(0)$ (d) $F(s) + f(0)$

11. If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(s-a)\} =$

(a) $e^{-at} f(t)$ (b) $e^{t} f(t)$ (c) $e^{at} f(t)$ (d) $e^{-t} f(t)$

12. If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(as)\} =$

(a) $\frac{1}{a} f\left(\frac{t}{a}\right)$ (b) $a f\left(\frac{t}{a}\right)$ (c) $\frac{1}{a} f(t)$ (d) $\frac{1}{a} f(at)$

13. $L^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} =$

(a) $\frac{t}{2}\sin t$ (b) $t\sin t$ (c) $t^2 \sin t$ (d) $\frac{t^2}{2}\cos t$

14. Using Laplace transform, the equation $(D^2 + 9)y = \cos 2t$ can be written as $(s^2+9)\,Y(s) - s\,y(0) - y'(0) =$

(a) $\frac{s}{s^2+2}$ (b) $\frac{s}{s^2+4}$ (c) $\frac{s}{s+2}$ (d) $\frac{s}{s+4}$

15. The value of $L\{e^{3t+3}\}$ is [Summer 2017]

(a) $\frac{e^3}{s+3}$ (b) $\frac{e^3}{s-3}$ (c) $\frac{e^3}{s}$ (d) $\frac{e^3}{s^2-3}$

Answers

1. (a)	2. (c)	3. (d)	4. (b)	5. (b)	6. (b)	7. (c)	8. (c)
9. (b)	10. (a)	11. (c)	12. (a)	13. (b)	14. (b)	15. (b)	

CHAPTER 3

Fourier Integral

Chapter Outline

3.1 INTRODUCTION

The generalized Fourier integral allows certain complex-valued functions $f(x)$ to be decomposed as the sum of integral-defined functions, each of which resembles the usual Fourier integral associated to $f(x)$ and maintains several key properties thereof. It is a formula for the decomposition of a nonperiodic function into harmonic components whose frequencies range over a continuous set of values.

3.2 FOURIER INTEGRAL

Let $f(x)$ be a function which is piecewise continuous in every finite interval in $(-\infty, \infty)$ and absolutely integrable in $(-\infty, \infty)$.

We know that the Fourier series of the function $f(x)$ in any interval $(-l, l)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad ...(3.1)$$

where $a_0 = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}\,dt$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\,dt$$

Substituting the values of a_0, a_n, and b_n in Eq. (3.1),

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{l} f(t)\cos\frac{n\pi t}{l}\cos\frac{n\pi x}{l}\,dt + \sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\sin\frac{n\pi x}{l}\,dt$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(t)\left[\cos\frac{n\pi t}{l}\cos\frac{n\pi x}{l} + \sin\frac{n\pi t}{l}\sin\frac{n\pi x}{l}\right]dt$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} f(t)\cos\frac{n\pi}{l}(t-x)\,dt$$

Putting $\omega_n = \frac{n\pi}{l}$ and $\Delta\omega_n = \omega_{n+1} - \omega_n = (n+1)\frac{\pi}{l} - \frac{n\pi}{l} = \frac{\pi}{l}$,

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{\Delta\omega_n}{\pi}\sum_{n=1}^{\infty}\int_{-l}^{l} f(t)\cos\omega_n\,(t-x)\,dt$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\int_{-l}^{l} f(t)\cos\omega_n(t-x)\,dt\right]\Delta\omega_n \quad ...(3.2)$$

As $l \to \infty, \frac{1}{l} \to 0$ and $\Delta\omega_n = \frac{\pi}{l} \to 0$, the infinite series in Eq. (3.2) becomes an integral from 0 to ∞.

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\omega(t-x)\,dt\right]d\omega \quad [\because\ l \to \infty,\ \Delta\omega_n \to d\omega] \quad ...(3.3)$$

Equation (3.3) is called *the Fourier integral of $f(x)$*.

Expanding $\cos\omega(t-x)$ in Eq. (3.3),

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(t)(\cos\omega t\cos\omega x + \sin\omega t\sin\omega x)\,dt\right]d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt\cos\omega x\,d\omega + \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt\sin\omega x\,d\omega$$

$$= \int_0^{\infty} A(\omega)\cos\omega x\,d\omega + \int_0^{\infty} B(\omega)\sin\omega x\,d\omega \quad ...(3.4)$$

where $A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\, dt$

and $B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt$

3.3 FOURIER COSINE INTEGRAL

When $f(x)$ is an even function,

$$A(\omega)=\frac{2}{\pi}\int_{0}^{\infty} f(t)\cos\omega t\, dt$$

$$B(\omega)=0$$

The Fourier integral of an even function $f(x)$ is given by

$$f(x)=\int_{0}^{\infty} A(\omega)\cos\omega x\, d\omega \quad \text{...(3.5)}$$

Equation (3.5) is called the Fourier cosine integral of $f(x)$.

3.4 FOURIER SINE INTEGRAL

When $f(x)$ is an odd function,

$$A(\omega)=0$$

$$B(\omega)=\frac{2}{\pi}\int_{0}^{\infty} f(t)\sin\omega t\, dt$$

The Fourier integral of an odd function $f(x)$ is given by

$$f(x)=\int_{0}^{\infty} B(\omega)\sin\omega x\, d\omega \quad \text{...(3.6)}$$

Equation (3.6) is called the Fourier sine integral of $f(x)$.

Example 1

Using Fourier integral representation, show that

$$\int_{0}^{\infty}\frac{\cos\omega x+\omega\sin\omega x}{1+\omega^2}d\omega = 0, \quad x<0$$

$$= \frac{\pi}{2}, \quad x=0$$

$$= \pi e^{-x} \quad x>0$$

[Winter 2014; Summer 2015]

Solution

Let
$$f(x) = 0 \qquad x < 0$$
$$= \frac{1}{2} \qquad x = 0$$
$$= e^{-x} \qquad x > 0$$

The Fourier integral of $f(x)$ is given by

$$f(x) = \int_0^{\infty} A(\omega)\cos\omega x \, d\omega + \int_0^{\infty} B(\omega)\sin\omega x \, d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t \, dt$$
$$= \frac{1}{\pi}\left[\int_{-\infty}^{0} 0\cdot\cos\omega t \, d\omega + \int_0^{\infty} e^{-t}\cos\omega t \, d\omega\right]$$
$$= \frac{1}{\pi}\left|\frac{e^{-t}}{1+\omega^2}(-\cos\omega t + \omega\sin\omega t)\right|_0^{\infty}$$
$$= \frac{1}{\pi(1+\omega^2)} \qquad [\because \cos 0 = 1, \sin 0 = 0]$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t \, dt$$
$$= \frac{1}{\pi}\left[\int_{-\infty}^{0} 0\cdot\sin\omega t \, d\omega + \int_0^{\infty} e^{-t}\sin\omega t \, d\omega\right]$$
$$= \frac{1}{\pi}\left|\frac{e^{-t}}{1+\omega^2}(-\sin\omega t - \omega\cos\omega t)\right|_0^{\infty}$$
$$= -\frac{1}{\pi(1+\omega^2)}(-\omega)$$
$$= \frac{\omega}{\pi(1+\omega^2)}$$

Hence,
$$f(x) = \frac{1}{\pi}\int_0^{\infty}\frac{1}{1+\omega^2}\cos\omega x \, d\omega + \frac{1}{\pi}\int_0^{\infty}\frac{\omega}{1+\omega^2}\sin\omega x \, d\omega$$
$$= \frac{1}{\pi}\int_0^{\infty}\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} \, d\omega$$

$$\int_0^{\infty}\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} \, d\omega = \pi f(x)$$

$$= \begin{cases} 0 & x < 0 \\ \frac{\pi}{2} & x = 0 \\ \pi e^{-x} & x > 0 \end{cases}$$

Example 2

Express the function $f(x) = 2 \quad |x| < 2$

$\qquad\qquad\qquad\qquad\quad = 0 \quad |x| > 2$

as Fourier integral.

[Summer 2017, 2016]

Solution

The function $f(x)$ is an even function. The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega) \cos \omega x \, d\omega$$

$$A(\omega) = \frac{2}{\pi} \int_0^\infty f(t) \cos \omega t \, dt$$

$$= \frac{2}{\pi} \int_0^2 2 \cdot \cos \omega t \, dt$$

$$= \frac{4}{\pi} \left| \frac{\sin \omega t}{\omega} \right|_0^2$$

$$= \frac{4}{\pi} \frac{\sin 2\omega}{\omega} \qquad [\because \sin 0 = 0]$$

Hence,
$$f(x) = \frac{4}{\pi} \int_0^\infty \frac{\sin 2\omega \cos \omega x}{\omega} \, d\omega$$

$$\int_0^\infty \frac{\sin 2\omega \cos \omega x}{\omega} \, d\omega = \frac{\pi}{4} f(x)$$

$$= \begin{cases} \frac{\pi}{2} & |x| < 2 \\ 0 & |x| > 2 \end{cases} \qquad \ldots(1)$$

At $|x| = 2$, i.e., $x = \pm 2$, $f(x)$ is discontinuous.

At $x = 2$,

$$f(x) = \frac{1}{2}\left[\lim_{x \to -2^-} f(x) + \lim_{x \to -2^+} f(x) \right]$$

$$= \frac{1}{2}[2 + 0]$$

$$= 1$$

At $x = -2$,

$$f(x) = \frac{1}{2}\left[\lim_{x \to -2^-} f(x) + \lim_{x \to -2^+} f(x) \right]$$

$$= \frac{1}{2}[0+2]$$
$$= 1$$

Hence, from Eq. (1),

$$\int_0^\infty \frac{\sin 2\omega \cos \omega x}{\omega} \, d\omega = \begin{cases} \frac{\pi}{2} & |x| < 2 \\ \frac{\pi}{4} & |x| = 2 \\ 0 & |x| > 2 \end{cases}$$

Example 3

Find the Fourier integral representation of the function

$$f(x) = 1 \qquad |x| < 1$$
$$= 0 \qquad |x| > 1$$

$|x| < 2$ 0 to 2

Hence, evaluate (i) $\int_0^\infty \frac{\sin \omega \cos \omega x}{\omega} d\omega$ (ii) $\int_0^\infty \frac{\sin \omega}{\omega} d\omega$

[Winter 2016, 2014, 2013]

Solution

The function $f(x)$ is an even function. The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega) \cos \omega x \, d\omega$$

Even cos
odd sin

$$A(\omega) = \frac{2}{\pi} \int_0^\infty f(t) \cos \omega t \, dt$$
$$= \frac{2}{\pi} \int_0^1 1 \cdot \cos \omega t \, dt$$
$$= \frac{2}{\pi} \left| \frac{\sin \omega t}{\omega} \right|_0^1$$
$$A(\omega) = \frac{2}{\pi} \frac{\sin \omega}{\omega} \qquad [\because \sin 0 = 0]$$

Hence, $f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin \omega \cos \omega x}{\omega} d\omega$

(i) $$\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\,d\omega = \frac{\pi}{2}f(x)$$

$$= \begin{cases} \frac{\pi}{2} & |x|<1 \\ 0 & |x|>1 \end{cases} \qquad \text{...(1)}$$

At $|x| = 1$, i.e., $x = \pm 1$, $f(x)$ is discontinuous.

At $x = 1$,

$$f(x) = \frac{1}{2}\left[\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right]$$

$$= \frac{1}{2}(1+0)$$

$$= \frac{1}{2}$$

At $x = -1$,

$$f(x) = \frac{1}{2}\left[\lim_{x\to -1^-} f(x) + \lim_{x\to -1^+} f(x)\right]$$

$$= \frac{1}{2}(0+1)$$

$$= \frac{1}{2}$$

Hence, from Eq. (1),

$$\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\,d\omega = \begin{cases} \frac{\pi}{2} & |x|<1 \\ \frac{\pi}{4} & |x|=1 \\ 0 & |x|>1 \end{cases}$$

(ii) Putting $x = 0$ in Eq. (1),

$$\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega = \frac{\pi}{2}f(0) = \frac{\pi}{2} \qquad [\because f(0)=1]$$

Example 4

Find the Fourier integral representation of the function

$$f(x) = 1 - x^2 \qquad |x| \le 1$$

$$= 0 \qquad |x| > 1$$

Solution

$f(x)$ is an even function. The Fourier cosine integral of $f(x)$ is given by

$$f(x)=\int_0^\infty A(\omega)\cos\omega x\,d\omega$$

$$A(\omega)=\frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\,dt$$

$$=\frac{2}{\pi}\int_0^\infty (1-t^2)\cos\omega t\,dt$$

$$=\frac{2}{\pi}\left|(1-t)^2\left(\frac{\sin\omega t}{\omega}\right)-(-2t)\left(-\frac{\cos\omega t}{\omega^2}\right)+(-2)\left(-\frac{\sin\omega t}{\omega^3}\right)\right|_0^1$$

$$=\frac{2}{\pi}\left(-\frac{2\cos\omega}{\omega^2}+\frac{2\sin\omega}{\omega^3}\right)\qquad [\because \sin 0=0]$$

$$=\frac{4}{\pi}\left(\frac{\sin\omega-\omega\cos\omega}{\omega^3}\right)$$

Hence, $$f(x)=\frac{4}{\pi}\int_0^\infty \frac{\sin\omega-\omega\cos\omega}{\omega^3}\cos\omega x\,d\omega$$

Example 5

Find the Fourier cosine integral of $f(x) = e^{-kx}$, where $x>0, k>0$ **[Winter 2016]**

Solution

The Fourier cosine integral of $f(x)$ is given by

$$f(x)=\int_0^\infty A(\omega)\cos\omega x\,d\omega$$

$$A(\omega)=\frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\,dt$$

$$=\frac{2}{\pi}\int_0^\infty e^{-kt}\cos\omega t\,dt$$

$$=\frac{2}{\pi}\left|\frac{e^{-kt}}{k^2+\omega^2}(-k\cos\omega t+\omega\sin\omega t)\right|_0^\infty$$

$$=\frac{2}{\pi}\left(\frac{k}{k^2+\omega^2}\right)\qquad [\because \cos 0=1, \sin 0=0]$$

Hence, $$f(x)=\frac{2k}{\pi}\int_0^\infty \frac{1}{k^2+\omega^2}\cos\omega x\,d\omega$$

$$\int_0^{\infty} \frac{\cos \omega x}{\omega^2 + k^2} d\omega = \frac{\pi}{2k} f(x)$$

$$= \frac{\pi}{2k} e^{-kx} \quad x > 0, \quad k > 0$$

Example 6

Find the Fourier cosine integral of $f(x) = \frac{\pi}{2} e^{-x}$, $x \geq 0$. **[Winter 2015]**

Solution

The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^{\infty} A(\omega) \cos \omega x \, d\omega$$

$$A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(t) \cos \omega t \, dt$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{\pi}{2} e^{-t} \cos \omega t \, dt$$

$$= \int_0^{\infty} e^{-t} \cos \omega t \, dt$$

$$= \left| \frac{e^{-t}}{1+\omega^2} (-\cos \omega t + \omega \sin \omega t) \right|_0^{\infty}$$

$$= \frac{1}{1+\omega^2} \quad [\because \cos 0 = 1, \sin 0 = 0]$$

Hence,
$$f(x) = \int_0^{\infty} \frac{1}{1+\omega^2} \cos \omega x \, d\omega$$

$$\int_0^{\infty} \frac{\cos \omega x}{1+\omega^2} d\omega = f(x)$$

$$\int_0^{\infty} \frac{\cos \omega x}{1+\omega^2} d\omega = \frac{\pi}{2} e^{-x}$$

Example 7

Find the Fourier cosine integral of the function $f(x) = \cos x \quad |x| < \frac{\pi}{2}$

$$= 0 \quad |x| > \frac{\pi}{2}$$

Solution

The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega)\cos\omega x \, d\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t \, dt$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos t \cos\omega t \, dt$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left[\cos(1+\omega)t + \cos(1-\omega)t\right]dt$$

$$= \frac{1}{\pi}\left|\frac{\sin(1+\omega)t}{1+\omega} + \frac{\sin(1-\omega)t}{1-\omega}\right|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left[\frac{\sin(1+\omega)\frac{\pi}{2}}{1+\omega} + \frac{\sin(1-\omega)\frac{\pi}{2}}{1-\omega}\right] \quad [\because \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[\frac{\cos\left(\frac{\pi\omega}{2}\right)}{1+\omega} + \frac{\cos\left(\frac{\pi\omega}{2}\right)}{1-\omega}\right]$$

$$= \frac{1}{\pi}\frac{2\cos\left(\frac{\pi\omega}{2}\right)}{1-\omega^2}$$

Hence, $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos\left(\frac{\pi\omega}{2}\right)}{1-\omega^2}\cos\omega x \, d\omega$

Example 8

Express the function $f(x) = 1 \quad 0 \le x < \pi$

$$= 0 \quad x > \pi$$

as a Fourier sine integral and hence, evaluate

$$\int_0^\infty \frac{1-\cos \pi\omega}{\omega} \sin \omega x \, d\omega$$

[Winter 2017]

Solution

The Fourier sine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty B(\omega) \sin \omega x \, d\omega$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t) \sin \omega t \, dt$$

$$= \frac{2}{\pi}\left[\int_0^\pi 1\cdot \sin \omega t \, dt + \int_\pi^0 0\cdot \sin \omega t \, dt\right]$$

$$= \frac{2}{\pi}\left| -\frac{\cos \omega t}{\omega}\right|_0^\pi$$

$$= \frac{2}{\pi}\left(\frac{-\cos \pi\omega + 1}{\omega}\right) \qquad [\because \cos 0 = 1]$$

$$= \frac{2}{\pi}\left(\frac{1-\cos \pi\omega}{\omega}\right)$$

Hence,

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos \pi\omega}{\omega} \sin \omega x \, d\omega$$

$$\int_0^\infty \frac{1-\cos \pi\omega}{\omega} \sin \omega x \, d\omega = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2} & 0 \le x < \pi \\ 0 & x > \pi \end{cases} \qquad \text{...(1)}$$

At $x = \pi$, $f(x)$ is discontinuous.

$$f(x) = \frac{1}{2}\left[\lim_{x\to\pi^-} f(x) + \lim_{x\to\pi^+} f(x)\right]$$

$$= \frac{1}{2}(1+0)$$

$$= \frac{1}{2}$$

Hence, from Eq. (1),

$$\int_0^\infty \frac{1-\cos \pi\omega}{\omega} \sin \omega x \, d\omega = \begin{cases} \frac{\pi}{2} & 0 \le x < \pi \\ \frac{\pi}{4} & x = \pi \\ 0 & x > \pi \end{cases}$$

Example 9

Express the function $f(x) = \sin x \qquad 0 \le x \le \pi$

$= 0 \qquad x > \pi$

as a Fourier sine integral and show that

$$\int_0^\infty \frac{\sin \omega x \sin \pi\omega}{1-\omega^2} d\omega = \frac{\pi}{2}\sin x \qquad 0 \le x \le \pi$$

Solution

The Fourier sine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty B(\omega) \sin \omega x \, d\omega$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t) \sin \omega t \, dt$$

$$= \frac{2}{\pi}\left[\int_0^\pi \sin t \sin \omega t \, dt + \int_\pi^\infty 0 \cdot \sin \omega t \, dt\right]$$

$$= \frac{2}{\pi}\int_0^\pi \frac{1}{2}\left[\cos(\omega-1)t - \cos(\omega+1)t\right] dt$$

$$= \frac{1}{\pi}\left|\frac{\sin(\omega-1)t}{\omega-1} - \frac{\sin(\omega+1)t}{\omega+1}\right|_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{\sin(\omega-1)\pi}{\omega-1} - \frac{\sin(\omega+1)\pi}{\omega+1}\right] \qquad [\because \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[-\frac{\sin \pi\omega}{\omega-1} + \frac{\sin \pi\omega}{\omega+1}\right]$$

$$= \frac{1}{\pi}\left(-\frac{2\sin \pi\omega}{\omega^2-1}\right)$$

$$= \frac{2}{\pi}\left(\frac{\sin \pi\omega}{1-\omega^2}\right)$$

Hence, $f(x) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\sin \pi\omega}{1-\omega^2} \sin \omega x \, d\omega, \qquad \omega \ne 1$

$$\int_0^\infty \frac{\sin \omega x \sin \pi\omega}{1-\omega^2} d\omega = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \dfrac{\pi}{2}\sin x & 0 \le x \le \pi \\ 0 & x > \pi \end{cases}$$

Example 10

Find the Fourier sine integral of $f(x) = e^{-bx}$.

Hence, show that $\frac{\pi}{2}e^{-bx} = \int_0^{\infty} \frac{\omega \sin \omega x}{b^2+\omega^2}\, d\omega$

Solution

The Fourier sine integral of $f(x)$ is given by

$$f(x) = \int_0^{\infty} B(\omega) \sin \omega x \, d\omega$$

$$B(\omega) = \frac{2}{\pi}\int_0^{\infty} f(t) \sin \omega t \, dt$$

$$= \frac{2}{\pi}\int_0^{\infty} e^{-bt} \sin \omega t \, dt$$

$$= \frac{2}{\pi}\left| \frac{e^{-bt}}{b^2+\omega^2}(-b \sin \omega t - \omega \cos \omega t) \right|_0^{\infty}$$

$$= \frac{2}{\pi}\left(\frac{\omega}{b^2+\omega^2}\right) \quad [\because \cos 0 = 1, \sin 0 = 0]$$

Hence,

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\omega \sin \omega x}{b^2+\omega^2}\, d\omega$$

$$\int_0^{\infty} \frac{\omega \sin \omega x}{b^2+\omega^2}\, d\omega = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} e^{-bx}$$

Example 11

Show that $\int_0^{\infty} \frac{\lambda^3 \sin \lambda x}{\lambda^4+4}\, d\lambda = \frac{\pi}{2} e^{-x} \cos x$, *where* $x > 0$. **[Winter 2015]**

Solution

$$f(x) = \frac{\pi}{2} e^{-x} \cos x, \quad x > 0$$

The Fourier sine integral of $f(x)$ is given by

$$f(x) = \int_0^{\infty} B(\lambda) \sin \lambda x \, d\lambda$$

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\sin\lambda x\,dx$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}e^{-x}\cos x\sin\lambda x\,dx$$

$$= \int_0^\infty e^{-x}\cos x\sin\lambda x\,dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x}(2\cos x\sin\lambda x)dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x}\left[\sin(\lambda+1)x+\sin(\lambda-1)x\right]dx$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-x}\sin(\lambda+1)x\,dx+\int_0^\infty e^{-x}\sin(\lambda-1)x\,dx\right]$$

$$= \frac{1}{2}\left[\left|\frac{e^{-x}}{1+(\lambda+1)^2}\{-\sin(\lambda+1)x-(\lambda+1)\cos(\lambda+1)x\}\right|_0^\infty\right.$$

$$\left.+\left|\frac{e^{-x}}{1+(\lambda-1)^2}\{-\sin(\lambda-1)x-(\lambda-1)\cos(\lambda-1)x\}\right|_0^\infty\right],\ (x>0)$$

$$= \frac{1}{2}\left[\frac{(\lambda+1)}{1+(\lambda+1)^2}+\frac{(\lambda-1)}{1+(\lambda-1)^2}\right]$$

$$= \frac{1}{2}\left[\frac{\lambda+1}{\lambda^2+2\lambda+2}+\frac{\lambda-1}{\lambda^2-2\lambda+2}\right]$$

$$= \frac{1}{2}\left[\frac{(\lambda+1)(\lambda^2-2\lambda+2)+(\lambda-1)(\lambda^2+2\lambda+2)}{(\lambda^2+2\lambda+2)(\lambda^2-2\lambda+2)}\right]$$

$$= \frac{1}{2}\left[\frac{\lambda^3-2\lambda^2+2\lambda+\lambda^2-2\lambda+2+\lambda^3+2\lambda^2+2\lambda-\lambda^2-2\lambda-2}{\lambda^4+4}\right]$$

$$= \frac{1}{2}\left[\frac{2\lambda^3}{\lambda^4+4}\right]$$

$$= \frac{\lambda^3}{\lambda^4+4}$$

Hence, $$f(x) = \int_0^{\infty} \frac{\lambda^3}{\lambda^4 + 4} \sin \lambda x \, d\lambda$$

$$\int_0^{\infty} \frac{\lambda^3}{\lambda^4 + 4} \sin \lambda x \, d\lambda = \frac{\pi}{2} e^{-x} \cos x$$

EXERCISE 3.1

1. Find the Fourier integral representations of the following functions:

(i) $f(x) = x \quad |x| < 1$
$\quad = 0 \quad |x| > 1$

(ii) $f(x) = -e^{ax} \quad x < 0$
$\quad = e^{-ax} \quad x > 0$

$$\left[\text{Ans.: (i) } \int_{-\infty}^{\infty} \frac{\sin\omega - \omega\cos\omega}{i\pi\omega^2} e^{i\omega x} \, d\omega \quad \text{(ii) } \frac{2}{\pi}\int_0^{\infty} \sin\omega x \frac{\omega}{a^2+\omega^2} d\omega\right]$$

2. Find the Fourier sine integral of $f(x) = e^{-ax} - e^{-bx}$.

$$\left[\text{Ans.: } \frac{2}{\pi}\int_0^{\infty} \frac{(b^2-a^2)\omega\sin\omega x}{(a^2+\omega^2)(b^2+\omega^2)} d\omega\right]$$

3. Find the Fourier cosine integral of $f(x) = e^{-x}\cos x$.

$$\left[\text{Ans.: } \frac{2}{\pi}\int_0^{\infty} \frac{\omega^2+2}{\omega^4+4} \cos\omega x \, d\omega\right]$$

4. Express the function

$$f(x) = \frac{\pi}{2} \quad 0 < x < \pi$$
$$= 0 \quad x < \pi$$

as the Fourier sine integral and show that

$$\int_0^{\infty} \frac{1-\cos\pi\omega}{\omega} \sin\omega x \, d\omega = \frac{\pi}{2}$$

$$\left[\text{Ans.: } \int_0^{-1} \frac{1-\cos\pi\omega}{\omega} \sin\omega x \, d\omega\right]$$

Points to Remember

Fourier Integral

$$f(x)=\int_0^{\infty} A(\omega)\cos\omega x\, d\omega+\int_0^{\infty} B(\omega)\sin\omega x\, d\omega$$

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\, dt$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt$$

Fourier Cosine Integral

$$f(x)=\int_0^{\infty} A(\omega)\cos\omega x\, d\omega$$

$$A(\omega)=\frac{2}{\pi}\int_0^{\infty} f(t)\cos\omega t\, dt$$

$$B(\omega)=0$$

Fourier Sine Integral

$$f(x)=\int_0^{\infty} B(\omega)\sin\omega x\, d\omega$$

$$A(\omega)=0$$

$$B(\omega)=\frac{2}{\pi}\int_0^{\infty} f(t)\sin\omega t\, dt$$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The integral $f(x)=\frac{2}{\pi}\int_0^{\infty}\left[\int_0^{\infty} f(u)\sin\lambda u\sin\lambda x\, du\right]dx$ is called

   (a) Fourier integral
   (b) Fourier sine integral
   (c) Fourier cosine integral
   (d) none of these

2. The integral $f(x)=\frac{2}{\pi}\int_0^{\infty}\left[\int_0^{\infty} f(u)\cos\lambda u\cos\lambda x\, du\right]dx$ is called

   (a) Fourier integral
   (b) Fourier sine integral
   (c) Fourier cosine integral
   (d) none of these

3. The integral $f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(t)\cos\omega(t-x)\,dt\right]d\omega$ is called

   (a) Fourier integral
   (b) Fourier sine integral
   (c) Fourier cosine integral
   (d) none of these

4. Fourier sine integral of e^{-bx} is

   (a) $\frac{2}{\pi}\int_0^\infty \frac{\omega\sin\omega x}{b^2+\omega^2}d\omega$
   (b) $\frac{2}{\pi}\int_0^\infty \frac{\omega\cos\omega x}{b^2+\omega^2}d\omega$
   (c) $\frac{1}{\pi}\int_0^\infty \frac{\omega\sin\omega x}{b^2+\omega^2}d\omega$
   (d) $\frac{1}{\pi}\int_0^\infty \frac{\omega\cos\omega x}{b^2+\omega^2}d\omega$

5. Fourier cosine integral of $\frac{\pi}{2}e^{-x}, x \geq 0$ is

   (a) $\int_0^\infty \frac{1}{1+\omega^2}\cos\omega x\,d\omega$
   (b) $\frac{2}{1+\omega^2}\cos\omega x\,d\omega$
   (c) $\int_0^\infty \frac{1}{1+\omega^2}\sin\omega x\,d\omega$
   (d) $\frac{1}{1+\omega^2}\sin\omega x\,d\omega$

Answers

1. (b) 2. (c) 3. (a) 4. (a) 5. (a)

CHAPTER 4

First Order Ordinary Differential Equations

Chapter Outline

4.1 INTRODUCTION

Differential equations are very important in engineering mathematics. A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and of its derivatives of various orders. It provides the medium for the interaction between mathematics and various branches of science and engineering. Most common differential equations are radioactive decay, chemical reactions, Newton's law of cooling, series *RL*, *RC*, and *RLC* circuits, simple harmonic motions, etc.

4.2 DIFFERENTIAL EQUATIONS

A differential equation is an equation which involves variables (dependent and independent) and their derivatives, e.g.,

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=e^x \qquad \dots(4.1)$$

$$\left(\frac{d^2y}{dx^2}\right)^2-\left[\left(\frac{dy}{dx}\right)^2+1\right]^3=0 \qquad \dots(4.2)$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=x+y \quad ...(4.3)$$

Equations (4.1) and (4.2) involve ordinary derivatives and, hence, are called *ordinary differential equations* whereas Eq. (4.3) involves partial derivatives and, hence, is called a *partial differential equation*.

4.2.1 Order

The order of a differential equation is the order of the highest derivative present in the equation, e.g., the order of Eqs (4.1) and (4.2) is 2.

4.2.2 Degree

The degree of a differential equation is the power of the highest order derivative after clearing the radical sign and fraction, e.g., the degree of Eq. (4.1) is 1 and the degree of Eq. (4.2) is 2.

4.2.3 Solution or Primitive

The solution of a differential equation is a relation between the dependent and independent variables (excluding derivatives), which satisfies the equation.

The solution of a differential equation is not always unique. It may have more than one solution or sometimes no solution.

The general solution of a differential equation of order n contains n arbitrary constants.

The particular solution of a differential equation is obtained from the general solution by giving particular values to the arbitrary constants.

4.2.4 Formation of Differential Equations

Ordinary differential equations are formed by elimination of arbitrary constants c_1, c_2,..., c_n from a relation like $f(x, y, c_1, c_2, ..., c_n) = 0$

Consider $\quad f(x, y, c_1, c_2, ..., c_n) = 0 \quad$ ...(4.4)

Differentiating Eq. (4.4) successively w.r.t. x, n times and eliminating n arbitrary constants $c_1, c_2, ..., c_n$ from the above $(n + 1)$ equations a differential equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots\right)=0$$

is obtained. Its general solution is given by Eq. (4.4) itself.

Example 1

Form the differential equation by eliminating arbitrary constants from $\log\left(\frac{y}{x}\right) = cx$.

Solution

$$\log\left(\frac{y}{x}\right) = cx \qquad ...(1)$$

Differentiating Eq.(1) w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} - \frac{1}{x} = c$$

Eliminating c from Eq. (1),

$$\log\left(\frac{y}{x}\right) = x\left(\frac{1}{y}\frac{dy}{dx} - \frac{1}{x}\right)$$

$$= \frac{x}{y}\frac{dy}{dx} - 1$$

which is the differential equation of first order.

Example 2

Find the differential equation of the family of circles of radius r whose centre lies on the x-axis. **[Winter 2014]**

Solution

Let $(a, 0)$ be the centre and r be the radius of the family of circles. The equation of the family of circles is

$$(x-a)^2 + (y-0)^2 = r^2$$

$$(x-a)^2 + y^2 = r^2 \qquad ...(1)$$

where a is an arbitrary constant.

Differentiating Eq. (1) w.r.t. x,

$$2x - 2a + 2y\frac{dy}{dx} = 0$$

$$x - a = -y\frac{dy}{dx} \qquad ...(2)$$

Eliminating a from Eqs (1) and (2),

$$\left(-y\frac{dy}{dx}\right)^2 + y^2 = r^2$$

$$y^2\left(\frac{dy}{dx}\right)^2 + y^2 = r^2$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]y^2 = r^2$$

which is the equation of the family of circles.

Example 3

Form the differential equation by eliminating arbitrary constants from $y = Ae^{-3x} + Be^{2x}$.

Solution

$$y = Ae^{-3x} + Be^{2x} \quad \text{...(1)}$$

Differentiating Eq. (1) twice w.r.t. x,

$$\frac{dy}{dx} = -3Ae^{-3x} + 2Be^{2x} \quad \text{...(2)}$$

$$\frac{d^2y}{dx^2} = 9Ae^{-3x} + 4Be^{2x} \quad \text{...(3)}$$

Eliminating A and B from Eqs (1), (2), and (3),

$$\begin{vmatrix} e^{-3x} & e^{2x} & y \\ -3e^{-3x} & 2e^{2x} & -\frac{dy}{dx} \\ 9e^{-3x} & 4e^{2x} & -\frac{d^2y}{dx^2} \end{vmatrix} = 0$$

$$(-1)e^{-3x}e^{2x}\begin{vmatrix} 1 & 1 & y \\ -3 & 2 & \frac{dy}{dx} \\ 9 & 4 & -\frac{d^2y}{dx^2} \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & y \\ -3 & 2 & \frac{dy}{dx} \\ 9 & 4 & \frac{d^2y}{dx^2} \end{vmatrix} = 0$$

$$1\left(2\frac{d^2y}{dx^2} - 4\frac{dy}{dx}\right) - 1\left(-3\frac{d^2y}{dx^2} - 9\frac{dy}{dx}\right) + y(-12-18) = 0$$

$$5\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 30y = 0$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

which is the differential equation of order two.

4.3 ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

A differential equation which contains first-order and first-degree derivatives of y (dependent variable) and known functions of x (independent variable) and y is known as an ordinary differential equation of first order and first degree. The general form of this equation can be written as

$$F\left(x, y, \frac{dy}{dx}\right) = 0$$

or in explicit form as

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad M(x, y)dx + N(x, y)dy = 0$$

Solution of the differential equation can be obtained by classifying them as follows:

(i) Variable separable
(ii) Homogeneous differential equations
(iii) Nonhomogeneous differential equations
(iv) Exact differential equations
(v) Non-exact differential equations reducible to exact form
(vi) Linear differential equations
(vii) Nonlinear differential equations reducible to linear form

4.3.1 Variable Separable

A differential equation of the form ...(4.5)

$$M(x)dx + N(y)dy = 0$$

where M (x) is the function of x only and N (y) is the function of y only, is called a differential equation with variables separable as in Eq. (4.5), the function of x and the function of y can be separated easily.

Integrating Eq. (4.5), we get the solution as

$$\int M(x)dx + \int N(y)dy = c$$

or

$$\int g(y)dy = \int f(x)dx + c$$

where c is the arbitrary constant.

Example 1

Solve $y(1+x^2)^{\frac{1}{2}} dy + x\sqrt{1+y^2}\, dx = 0$.

Solution

$$y(1+x^2)^{\frac{1}{2}} dy = -x\sqrt{1+y^2}\, dx$$

$$\frac{y}{\sqrt{1+y^2}} dy = -\frac{x}{\sqrt{1+x^2}} dx$$

Integrating both the sides,

$$\int \frac{y}{\sqrt{1+y^2}} dy = -\int \frac{x}{\sqrt{1+x^2}} dx$$

$$\frac{1}{2}\int (1+y^2)^{-\frac{1}{2}}(2y)dy = -\frac{1}{2}\int (1+x^2)^{-\frac{1}{2}}(2x)dx$$

$$\frac{1}{2}\cdot\frac{(1+y^2)^{\frac{1}{2}}}{\frac{1}{2}} = -\frac{1}{2}\cdot\frac{(1+x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c \qquad \left[\because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$\sqrt{1+x^2} + \sqrt{1+y^2} = c$$

Example 2

Solve $9yy' + 4x = 0$. **[Summer 2016]**

Solution

$$9yy' = -4x$$

$$9y\frac{dy}{dx} = -4x$$

$$9y\, dy = -4xdy$$

Integrating both the sides,

$$9\frac{y^2}{2} = -\frac{4x^2}{2} + c$$

$$9y^2 + 4x^2 = 2c = c' \qquad \text{where } c' = 2c$$

Example 3

Solve $3e^x \tan y \, dx + (1 + e^x)\sec^2 y \, dy = 0.$ **[Winter 2017]**

Solution

$$3e^x \tan y \, dx = -(1 + e^x)\sec^2 y \, dy$$

$$\frac{3e^x}{1+e^x} dx = -\frac{\sec^2 y}{\tan y} dy$$

Integrating both the sides,

$$\int \frac{3e^x}{1+e^x} dx = -\int \frac{\sec^2 y}{\tan y} dy$$

$$3\log(1+e^x) = -\log \tan y + \log c \qquad \left[\because \int \frac{f'(x)}{f(x)} dx = \log|f(x)|\right]$$

$$\log(1+e^x)^3 = \log \frac{c}{\tan y}$$

$$(1+e^x)^3 = \frac{c}{\tan y}$$

$$(1+e^x)^3 \tan y = c$$

Example 4

Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}.$ **[Summer 2018]**

Solution

$$e^y \frac{dy}{dx} = e^x + x^2$$

$$e^y dy = (e^x + x^2) dx$$

Integrating both the sides,

$$\int e^y dy = \int (e^x + x^2) dx$$

$$e^y = e^x + \frac{x^3}{3} + c$$

4.3.2 Homogeneous Differential Equations

A differential equation of the form

$$\frac{dy}{dx}=\frac{M(x,y)}{N(x,y)} \quad \ldots(4.6)$$

is called a homogeneous equation if M(x, y) and N(x, y) are homogeneous functions of the same degree, i.e., degree of the RHS of Eq. (4.6) is zero.

Equation (4.6) can be reduced to variable-separable form by putting $y = vx$.

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Equation (4.6) reduces to

$$v+x\frac{dv}{dx}=\frac{M(x,vx)}{N(x,vx)}=g(v)$$

$$x\frac{dv}{dx}=g(v)-v$$

$$\frac{dv}{g(v)-v}=\frac{dx}{x}$$

This equation is in variable-separable form and can be solved by integrating

$$\int\frac{dv}{g(v)-v}=\int\frac{dx}{x}+c$$

After integrating and replacing v by $\frac{y}{x}$, we get the solution of Eq. (4.6).

Note: *Homogeneous functions*: A function $f(x, y, z)$ is said to be a homogeneous function of degree n, if for any positive number t,

$$f(xt,yt,zt)=t^n f(x,y,z),$$

where n is a real number.

Example 1

Solve $x(x-y)dy+y^2dx=0.$

Solution

$$\frac{dy}{dx}=\frac{-y^2}{x^2-xy}=\frac{M(x,y)}{N(x,y)} \quad \ldots(1)$$

The equation is homogeneous since M and N are of the same degree 2.

Let

$$y=vx$$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v+x\frac{dv}{dx}=\frac{-v^2x^2}{x^2(1-v)}=\frac{-v^2}{1-v}$$

$$x\frac{dv}{dx}=\frac{-v^2}{1-v}-v=\frac{-v}{1-v}$$

$$\left(\frac{v-1}{v}\right)dv=\frac{dx}{x}$$

$$\left(1-\frac{1}{v}\right)dv=\frac{dx}{x}$$

Integrating both the sides,

$$\int\left(1-\frac{1}{v}\right)dv=\int\frac{dx}{x}$$

$$v-\log v=\log x+\log c$$

$$v=\log v+\log cx=\log cxv$$

$$\frac{y}{x}=\log cy$$

$$y=x\log cy$$

Example 2

Solve $\frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x}$.

Solution

$$\frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x}=\frac{M(x,y)}{N(x,y)} \quad \text{... (1)}$$

The equation is homogeneous since M and N are of the same degree 1.

Let

$$y=vx$$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v+x\frac{dv}{dx}=\frac{vx+\sqrt{x^2+v^2x^2}}{x}=v+\sqrt{1+v^2}$$

$$x\frac{dv}{dx}=\sqrt{1+v^2}$$

$$\frac{dv}{\sqrt{1+v^2}}=\frac{dx}{x}$$

Integrating both the sides,

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\log\left(v+\sqrt{v^2+1}\right) = \log x + \log c = \log cx$$

$$v+\sqrt{v^2+1} = cx$$

$$\frac{y}{x}+\sqrt{\frac{y^2}{x^2}+1} = cx$$

$$y+\sqrt{y^2+x^2} = cx^2$$

Example 3

Solve $x^2y\,dx-(x^3+xy^2)\,dy=0.$ [Winter 2012]

Solution

$$\frac{dy}{dx} = \frac{x^2y}{x^3+xy^2} = \frac{xy}{x^2+xy^2} = \frac{M(x, y)}{N(x, y)} \quad \text{...(1)}$$

The equation is homogeneous since M and N are of the same degree 2.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x\frac{dv}{dx} = \frac{x(vx)}{x^2+x(vx)} = \frac{v}{1+v}$$

$$x\frac{dv}{dx} = \frac{v}{1+v} - v = \frac{v-v-v^2}{1+v}$$

$$\frac{1+v}{v^2}dv = -\frac{dx}{x}$$

$$\left(\frac{1}{v^2}+\frac{1}{v}\right)dv = -\frac{dx}{x}$$

Integrating both the sides,

$$-\frac{1}{v} + \log v = -\log x + \log c$$

$$\log v + \log x = \frac{1}{v} + \log c$$

$$= \frac{1}{v} \log e + \log c$$

$$= \log e^{\frac{1}{v}} + \log c$$

$$\log vx = \log c\, e^{\frac{1}{v}}$$

$$vx = ce^{\frac{1}{v}}$$

$$y = ce^{\frac{x}{y}}$$

4.3.3 Nonhomogeneous Differential Equations

A differential equation of the form

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \qquad \text{...(4.7)}$$

is called a nonhomogeneous equation where $a_1, b_1, c_1, a_2, b_2, c_2$ are all constants. These equations are classified into two parts and can be solved by the following methods:

Case I If

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = m$$

$$a_1 = a_2 m,\ b_1 = b_2 m,$$

then Eq. (4.7) reduces to

$$\frac{dy}{dx} = \frac{m(a_2 x + b_2 y) + c_1}{a_2 x + b_2 y + c_2} \qquad \text{...(4.8)}$$

Putting $a_2 x + b_2 y = t,\ a_2 + b_2 \frac{dy}{dx} = \frac{dt}{dx}$, Eq. (4.8) reduces to variable-separable form and can be solved using the method of variable-separable equation.

Case II If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then substituting

$x = X + h,\ y = Y + k$ in Eq. (4.7),

$$\frac{dy}{dx} = \frac{dY}{dX}$$

$$= \frac{a_1(X+h) + b_1(Y+k) + c_1}{a_2(X+h) + b_2(Y+k) + c_2}$$

$$= \frac{(a_1 X + b_1 Y) + (a_1 h + b_1 k + c_1)}{(a_2 X + b_2 Y) + (a_2 h + b_2 k + c_2)} \qquad \text{...(4.9)}$$

Choosing h, k such that

$$a_1h + b_1k + c_1 = 0,\ a_2h + b_2k + c_2 = 0,$$

then Eq. (4.9) reduces to

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

which is a homogeneous equation and can be solved using the method of homogeneous equation. Finally, substituting $X = x - h$, $Y = y - k$, we get the solution of Eq. (4.7).

Problems Based on Case I: $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

Example 1

Solve $(x+y-1)\,dx + (2x+2y-3)\,dy = 0.$

Solution

$$\frac{dy}{dx} = -\frac{x+y-1}{2x+2y-3} = \frac{-x-y+1}{2x+2y-3} \quad \ldots(1)$$

The equation is nonhomogeneous and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = -\frac{1}{2}$

Let $x + y = t$

$$1 + \frac{dy}{dx} = \frac{dt}{dx}, \quad \frac{dy}{dx} = \frac{dt}{dx} - 1$$

Substituting in Eq. (1),

$$\frac{dt}{dx} - 1 = \frac{-t+1}{2t-3}$$

$$\frac{dt}{dx} = \frac{-t+1}{2t-3} + 1$$

$$= \frac{-t+1+2t-3}{2t-3}$$

$$= \frac{t-2}{2t-3}$$

$$\left(\frac{2t-3}{t-2}\right)dt = dx$$

$$\left(2 + \frac{1}{t-2}\right)dt = dx$$

Integrating both the sides,

$$\int\left(2+\frac{1}{t-2}\right)dt = \int dx$$

$$2t+\log(t-2) = x+c$$

$$2(x+y)+\log(x+y-2) = x+c$$

$$x+2y+\log(x+y-2) = c$$

Example 2

Solve $(x+y)\,dx+(3x+3y-4)\,dy = 0,\ y(1) = 0.$

Solution

$$\frac{dy}{dx} = \frac{-x-y}{3x+3y-4} \quad \ldots(1)$$

The equation is nonhomogeneous and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{-1}{3}$

Let $x+y=t$

$$1+\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx}-1$$

Substituting in Eq. (1),

$$\frac{dt}{dx}-1 = \frac{-t}{3t-4}$$

$$\frac{dt}{dx} = \frac{-t}{3t-4}+1 = \frac{-t+3t-4}{3t-4} = \frac{2t-4}{3t-4}$$

$$\left(\frac{3t-4}{2t-4}\right)dt = dx$$

$$\frac{1}{2}\left(3+\frac{2}{t-2}\right)dt = dx$$

Integrating both the sides,

$$\frac{1}{2}\int\left(3+\frac{2}{t-2}\right)dt = \int dx$$

$$\frac{1}{2}[3t+2\log|(t-2)|] = x+c$$

$$3(x+y)+2\log|(x+y-2)| = 2x+2c$$

$$x+3y+2\log|(x+y-2)| = k, \text{ where } 2c = k$$

Given $y(1) = 0$
Putting $x = 1, y = 0$ in the above equation,

$$1 + 2\log|-1| = k$$
$$1 + 2\log 1 = k$$
$$k = 1$$

Hence, the solution is

$$x + 3y + 2\log|x + y - 2| = 1$$

Problems Based on Case II: $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Example 1

Solve $(x+2y)\,dx + (y-1)\,dy = 0.$

Solution

$$\frac{dy}{dx} = \frac{-x-2y}{y-1}$$

The equation is nonhomogeneous and $\frac{-1}{0} \neq \frac{-2}{1}$

Let $x = X + h, \quad y = Y + k$

$dx = dX, \quad dy = dY$

$$\frac{dy}{dx} = \frac{dY}{dX}$$

Substituting in Eq. (1),

$$\frac{dY}{dX} = \frac{-(X+h) - 2(Y+k)}{(Y+k)-1}$$
$$= \frac{(-X-2Y) + (-h-2k)}{Y + (k-1)}$$

Choosing h, k such that

$$-h - 2k = 0, \; k - 1 = 0$$

Solving these equations,

$$k = 1, h = -2$$

Substituting Eq. (3) in Eq. (2),

$$\frac{dY}{dX} = \frac{-X-2Y}{Y}$$

which is a homogeneous equation.
Let $\quad Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

Substituting in Eq. (4),

$$v + X\frac{dv}{dX} = \frac{-X-2vX}{vX}$$

$$= \frac{-1-2v}{v}$$

$$X\frac{dv}{dX} = \frac{-1-2v}{v} - v$$

$$= \frac{-1-2v-v^2}{v}$$

$$= \frac{-(v+1)^2}{v}$$

$$\frac{v}{(v+1)^2}dv = -\frac{dX}{X}$$

$$\left[\frac{1}{v+1} - \frac{1}{(v+1)^2}\right]dv = -\frac{dX}{X}$$

Integrating both the sides,

$$\int\frac{1}{v+1}dv - \int\frac{1}{(v+1)^2}dv = -\int\frac{dX}{X}$$

$$\log(v+1) + \frac{1}{v+1} = -\log X + c$$

$$\log\left(\frac{Y}{X}+1\right) + \frac{1}{\frac{Y}{X}+1} = -\log X + c$$

$$\log\left(\frac{Y+X}{X}\right) + \frac{X}{Y+X} = -\log X + c$$

$$\log(Y+X) - \log X + \frac{X}{Y+X} = -\log X + c$$

$$\log(Y+X) + \frac{X}{Y+X} = c$$

Now,

$$X = x - h = x + 2$$
$$Y = y - k = y - 1$$

Hence, the general solution is

$$\log(x+y+1) + \left(\frac{x+2}{x+y+1}\right) = c$$

Example 2

Solve $\frac{dy}{dx} = \frac{2x-5y+3}{2x+4y-6}$.

Solution

The equation is nonhomogeneous and $\frac{2}{2} \neq \frac{-5}{4}$

Let $x = X + h, \quad y = Y + k$

$$dx = dX, \quad dy = dY$$

$$\frac{dy}{dx} = \frac{dY}{dX}$$

Substituting in the given equation,

$$\frac{dY}{dX} = \frac{2(X+h)-5(Y+k)+3}{2(X+h)+4(Y+k)-6}$$

$$= \frac{(2X-5Y)+(2h-5k+3)}{(2X+4Y)+(2h+4k-6)} \quad ...(1)$$

Choosing h, k such that

$$2h - 5k + 3 = 0, \qquad 2h + 4k - 6 = 0 \quad ...(2)$$

Solving the equations,

$$h = k = 1$$

Substituting Eq. (2) in Eq. (1),

$$\frac{dY}{dX} = \frac{2X-5Y}{2X+4Y} \quad ...(3)$$

which is a homogeneous equation.

Let $Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

Substituting in Eq. (3),

$$v + X\frac{dv}{dX} = \frac{2X-5vX}{2X+4vX} = \frac{2-5v}{2+4v}$$

$$X\frac{dv}{dX} = \frac{2-5v}{2+4v} - v$$

$$= \frac{2-5v-2v-4v^2}{2+4v}$$

$$= \frac{-4v^2-7v+2}{2+4v}$$

$$\frac{2+4v}{4v^2+7v-2}\,dv=-\frac{dX}{X}$$

$$\frac{2+4v}{(4v-1)(v+2)}\,dv=-\frac{dX}{X} \qquad \ldots(4)$$

$$\frac{2+4v}{(4v-1)(v+2)}=\frac{A}{4v-1}+\frac{B}{v+2}$$

Now,

$$2+4v=A(v+2)+B(4v-1)$$

$$=(A+4B)v+(2A-B)$$

Comparing coefficients on both the sides,

$$A+4B=4, \qquad 2A-B=2$$

$$A=\frac{4}{3}, \qquad B=\frac{2}{3}$$

$$\frac{2+4v}{(4v-1)(v+2)}=\frac{4}{3(4v-1)}+\frac{2}{3(v+2)}$$

Substituting in Eq. (4),

$$\left[\frac{4}{3(4v-1)}+\frac{2}{3(v+2)}\right]dv=-\frac{dX}{X}$$

Integrating both the sides,

$$\int\left[\frac{4}{3(4v-1)}+\frac{2}{3(v+2)}\right]dv=-\int\frac{dX}{X}$$

$$\frac{4}{3}\frac{\log(4v-1)}{4}+\frac{2}{3}\log(v+2)=-\log X+\log c$$

$$\frac{1}{3}\log(4v-1)(v+2)^2=\log\frac{c}{X}$$

$$\log(4v-1)^{\frac{1}{3}}(v+2)^{\frac{2}{3}}=\log\frac{c}{X}$$

$$(4v-1)^{\frac{1}{3}}(v+2)^{\frac{2}{3}}=\frac{c}{X}$$

$$\left(\frac{4Y}{X}-1\right)^{\frac{1}{3}}\left(\frac{Y}{X}+2\right)^{\frac{2}{3}}=\frac{c}{X}$$

$$(4Y-X)^{\frac{1}{3}}(Y+2X)^{\frac{2}{3}}=c$$

$$(4Y-X)(Y+2X)^2=c^3=k$$

$$X = x - h = x - 1$$

Now.

$$Y = y - k = y - 1$$

Hence, the general solution is

$$(4y - x - 3)(y + 2x - 3)^2 = k$$

4.3.4 Exact Differential Equations

Any first-order differential equation which is obtained by differentiation of its general solution without any elimination or reduction of terms is known as exact differential equation.

If $f(x, y) = c$ is the general solution then

$$df = 0$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$M(x, y)dx + N(x, y)dy = 0 \qquad \text{...(4.10)}$$

represents an exact differential equation,

where $M(x, y) = \frac{\partial f}{\partial x}$, $N(x, y) = \frac{\partial f}{\partial y}$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}, \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

But

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Therefore,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus, the necessary condition for a differential equation to be an exact differential equation is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The solution of Eq. (4.10) can be written as

$$\int_{y \text{ constant}} M(x, y)dx + \int (\text{terms of N not containing } x)dy = c$$

Sometimes, integration of M w.r.t. x is tedious whereas N can be integrated easily w.r.t. y. In this case, the solution can be written as

$$\int (\text{terms of M not containing } y)dx + \int_{x \text{ constant}} N(x, y)dy = c$$

Example 1

Check whether the given differential equation is exact or not

$$(x^4 - 2xy^2 + y^4)\, dx - (2x^2y - 4xy^3 + \sin y)\, dy = 0$$

Hence, find the general solution.

[Winter 2017]

Solution

$$M = x^4 - 2xy^2 + y^4, \quad N = -2x^2y + 4xy^3 - \sin y$$

$$\frac{\partial M}{\partial y} = -4xy + 4y^3, \quad \frac{\partial N}{\partial x} = -4xy + 4y^3$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{ constant}} M\,dx + \int (\text{terms of N not containing } x)\, dy = c$$

$$\int (x^4 - 2xy^2 + y^4)\, dx + \int (-\sin y)\, dy = c$$

$$\frac{x^5}{5} - 2\frac{x^2}{2}y^2 + xy^4 + \cos y = c$$

$$\frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c$$

Example 2

Solve $(y^2 - x^2)dx + 2xydy = 0$.

Solution

$$M = y^2 - x^2, \quad N = 2xy$$

$$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 2y$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{ constant}} M\,dx + \int (\text{terms of N not containing } x)\, dy = c$$

$$\int (y^2 - x^2)\,dx + \int 0\,dy = c$$

$$xy^2 - \frac{x^3}{3} = c$$

Example 3

Solve $(x^3 + 3xy^2)\,dx + (3x^2y + y^3)\,dy = 0.$ [Winter 2014]

Solution

$$M = x^3 + 3xy^2, \qquad N = 3x^2y + y^3$$

$$\frac{\partial M}{\partial y} = 3x(2y), \qquad \frac{\partial N}{\partial x} = 3y(2x)$$

$$= 6xy, \qquad = 6xy$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{ constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int (x^3 + 3xy^2)\,dx + \int y^3\,dy = c$$

$$\frac{x^4}{4} + 3y^2\frac{x^2}{2} + \frac{y^4}{4} = c$$

Example 4

Solve $(2xy\cos x^2 - 2xy + 1)dx + (\sin x^2 - x^2)dy = 0.$

Solution

$$M = 2xy\cos x^2 - 2xy + 1, \qquad N = \sin x^2 - x^2$$

$$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x, \qquad \frac{\partial N}{\partial x} = (\cos x^2)(2x) - 2x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int(2xy\cos x^2 - 2xy + 1)\,dx + \int 0\,dy = c$$

$$y\sin x^2 - x^2 y + x = c \quad \left[\because \int\{\cos f(x)\}f'(x)\,dx = \sin f(x)\right]$$

Example 5

Solve $ye^x dx + (2y + e^x)\,dy = 0$. **[Summer 2015]**

Solution

$M = ye^x,$ $\quad N = 2y + e^x$

$\frac{\partial M}{\partial y} = e^x,$ $\quad \frac{\partial N}{\partial x} = e^x$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{ constant}} M\,dx + \int(\text{terms of N not containg } x)\,dy = c$$

$$\int ye^x\,dx + \int 2y\,dy = c$$

$$ye^x + 2\frac{y^2}{2} = c$$

$$ye^x + y^2 = c$$

Example 6

Solve $\frac{dy}{dx} = \frac{x^2 - x - y^2}{2xy}$. **[Winter 2015]**

Solution

$$(x^2 - x - y^2)dx = 2xy\,dy$$

$$(x^2 - x - y^2)dx - 2xy\,dy = 0$$

$M = x^2 - x - y^2,$ $\quad N = -2xy$

$\frac{\partial M}{\partial y} = -2y,$ $\quad \frac{\partial N}{\partial x} = -2y$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{ constant}} M\,dx + \int(\text{terms of N not containing } x)dy = c$$

$$\int (x^2 - x - y^2)\,dx + \int 0\,dy = c$$

$$\frac{x^3}{3} - \frac{x^2}{2}y - xy^2 = c$$

Example 7

Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$ **[Winter 2016; Summer 2013]**

Solution

$$(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$$

$$M = y\cos x + \sin y + y, \qquad N = \sin x + x\cos y + x$$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1, \qquad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int (y\cos x + \sin y + y)dx + \int 0\,dy = c$$

$$y\sin x + x(\sin y + y) = c$$

Example 8

Solve $[(x+1)e^x - e^y]\,dx - xe^y\,dy = 0,\ y(1) = 0.$ **[Summer 2014]**

Solution

$$M = (x+1)e^x - e^y, \quad N = -xe^y$$

$$\frac{\partial M}{\partial y} = -e^y, \qquad \frac{\partial N}{\partial x} = -e^y$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int\left[(x+1)e^x - e^y\right]dx + \int 0\,dy = c$$

$$\int (x+1)e^x dx - e^y \int dx = c$$

$$(x+1)e^x - e^x - xe^y = c$$

$$xe^x - xe^y = c \quad ...(1)$$

Given $y(1) = 0$

Substituting $x = 1, y = 0$ in Eq. (1),

$$e - 1 = c$$

Hence, the solution is

$$xe^x - xe^y = e - 1$$

Example 9

Solve $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0,\ y(0) = 4.$

Solution

$$M = 1+e^{\frac{x}{y}}, \qquad N = e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right), \qquad \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{1}{y}\right)\left(1-\frac{x}{y}\right) + e^{\frac{x}{y}}\left(-\frac{1}{y}\right)$$

$$= \frac{-x}{y^2}e^{\frac{x}{y}}, \qquad = -\frac{x}{y^2}e^{\frac{x}{y}}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int\left(1+e^{\frac{x}{y}}\right)dx + \int 0\,dy = c$$

$$x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c$$

$$x + ye^{\frac{x}{y}} = c \quad ...(1)$$

Given $y(0) = 4$

Substituting in Eq. (1),

$$0 + 4e^0 = c$$
$$4 = c$$

Hence, the solution is

$$x + ye^{\frac{x}{y}} = 4$$

Example 10

Solve $\left[\log(x^2 + y^2) + \dfrac{2x^2}{x^2 + y^2}\right]dx + \dfrac{2xy}{x^2 + y^2}dy = 0.$

Solution

$$M = \left[\log(x^2 + y^2) + \frac{2x^2}{x^2 + y^2}\right], \qquad N = \frac{2xy}{x^2 + y^2}$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2 + y^2}\cdot 2y - \frac{2x^2}{(x^2 + y^2)^2}\cdot 2y, \qquad \frac{\partial N}{\partial x} = \frac{2y}{x^2 + y^2} - \frac{2xy}{(x^2 + y^2)^2}\cdot 2x$$

$$= \frac{2y}{x^2 + y^2} - \frac{4x^2 y}{(x^2 + y^2)^2} \qquad = \frac{2y}{x^2 + y^2} - \frac{4x^2 y}{(x^2 + y^2)^2}$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int (\text{terms of M not containing } y)dx + \int_{x\text{ constant}} N\,dy = c$$

$$\int 0\,dx + \int \frac{2xy}{x^2 + y^2}dy = c$$

$$x\log(x^2 + y^2) = c$$

Example 11

For what values of a and b is the differential equation $(y + x^3)dx + (ax + by^3)dy = 0$ *exact? Also, find the solution of the equation.*

Solution

$$M = y + x^3, \qquad N = ax + by^3$$

$$\frac{\partial M}{\partial y} = 1, \qquad \frac{\partial N}{\partial x} = a$$

The equation will be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$1 = a$$

Hence, the equation is exact for $a = 1$ and for all values of b.
Substituting $a = 1$ in the equation, $(y + x^3)\,dx + (x + by^3)\,dy = 0$, which is exact.
Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int (y + x^3)\,dx + \int by^3\,dy = c$$

$$xy + \frac{x^4}{4} + \frac{by^4}{4} = c$$

Example 12

Solve $(\cos x + y\sin x)\,dx = (\cos x)dy,\ y(\pi) = 0.$

Solution

$$(\cos x + y\sin x)\,dx - (\cos x)\,dy = 0$$

$$M = \cos x + y\sin x, \qquad N = -\cos x$$

$$\frac{\partial M}{\partial y} = \sin x, \qquad \frac{\partial N}{\partial x} = \sin x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int (\cos x + y\sin x)\,dx + \int 0\,dy = c$$

$$\sin x - y\cos x = c \qquad \ldots(1)$$

Given $y(\pi) = 0$

Substituting $x = \pi$, $y = 0$ in Eq. (1),

$$\sin \pi - 0 = c$$

$$0 = c$$

Hence, the solution is

$$\sin x - y\cos x = 0$$

$$y = \tan x$$

Example 13

Solve $(ye^{xy} + 4y^3)\,dx + (xe^{xy} + 12xy^2 - 2y)\,dy = 0,\; y(0) = 2.$

Solution

$$M = ye^{xy} + 4y^3, \qquad N = xe^{xy} + 12xy^2 - 2y$$

$$\frac{\partial M}{\partial y} = e^{xy} + ye^{xy}\cdot x + 12y^2, \qquad \frac{\partial N}{\partial x} = e^{xy} + xe^{xy}\cdot y + 12y^2$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int (ye^{xy} + 4y^3)\,dx + \int -2y\,dy = c$$

$$y\frac{e^{xy}}{y} + 4y^3x - y^2 = c$$

$$e^{xy} + 4xy^3 - y^2 = c \qquad \text{...(1)}$$

Given $y(0) = 2$

Substituting $x = 0, y = 2$ in Eq. (1),

$$e^0 + 0 - 4 = c, \qquad -3 = c$$

Hence, the solution is

$$e^{xy} + 4xy^3 - y^2 = -3$$

EXERCISE 4.1

Solve the following differential equations:

1. $(2x^3 + 3y)\,dx + (3x + y - 1)\,dy = 0$

$$\left[\textbf{Ans.: } x^4 + 6xy + y^2 - 2y = c\right]$$

2. $(1 + e^x)\,dx + y\,dy = 0$

$$\left[\textbf{Ans.: } x + e^x + \frac{y^2}{2} = c\right]$$

3. $\sinh x \cos y\, dx - \cosh x \sin y\, dy = 0$

$$\left[\text{Ans.: } \cosh x \cos y = c\right]$$

4. $xe^{x^2+y^2} dx + y(1+e^{x^2+y^2})dy = 0,\ y(0) = 0$

$$\left[\text{Ans.: } y^2 + e^{x^2+y^2} = 1\right]$$

5. $\left(4x^3y^3 + \frac{1}{x}\right)dx + \left(3x^4y^2 - \frac{1}{y}\right)dy = 0,\ y(1) = 1$

$$\left[\text{Ans.: } x^4y^3 + \log\left(\frac{x}{y}\right) = 1\right]$$

6. $(4x^3y^3 dx + 3x^4y^2 dy) - (2xy\, dx + x^2 dy) = 0$

$$\left[\text{Ans.: } x^4y^3 - x^2y = c\right]$$

7. $2x(ye^{x^2} - 1)dx + e^{x^2} dy = 0$

$$\left[\text{Ans.: } ye^{x^2} - x^2 = c\right]$$

8. $(1 + x^2\sqrt{y})y\, dx + (x^2\sqrt{y} + 2)x\, dy = 0$

$$\left[\text{Ans.: } 2xy + \frac{2}{3}x^3y^{\frac{3}{2}} = c\right]$$

9. $(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$

$$\left[\text{Ans.: } \sin x(e^y + 1) = c\right]$$

10. $(x^2 + 1)\frac{dy}{dx} = x^3 - 2xy + x$

$$\left[\text{Ans.: } x^4 - 4x^2y + 2x^2 - 4y = c\right]$$

11. $\frac{dy}{dx} = \frac{x^2 - 2xy}{x^2 - \sin y}$

$$\left[\text{Ans.: } x^3 - 3(x^2y + \cos y) = c\right]$$

12. $\frac{dy}{dx} = \frac{y+1}{(y+2)e^y - x}$

$$\left[\text{Ans.: } (y+1)(x - e^y) = c\right]$$

13. $(x - y\cos x)dx - \sin x\, dy = 0,\ y\left(\frac{\pi}{2}\right) = 1$

$\left[\textbf{Ans.: } x^2 - 2y\sin x = \frac{\pi^2}{4} - 2\right]$

14. $(2xy + e^y)dx + (x^2 + xe^y)dy = 0,\ y(1) = 1$

$\left[\textbf{Ans.: } x^2y + xe^y = e+1\right]$

4.3.5 Non-Exact Differential Equations Reducible to Exact Form

Sometimes, a differential equation is not exact but can be made exact by multiplying with a suitable function. This function is known as Integrating factor (IF). There may exists more than one integrating factor to a differential equation.
Here, we will discuss different methods to find an IF to a non-exact differential equation,

$$\mathrm{M}\,dx + \mathrm{N}\,dy = 0$$

Case I

If $\dfrac{\dfrac{\partial \mathrm{M}}{\partial y} - \dfrac{\partial \mathrm{N}}{\partial x}}{N} = f(x)$, (function of x alone) then $\mathrm{IF} = e^{\int f(x)dx}$

After multiplication with the IF, the equation becomes exact and can be solved using the method of exact differential equations.

Example 1

Solve $(x^2 + y^2 + 3)\,dx - 2xy\,dy = 0.$ **[Summer 2017]**

Solution

$\mathrm{M} = x^2 + y^2 + 3, \qquad \mathrm{N} = -2xy$

$$\frac{\partial \mathrm{M}}{\partial y} = 2y, \qquad \frac{\partial \mathrm{N}}{\partial x} = -2y$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} \neq \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is not exact.

$$\frac{\dfrac{\partial \mathrm{M}}{\partial y} - \dfrac{\partial \mathrm{N}}{\partial x}}{\mathrm{N}} = \frac{2y - (-2y)}{-2xy} = -\frac{2}{x}$$

$$\text{IF} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying the DE by the IF,

$$\frac{1}{x^2}(x^2 + y^2 + 3)dx - \frac{1}{x^2}2xy\,dy = 0$$

$$\left(1 + \frac{y^2+3}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

$$M_1 = 1 + \frac{y^2+3}{x^2}, \qquad N_1 = -\frac{2y}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{2y}{x^2}, \qquad \frac{\partial N_1}{\partial x} = \frac{2y}{x^2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{ constant}} M_1 dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$$

$$\int\left(1 + \frac{y^2+3}{x^2}\right)dx + \int 0\,dy = c$$

$$x - \frac{y^2+3}{x} = c$$

$$x^2 - y^2 - 3 = cx$$

Example 2

Solve $\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2y\,dy = 0.$

Solution

$$M = xy^2 - e^{\frac{1}{x^3}}, \qquad N = -x^2y$$

$$\frac{\partial M}{\partial y} = 2xy, \qquad \frac{\partial N}{\partial x} = -2xy$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2xy-(-2xy)}{-x^2y}=-\frac{4}{x}$$

$$\text{IF}=e^{\int -\frac{4}{x}dx}=e^{-4\log x}=e^{\log x^{-4}}=x^{-4}=\frac{1}{x^4}$$

Multiplying the DE by the IF,

$$\frac{1}{x^4}(xy^2-e^{\frac{1}{x^3}})\mathrm{d}x-\frac{1}{x^4}(x^2y)\mathrm{d}y=0$$

$$\left(\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4}\right)\mathrm{d}x-\frac{y}{x^2}\mathrm{d}y=0$$

$$M_1=\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4},\qquad N_1=-\frac{y}{x^2}$$

$$\frac{\partial M_1}{\partial y}=\frac{2y}{x^3},\qquad \frac{\partial N_1}{\partial x}=\frac{2y}{x^3}$$

Since $\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\,\text{constant}} M_1\mathrm{d}x+\int(\text{terms of } N_1 \text{ not containing } x)\mathrm{d}y=c$$

$$\int\left(\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4}\right)\mathrm{d}x+\int 0\,\mathrm{d}y=c$$

$$-\frac{y^2}{2x^2}+\frac{1}{3}\int e^{\frac{1}{x^3}}\left(-\frac{3}{x^4}\right)\mathrm{d}x=c$$

$$-\frac{y^2}{2x^2}+\frac{1}{3}e^{\frac{1}{x^3}}=c \qquad \left[\because \int e^{f(x)}f'(x)\mathrm{d}x=e^{f(x)}+c\right]$$

Example 3

Solve $(2x\log x-xy)\,\mathrm{d}y+2y\,\mathrm{d}x=0.$

Solution

$$2y\,\mathrm{d}x+(2x\log x-xy)\mathrm{d}y=0$$

$$M = 2y, \qquad N = 2x\log x - xy$$

$$\frac{\partial M}{\partial y} = 2, \qquad \frac{\partial N}{\partial x} = 2\log x + 2 - y$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2-(2\log x + 2 - y)}{2x\log x - xy}$$

$$= \frac{-(2\log x - y)}{x(2\log x - y)} = -\frac{1}{x}$$

$$\text{IF} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

Multiplying the DE by the IF,

$$\frac{1}{x}(2y)\,dx + \frac{1}{x}(2x\log x - xy)\,dy = 0$$

$$\frac{2y}{x}dx + (2\log x - y)dy = 0$$

$$M_1 = \frac{2y}{x}, \qquad N_1 = 2\log x - y$$

$$\frac{\partial M_1}{\partial y} = \frac{2}{x}, \qquad \frac{\partial N_1}{\partial x} = \frac{2}{x}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$$

$$\int \frac{2y}{x}dx + \int (-y)\,dy = c$$

$$2y\log x - \frac{y^2}{2} = c$$

Example 4

Solve $x\sin x\frac{dy}{dx} + y(x\cos x - \sin x) = 2.$

Solution

$$x\sin x\,dy + (xy\cos x - y\sin x - 2)\,dx = 0$$

$$(xy\cos x - y\sin x - 2)\,dx + x\sin x\,dy = 0$$

$$M = xy\cos x - y\sin x - 2 \qquad N = x\sin x$$

$$\frac{\partial M}{\partial y} = x\cos x - \sin x \qquad \frac{\partial N}{\partial x} = \sin x + x\cos x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(x\cos x - \sin x) - (\sin x + x\cos x)}{x\sin x}$$

$$= -\frac{2\sin x}{x\sin x} = -\frac{2}{x}$$

$$\text{IF} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying the DE by the IF,

$$\frac{1}{x^2}(xy\cos x - y\sin x - 2)\,dx + \frac{1}{x^2}(x\sin x)\,dy = 0$$

$$\left(\frac{y}{x}\cos x - \frac{y}{x^2}\sin x - \frac{2}{x^2}\right)dx + \frac{1}{x}\sin x\,dy = 0$$

$$M_1 = \frac{y}{x}\cos x - \frac{y}{x^2}\sin x - \frac{2}{x^2}, \qquad N_1 = \frac{\sin x}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{\cos x}{x} - \frac{\sin x}{x^2}, \qquad \frac{\partial N_1}{\partial x} = \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int (\text{terms of } M_1 \text{ not containing } y)\,dx + \int N_1\,dy = c$$

$$\int -\frac{2}{x^2}dx + \int \frac{\sin x}{x}dy = c$$

$$\frac{2}{x} + \left(\frac{\sin x}{x}\right)y = c$$

$$\frac{2}{x} + \frac{y\sin x}{x} = c$$

Case II

If $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=f(y)$, (function of y alone), then $\text{IF}=e^{\int f(y)dy}$

After multiplying with the IF, the equation becomes exact and can be solved using the method of exact differential equation.

Example 1

Solve $(y^4+2y)\,dx+(xy^3+2y^4-4x)\,dy=0.$

Solution

$$M=y^4+2y, \qquad N=xy^3+2y^4-4x$$

$$\frac{\partial M}{\partial y}=4y^3+2, \qquad \frac{\partial N}{\partial x}=y^3-4$$

Since $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$, equation is not exact.

$$\frac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=\frac{y^3-4-(4y^3+2)}{y^4+2y}=\frac{-3(y^3+2)}{y(y^3+2)}=-\frac{3}{y}$$

$$\text{IF}=e^{\int -\frac{3}{y}dy}=e^{-3\log y}=e^{\log y^{-3}}=y^{-3}=\frac{1}{y^3}$$

Multiplying the DE by the IF,

$$\frac{1}{y^3}(y^4+2y)dx+\frac{1}{y^3}(xy^3+2y^4-4x)dy=0$$

$$\left(y+\frac{2}{y^2}\right)dx+\left(x+2y-\frac{4x}{y^3}\right)dy=0$$

$$M_1=y+\frac{2}{y^2}, \qquad N_1=x+2y-\frac{4x}{y^3}$$

$$\frac{\partial M_1}{\partial y}=1-\frac{4}{y^3}, \qquad \frac{\partial N_1}{\partial x}=1-\frac{4}{y^3}$$

Since, $\dfrac{\partial M_1}{\partial y}=\dfrac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 dx+\int(\text{terms of } N_1 \text{ not containing } x)dy=c$$

$$\int\left(y+\frac{2}{y^2}\right)dx+\int 2y\,dy=c$$

$$\left(y+\frac{2}{y^2}\right)x+y^2=c$$

Example 2

Solve $(2xy^4e^y+2xy^3+y)\,dx+(x^2y^4e^y-x^2y^2-3x)\,dy=0.$

Solution

$$M=2xy^4e^y+2xy^3+y, \qquad N=x^2y^4e^y-x^2y^2-3x$$

$$\frac{\partial M}{\partial y}=2x(y^4e^y+4y^3e^y+3y^2)+1, \qquad \frac{\partial N}{\partial x}=2xy^4e^y-2xy^2-3$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, the equation is not exact.

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\frac{(2xy^4e^y-2xy^2-3)-(2xy^4e^y+8xy^3e^y+6xy^2+1)}{2xy^4e^y+2xy^3+y}$$

$$=\frac{-4(2xy^3e^y+2xy^2+1)}{y(2xy^3e^y+2xy^2+1)}=-\frac{4}{y}$$

$$\text{IF}=e^{\int -\frac{4}{y}dy}=e^{-4\log y}=e^{\log y^{-4}}=y^{-4}=\frac{1}{y^4}$$

Multiplying the DE by the IF,

$$\frac{1}{y^4}(2xy^4e^y+2xy^3+y)dx+\frac{1}{y^4}(x^2y^4e^y-x^2y^2-3x)dy=0$$

$$\left(2xe^y+\frac{2x}{y}+\frac{1}{y^3}\right)dx+\left(x^2e^y-\frac{x^2}{y^2}-\frac{3x}{y^4}\right)dy=0$$

$$M_1=2xe^y+\frac{2x}{y}+\frac{1}{y^3}, \qquad N_1=x^2e^y-\frac{x^2}{y^2}-\frac{3x}{y^4}$$

$$\frac{\partial M_1}{\partial y}=2xe^y-\frac{2x}{y^2}-\frac{3}{y^4}, \qquad \frac{\partial N_1}{\partial x}=2xe^y-\frac{2x}{y^2}-\frac{3}{y^4}$$

Since $\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 dx + \int (\text{terms of } N_1 \text{ not containing } x)\, dy = c$$

$$\int\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \int 0\, dy = c$$

$$x^2 e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$$

Example 3

Solve $xe^x(dx - dy) + e^x dx + ye^y dy = 0.$

Solution

$$(xe^x + e^x)dx + (ye^y - xe^x)dy = 0$$

$$M = xe^x + e^x, \qquad N = ye^y - xe^x$$

$$\frac{\partial M}{\partial y} = 0, \qquad \frac{\partial N}{\partial x} = -e^x - xe^x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-e^x(x+1) - 0}{e^x(x+1)} = -1$$

$$\text{IF} = e^{\int -dy} = e^{-y}$$

Multiplying the DE by the IF,

$$e^{-y}(xe^x + e^x)dx + e^{-y}(ye^y - xe^x) = 0$$

$$M_1 = e^{-y}(xe^x + e^x), \qquad N_1 = y - xe^{x-y}$$

$$\frac{\partial M_1}{\partial y} = -e^{-y}(xe^x + e^x), \qquad \frac{\partial N_1}{\partial x} = -e^{-y}(xe^x + e^x)$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 dx + \int (\text{terms of } N_1 \text{ not containing } x)\, dy = c$$

$$\int e^{-y}(xe^x + e^x)dx + \int y\, dy = c$$

$$e^{-y}(xe^x - e^x + e^x) + \frac{y^2}{2} = c$$

$$xe^{x-y} + \frac{y^2}{2} = c$$

Example 4

Solve $\left(\frac{y}{x}\sec y - \tan y\right)dx + (\sec y \log x - x)dy = 0.$

Solution

$$M = \frac{y}{x}\sec y - \tan y, \qquad N = \sec y \log x - x$$

$$\frac{\partial M}{\partial y} = \frac{1}{x}\sec y + \frac{y}{x}\sec y \tan y - \sec^2 y, \qquad \frac{\partial N}{\partial x} = \frac{\sec y}{x} - 1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{\frac{\sec y}{x} - 1 - \frac{\sec y}{x} - \frac{y}{x}\sec y \tan y + \sec^2 y}{\frac{y}{x}\sec y - \tan y}$$

$$= \frac{-\frac{y}{x}\sec y \tan y + \tan^2 y}{\frac{y}{x}\sec y - \tan y}$$

$$= -\tan y$$

$$\text{IF} = e^{\int -\tan y\, dy} = e^{-\log \sec y} = e^{\log(\sec y)^{-1}} = (\sec y)^{-1} = \cos y$$

Multiplying the DE by the IF,

$$\cos y\left(\frac{y}{x}\sec y - \tan y\right)dx + \cos y(\sec y \log x - x)dy = 0$$

$$\left(\frac{y}{x} - \sin y\right)dx + (\log x - x\cos y)dy = 0$$

$$M_1 = \frac{y}{x} - \sin y, \qquad N_1 = \log x - x\cos y$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{x} - \cos y, \qquad \frac{\partial N_1}{\partial x} = \frac{1}{x} - \cos y$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 dx + \int (\text{terms of } N_1 \text{ not containing } x)\, dy = c$$

$$\int \left(\frac{y}{x} - \sin y \right) dx + \int 0\, dy = c$$

$$y \log x - x \sin y = c$$

Case III

If the differential equation is of the form $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$ then IF$= \frac{1}{Mx - Ny}$, where $M = f_1(xy)y$, $N = f_2(xy)x$ provided $Mx - Ny \neq 0$.

After multiplying with the IF, the equation becomes exact and can be solved using the method of exact differential equations.

Example 1

Solve $(x^2y^2 + 2)\, y\,dx + (2 - x^2y^2)\, x\,dy = 0$. **[Winter 2014]**

Solution

$$M = x^2y^3 + 2y,\ N = 2x - x^3y^2$$

The equation is of the form

$$f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0$$

$$\text{IF} = \frac{1}{Mx - Ny} = \frac{1}{x^3y^3 + 2xy - 2yx + x^3y^3} = \frac{1}{2x^3y^3}$$

Multiplying the DE by the IF,

$$(x^2y^2 + 2)y\left(\frac{1}{2x^3y^3}\right)dx + (2 - x^2y^2)x\left(\frac{1}{2x^3y^3}\right)dy = 0$$

$$\frac{1}{2}\left(\frac{1}{x} + \frac{2}{x^3y^2}\right)dx + \frac{1}{2}\left(\frac{2}{x^2y^3} - \frac{1}{y}\right)dy = 0$$

$$M_1 = \frac{1}{2}\left(\frac{1}{x} + \frac{2}{x^3y^2}\right), \quad N_1 = \frac{1}{2}\left(\frac{2}{x^2y^3} - \frac{1}{y}\right)$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{2}\left[2\left(\frac{-2}{x^3y^3}\right)\right], \qquad \frac{\partial N_1}{\partial x} = \frac{1}{2}\left[2\left(\frac{-2}{x^3y^3}\right)\right]$$

$$= -\frac{2}{x^3y^3}, \qquad = -\frac{2}{x^3y^3}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 dx + \int (\text{terms of } N_1 \text{ not containing } x)\, dy = c$$

$$\int \frac{1}{2}\left(\frac{1}{x} + \frac{2}{x^3y^2}\right)dx + \int \frac{1}{2}\left(-\frac{1}{y}\right)dy = c$$

$$\frac{1}{2}\left(\log x - \frac{1}{x^2y^2}\right) - \frac{1}{2}\log y = c$$

$$\log x - \log y - \frac{1}{x^2y^2} = c$$

$$\log\left(\frac{x}{y}\right) - \frac{1}{x^2y^2} = c$$

Example 2

Solve $y(1 + xy + x^2y^2)dx + x(1 - xy + x^2y^2)dy = 0.$

Solution

The equation is of the form

$$f_1(xy)y\,dx + f_2(xy)x\,dy = 0$$

$$\text{IF} = \frac{1}{Mx - Ny} = \frac{1}{(xy + x^2y^2 + x^3y^3) - (xy - x^2y^2 + x^3y^3)} = \frac{1}{2x^2y^2}$$

Multiplying the DE by the IF,

$$\frac{y}{2x^2y^2}(1 + xy + x^2y^2)dx + \frac{x}{2x^2y^2}(1 - xy + x^2y^2)dy = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2}\right)dy = 0$$

$$M_1 = \frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2}, \qquad N_1 = \frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2}$$

$$\frac{\partial M_1}{\partial y} = -\frac{1}{2x^2y^2} + \frac{1}{2}, \qquad \frac{\partial N_1}{\partial x} = -\frac{1}{2x^2y^2} + \frac{1}{2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 dx + \int (\text{terms of } N_1 \text{ not containing } x)\, dy = c$$

$$\int \left(\frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2} \right) dx + \int -\frac{1}{2y}\, dy = c$$

$$-\frac{1}{2xy} + \frac{1}{2}\log x + \frac{xy}{2} - \frac{1}{2}\log y = c$$

$$-\frac{1}{2xy} + \frac{xy}{2} + \frac{1}{2}\log \frac{x}{y} = c$$

Example 3

Solve $(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0.$

Solution

$M = xy^2 \sin xy + y\cos xy, \qquad N = x^2y \sin xy - x\cos xy$

The equation is in the form

$$f_1(xy)y\,dx + f_2(xy)x\,dy = 0$$

$$\text{IF} = \frac{1}{Mx - Ny} = \frac{1}{x^2y^2 \sin xy + xy\cos xy - x^2y^2 \sin xy + xy\cos xy}$$

$$= \frac{1}{2xy\cos xy}$$

Multiplying the DE by the IF,

$$\frac{1}{2xy\cos xy}(xy\sin xy + \cos xy)y\,dx + \frac{1}{2xy\cos xy}(xy\sin xy - \cos xy)x\,dy = 0$$

$$\left(\frac{y\tan xy}{2} + \frac{1}{2x} \right) dx + \left(\frac{x\tan xy}{2} - \frac{1}{2y} \right) dy = 0$$

$$M_1 = \frac{y\tan xy}{2} + \frac{1}{2x}, \qquad N_1 = \frac{x\tan xy}{2} - \frac{1}{2y}$$

$$\frac{\partial M_1}{\partial y} = \frac{\tan xy + xy\sec^2 xy}{2}, \qquad \frac{\partial N_1}{\partial x} = \frac{\tan xy + xy\sec^2 xy}{2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 \mathrm{d}x + \int (\text{terms of } N_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\frac{1}{2}\int\left(y\tan xy + \frac{1}{x}\right)\mathrm{d}x + \int -\frac{1}{2y}\mathrm{d}y = c$$

$$\frac{1}{2}\left(\frac{y}{y}\log\sec xy + \log x\right) - \frac{1}{2}\log y = c$$

$$\log(x\sec xy) - \log y = 2c$$

$$\log\left(\frac{x}{y}\sec xy\right) = 2c$$

$$\frac{x}{y}\sec xy = e^{2c} = k, \quad \frac{x}{y}\sec xy = k$$

Case IV

If the differential equation $M\mathrm{d}x + N\mathrm{d}y = 0$ is a homogeneous equation in x and y (degree of each term is same) then $\mathrm{IF} = \frac{1}{Mx + Ny}$ provided $Mx + Ny \neq 0$.

After multiplying with the IF, the equation becomes exact and can be solved using the method of exact differential equations.

Example 1

Solve $(x^4 + y^4)\mathrm{d}x - xy^3\,\mathrm{d}y = 0.$ **[Summer 2018]**

Solution

$$M = x^4 + y^4, \qquad N = -xy^3$$

The differential equation is homogeneous as each term is of degree 4.

$$\mathrm{IF} = \frac{1}{Mx + Ny} = \frac{1}{x^5 + xy^4 - xy^4} = \frac{1}{x^5}$$

Multiplying the DE by the IF,

$$\frac{1}{x^5}(x^4+y^4)\,dx-\frac{1}{x^5}(xy^3)\,dy=0$$

$$\left(\frac{1}{x}+\frac{y^4}{x^5}\right)dx-\frac{y^3}{x^4}\,dy=0$$

$$M_1=\frac{1}{x}+\frac{y^4}{x^5},\qquad N_1=-\frac{y^3}{x^4}$$

$$\frac{\partial M_1}{\partial y}=\frac{4y^3}{x^5},\qquad \frac{\partial N_1}{\partial x}=\frac{4y^3}{x^5}$$

Since $\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{ constant}} M_1\,dx+\int(\text{terms of } N_1 \text{ not containing } x)\,dy=c$$

$$\int\left(\frac{1}{x}+\frac{y^4}{x^5}\right)dx+\int 0\,dy=c$$

$$\log x-\frac{y^4}{4x^4}=c$$

Example 2

Solve $x^2y\,dx-(x^3+xy)^2\,dy=0.$ **[Winter 2014]**

Solution

$$M=x^2y,\qquad N=-x^3-xy^2$$

The differential equation is homogeneous as each term is of degree 3.

$$IF=\frac{1}{Mx+Ny}=\frac{1}{x^3y-x^3y-xy^3}=-\frac{1}{xy^3}$$

Multiplying the DE by the IF,

$$-\frac{1}{xy^3}(x^2y)\,dx-\left(-\frac{1}{xy^3}\right)(x^3+xy^2)\,dy=0$$

$$-\frac{x}{y^2}\,dx+\left(\frac{x^2}{y^3}+\frac{1}{y}\right)dy=0$$

$$M_1 = -\frac{x}{y^2}, \qquad N_1 = \frac{x^2}{y^3} + \frac{1}{y}$$

$$\frac{\partial M_1}{\partial y} = \frac{2x}{y^3}, \qquad \frac{\partial N_1}{\partial x} = \frac{2x}{y^3}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{ constant}} M_1 dx + \int (\text{terms of } N_1 \text{ not containing } x)\, dy = c$$

$$\int -\frac{x}{y^2} dx + \int \frac{1}{y} dy = c$$

$$-\frac{x^2}{2y^2} + \log y = c$$

Example 3

Solve $(xy - 2y^2)\,dx - (x^2 - 3xy)\,dy = 0.$ **[Winter 2013]**

Solution

$$M = xy - 2y^2, \qquad N = -x^2 + 3xy$$

The differential equation is homogeneous as each term is of degree 2.

$$\text{IF} = \frac{1}{Mx + Ny} = \frac{1}{x^2y - 2xy^2 - x^2y + 3xy^2} = \frac{1}{xy^2}$$

Multiplying the DE by the IF,

$$\frac{1}{xy^2}(xy - 2y^2)dx - \frac{1}{xy^2}(x^2 - 3xy)dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(-\frac{x}{y^2} + \frac{3}{y}\right)dy = 0$$

$$M_1 = \frac{1}{y} - \frac{2}{x}, \qquad N_1 = -\frac{x}{y^2} + \frac{3}{y}$$

$$\frac{\partial M_1}{\partial y} = -\frac{1}{y^2}, \qquad \frac{\partial N_1}{\partial x} = -\frac{1}{y^2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 dx + \int (\text{terms of } N_1 \text{ not containing } x) dy = c$$

$$\int \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \frac{3}{y} dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

Example 4

Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$

Solution

$$M = x^2y - 2xy^2, \quad N = -x^3 + 3x^2y$$

The differential equation is homogeneous as each term is of degree 3.

$$IF = \frac{1}{Mx + Ny} = \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2} = \frac{1}{x^2y^2}$$

Multiplying the DE by the IF,

$$\frac{1}{x^2y^2}(x^2y - 2xy^2)dx - \frac{1}{x^2y^2}(x^3 - 3x^2y)dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

$$M_1 = \frac{1}{y} - \frac{2}{x}, \quad N_1 = -\frac{x}{y^2} + \frac{3}{y}$$

$$\frac{\partial M_1}{\partial y} = -\frac{1}{y^2}, \quad \frac{\partial N_1}{\partial x} = -\frac{1}{y^2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 dx + \int (\text{terms of } N_1 \text{ not containing } x) dy = c$$

$$\int \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \frac{3}{y} dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} + \log\frac{y^3}{x^2} = c$$

Example 5

Solve $x\dfrac{dy}{dx} + \dfrac{y^2}{x} = y.$

Solution

$$x^2 dy + y^2 dx = xy\,dx$$

$$(y^2 - xy)dx + x^2 dy = 0$$

$$M = y^2 - xy, \qquad N = x^2$$

The differential equation is homogeneous as each term is of degree 2.

$$\text{IF} = \frac{1}{Mx + Ny} = \frac{1}{xy^2 - x^2y + x^2y} = \frac{1}{xy^2}$$

Multiplying the DE by the IF,

$$\frac{1}{xy^2}(y^2 - xy)dx + \frac{x^2}{xy^2}dy = 0$$

$$\left(\frac{1}{x} - \frac{1}{y}\right)dx + \frac{x}{y^2}dy = 0$$

$$M_1 = \frac{1}{x} - \frac{1}{y}, \qquad N_1 = \frac{x}{y^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{y^2}, \qquad \frac{\partial N_1}{\partial x} = \frac{1}{y^2}$$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$$

$$\int\left(\frac{1}{x} - \frac{1}{y}\right)dx + \int 0\,dy = c$$

$$\log x - \frac{x}{y} = c$$

Example 6

Solve $3x^2y^4dx + 4x^3y^3dy = 0,\ y(1) = 1.$

Solution

$$M = 3x^2y^4, \qquad N = 4x^3y^3$$

The differential equation is homogeneous as each term is of degree 6.

$$IF = \frac{1}{Mx + Ny} = \frac{1}{3x^3y^4 + 4x^3y^4} = \frac{1}{7x^3y^4}$$

Multiplying the DE by the IF,

$$\frac{1}{7x^3y^4}(3x^2y^4)dx + \frac{1}{7x^3y^4}(4x^3y^3)dy = 0$$

$$\frac{3}{7x}dx + \frac{4}{7y}dy = 0$$

$$M_1 = \frac{3}{7x}, \qquad N_1 = \frac{4}{7y}$$

$$\frac{\partial M_1}{\partial y} = 0, \qquad \frac{\partial N_1}{\partial x} = 0$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 dx + \int (\text{terms of } N_1 \text{ not containing } x)\, dy = c$$

$$\int \frac{3}{7x}dx + \int \frac{4}{7y}dy = \log c$$

$$\frac{3}{7}\log x + \frac{4}{7}\log y = \log c$$

$$\log x^{\frac{3}{7}} + \log y^{\frac{4}{7}} = \log c$$

$$\log\left(x^{\frac{3}{7}}y^{\frac{4}{7}}\right) = \log c$$

$$x^{\frac{3}{7}}y^{\frac{4}{7}} = c \qquad \ldots(1)$$

Given $y(1) = 1$

Substituting $x = 1, y = 1$ in Eq. (1),

$$(1)^{\frac{3}{7}} \cdot (1)^{\frac{4}{7}} = c, \quad 1 = c$$

Hence, the solution is

$$x^{\frac{3}{7}} y^{\frac{4}{7}} = 1$$

Case V

If the differential equation is of the type

$$x^{m_1} y^{n_1} (a_1 y\,dx + b_1 x\,dy) + x^{m_2} y^{n_2} (a_2 y\,dx + b_2 x\,dy) = 0$$

then $\text{IF} = x^h y^k$

where $$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

and $$\frac{m_2 + h + 1}{a_2} = \frac{n_2 + k + 1}{b_2}$$

Solving these two equations, we get the values of h and k.

Example 1

Solve $x(4y\,dx + 2x\,dy) + y^3(3y\,dx + 5x\,dy) = 0.$

Solution

$$xy^0(4y\,dx + 2x\,dy) + x^0 y^3 (3y\,dx + 5x\,dy) = 0$$

$$m_1 = 1, n_1 = 0, a_1 = 4, b_1 = 2, m_2 = 0, n_2 = 3, a_2 = 3, b_2 = 5$$

$$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

$$\frac{1 + h + 1}{4} = \frac{0 + k + 1}{2}$$

$$2h + 4 = 4k + 4$$

$$h = 2k \quad \text{...(1)}$$

and $$\frac{m_2 + h + 1}{a_2} = \frac{n_2 + k + 1}{b_2}$$

$$\frac{0 + h + 1}{3} = \frac{3 + k + 1}{5}$$

$$5h + 5 = 3k + 12$$

$$5h - 3k = 7 \quad \text{...(2)}$$

Solving Eqs (1) and (2),

$$h = 2,\ k = 1$$

$$\text{IF} = x^2 y$$

Multiplying the DE by the IF,

$$x^3 y(4y\,dx + 2x\,dy) + x^2 y^4 (3y\,dx + 5x\,dy) = 0$$

$$(4x^3y^2 + 3x^2y^5)\,dx + (2x^4 y + 5x^3y^4)\,dy = 0$$

$$M = 4x^3y^2 + 3x^2y^5, \quad N = 2x^4y + 5x^3y^4$$

$$\frac{\partial M}{\partial y} = 8x^3y + 15x^2y^4, \quad \frac{\partial N}{\partial x} = 8x^3y + 15x^2y^4$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int (4x^3y^2 + 3x^2y^5)\,dx + \int 0\,dy = c$$

$$x^4y^2 + x^3y^5 = c$$

Example 2

Solve $(x^7y^2 + 3y)\,dx + (3x^8y - x)\,dy = 0.$

Solution

$$M = x^7y^2 + 3y, \quad N = 3x^8y - x$$

$$\frac{\partial M}{\partial y} = 2x^7y + 3, \quad \frac{\partial N}{\partial x} = 24x^7y - 1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

Rewriting the equation,

$$x^7y^2\,dx + 3x^8y\,dy + 3y\,dx - x\,dy = 0$$

$$x^7 y(y\,dx + 3x\,dy) + (3y\,dx - x\,dy) = 0$$

$$m_1 = 7,\ n_1 = 1,\ a_1 = 1,\ b_1 = 3,\ m_2 = 0,\ n_2 = 0,\ a_2 = 3,\ b_2 = -1$$

$$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

$$\frac{7+h+1}{1}=\frac{1+k+1}{3}$$

$$3h+24=k+2$$

$$3h-k=-22 \quad \ldots(1)$$

and

$$\frac{m_2+h+1}{a_2}=\frac{n_2+k+1}{b_2}$$

$$\frac{0+h+1}{3}=\frac{0+k+1}{-1}$$

$$-h-1=3k+3$$

$$h+3k=-4 \quad \ldots(2)$$

Solving Eqs (1) and (2),

$$h=-7,\ k=1$$

$$\therefore \quad \text{IF}=x^{-7}y$$

Multiplying the DE by the IF,

$$x^{-7}y(x^7y^2+3y)\text{d}x+x^{-7}y(3x^8y-x)\text{d}y=0$$

$$(y^3+3x^{-7}y^2)\text{d}x+(3xy^2-x^{-6}y)\text{d}y=0$$

$$\text{M}_1=y^3+3x^{-7}y^2, \qquad \text{N}_1=3xy^2-x^{-6}y$$

$$\frac{\partial \text{M}_1}{\partial y}=3y^2+6x^{-7}y, \qquad \frac{\partial \text{N}_1}{\partial x}=3y^2+6x^{-7}y$$

Since $\frac{\partial \text{M}_1}{\partial y}=\frac{\partial \text{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\,\text{constant}} \text{M}_1\text{d}x+\int(\text{terms of N}_1\text{ not containing }x)\,\text{d}y=c$$

$$\int(y^3+3x^{-7}y^2)\text{d}x+\int 0\,\text{d}y=c$$

$$xy^3+\frac{3x^{-6}y^2}{-6}=c$$

$$xy^3-\frac{x^{-6}y^2}{2}=c$$

Case VI ***Integrating Factors by Inspection*** Sometimes, the integrating factor can be identified by regrouping the terms of the differential equation. The following table helps in identifying the IF after regrouping the terms.

Sr. No.	Group of Terms	Integrating Factor	Exact Differential Equation
1.	$dx \pm dy$	$\frac{1}{x \pm y}$	$\frac{dx \pm dy}{x \pm y} = d[\log(x \pm y)]$
2.	$y\,dx + x\,dy$	$\frac{1}{2xy}$	$y\,dx + x\,dy = d(xy)$ $2x^2 y\,dy + 2xy^2 dx = d(x^2 y^2)$
		$\frac{1}{xy}$	$\frac{y\,dx + x\,dy}{xy} = d[\log(xy)]$
		$\frac{1}{(xy)^n}$	$\frac{y\,dx + x\,dy}{(xy)^n} = d\left[\frac{(xy)^{1-n}}{1-n}\right], n \neq 1$
3.	$y\,dx - x\,dy$	$\frac{1}{y^2}$	$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$
		$\frac{1}{x^2 + y^2}$	$\frac{y\,dx - x\,dy}{x^2 + y^2} = d\left[\tan^{-1}\left(\frac{x}{y}\right)\right]$
		$\frac{1}{x^2}$	$\frac{y\,dx - x\,dy}{x^2} = d\left(-\frac{y}{x}\right)$
		$\frac{1}{xy}$	$\frac{y\,dx - x\,dy}{xy} = d\left[\log\left(\frac{x}{y}\right)\right]$
4.	$x\,dx \pm y\,dy$	2	$2x\,dx \pm 2y\,dy = d(x^2 \pm y^2)$
		$\frac{1}{(x^2 \pm y^2)}$,	$\frac{2x\,dx \pm 2y\,dy}{x^2 \pm y^2} = d[\log(x^2 \pm y^2)]$
		$\frac{1}{(x^2 \pm y^2)^n}$	$\frac{2x\,dx \pm 2y\,dy}{(x^2 \pm y^2)^n} = d\left[\frac{(x^2 \pm y^2)^{1-n}}{2(1-n)}\right]$
5.	$2y\,dx + x\,dy$	x	$2xy\,dx + x^2\,dy = d(x^2 y)$
6.	$y\,dx + 2x\,dy$	y	$y^2\,dx + 2xy\,dy = d(xy^2)$
7.	$2y\,dx - x\,dy$	$\frac{x}{y^2}$	$\frac{2xy\,dx - x^2 dy}{y^2} = d\left(\frac{x^2}{y}\right)$
8.	$2x\,dy - y\,dx$	$\frac{y}{x^2}$	$\frac{2xy\,dy - y^2 dx}{x^2} = d\left(\frac{y^2}{x}\right)$

Example 1

Solve $x\,dy - y\,dx + 2x^3 dx = 0.$

Solution

Dividing the equation by x^2,

$$\frac{x\,dy - y\,dx}{x^2} + 2x\,dx = 0$$

$$d\left(\frac{y}{x}\right) + d(x^2) = 0$$

Integrating both the sides,

$$\frac{y}{x} + x^2 = c$$

Example 2

Solve $x\,dx + y\,dy + 2(x^2 + y^2)\,dx = 0.$

Solution

Dividing the equation by $x^2 + y^2$,

$$\frac{x\,dx + y\,dy}{x^2 + y^2} + 2\,dx = 0$$

$$\frac{1}{2} d[\log(x^2 + y^2)] + 2\,dx = 0$$

Integrating both the sides,

$$\frac{1}{2}\log(x^2 + y^2) + 2x = c$$

Example 3

Solve $(1 + xy)y\,dx + (1 - xy)x\,dy = 0.$

Solution

$$y\,dx + xy^2\,dx + x\,dy - x^2 y\,dy = 0$$

Regrouping the terms,

$$(y\,dx + x\,dy) + (xy^2 dx - x^2 y\,dy) = 0$$

Dividing the equation by x^2y^2,

$$\frac{y\,dx + x\,dy}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0$$

$$d\left(-\frac{1}{xy}\right) + \frac{dx}{x} - \frac{dy}{y} = 0$$

Integrating both the sides,

$$-\frac{1}{xy} + \log x - \log y = c$$

$$-\frac{1}{xy} + \log\frac{x}{y} = c$$

Example 4

Solve $x\,dy - y\,dx = 3x^2(x^2 + y^2)\,dx.$

Solution

Dividing the equation by $(x^2 + y^2)$,

$$\frac{x\,dy - y\,dx}{x^2 + y^2} = 3x^2 dx$$

$$d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = d(x^3)$$

Integrating both the sides,

$$\tan^{-1}\left(\frac{y}{x}\right) = x^3 + c$$

Example 5

Solve $(xy - 2y^2)\,dx - (x^2 - 3xy)\,dy = 0.$

Solution

$$xy\,dx - 2y^2dx - x^2dy + 3xy\,dy = 0$$

Regrouping the terms,

$$x(y\,dx - x\,dy) - 2y^2\,dx + 3xy\,dy = 0$$

Dividing the equation by xy^2,

$$\frac{y\,dx - x\,dy}{y^2} - \frac{2}{x}dx + \frac{3}{y}dy = 0$$

$$d\left(\frac{x}{y}\right) - \frac{2}{x}dx + \frac{3}{y}dy = 0$$

Integrating both the sides,

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} - \log x^2 + \log y^3 = c$$

$$\frac{x}{y} + \log\frac{y^3}{x^2} = c$$

Example 6

Solve $y(2xy + e^x)\,dx = e^x dy.$

Solution

$$2xy^2 dx + e^x y\,dx - e^x dy = 0$$

Dividing the equation by y^2,

$$2x\,dx + \frac{ye^x dx - e^x dy}{y^2} = 0$$

$$2x\,dx + d\left(\frac{e^x}{y}\right) = 0$$

Integrating both the sides,

$$x^2 + \frac{e^x}{y} = c$$

Example 7

Solve $y\,dx + x(x^2y - 1)\,dy = 0.$

Solution

$$y\,dx + x^3y\,dy - x\,dy = 0$$

Regrouping the terms,

$$y\,dx - x\,dy + x^3y\,dy = 0$$

Dividing the equation by $\frac{x^3}{y}$,

$$\frac{y^2 dx - xy\,dy}{x^3} + y^2 dy = 0$$

$$\frac{1}{2}\left(\frac{y^2 \cdot 2x\,dx - x^2 \cdot 2y\,dy}{x^4}\right) + y^2 dy = 0$$

$$\frac{1}{2} d\left(-\frac{y^2}{x^2}\right) + y^2 dy = 0$$

Integrating both the sides,

$$-\frac{1}{2}\frac{y^2}{x^2} + \frac{y^3}{3} = c$$

$$-\frac{y^2}{2x^2} + \frac{y^3}{3} = c$$

Example 8

Solve $y(x^3 e^{xy} - y)dx + x(y + x^3 e^{xy})dy = 0.$

Solution

$$x^3 y e^{xy} dx - y^2 dx + xy\,dy + x^4 e^{xy} dy = 0$$

Regrouping the terms,

$$x^3 y e^{xy} dx + x^4 e^{xy} dy - y^2 dx + xy\,dy = 0$$

Dividing the equation by x^3,

$$ye^{xy} dx + xe^{xy} dy - \frac{1}{2}\left(\frac{y^2 \cdot 2x\,dx - x^2 \cdot 2y\,dy}{x^4}\right) = 0$$

$$d(e^{xy}) + \frac{1}{2} d\left(\frac{y^2}{x^2}\right) = 0$$

Integrating both the sides,

$$e^{xy} + \frac{1}{2}\frac{y^2}{x^2} = c$$

Example 9

If x^n is an integrating factor of $(y-2x^3)\,dx - x(1-xy)\,dy = 0$ then find n and solve the equation.

Solution

If x^n is an IF then after multiplication with x^n, the equation becomes exact.

If x^n is an IF then $(x^n y - 2x^{n+3})dx - x^{n+1}(1-xy)dy = 0$ is an exact DE

$$(x^n y - 2x^{n+3})dx - x^{n+1}(1-xy)dy = 0$$

where

$$M = x^n y - 2x^{n+3}, \quad N = -x^{n+1} + x^{n+2}y$$

and

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$x^n = -(n+1)x^n + (n+2)x^{n+1}y$$

$$(n+2)x^n(1+xy) = 0$$

$$n+2 = 0$$

$$n = -2$$

Putting $n = -2$ in the equation,

$$(x^{-2}y - 2x)dx - x^{-1}(1-xy)dy = 0$$

$$\left(\frac{y}{x^2} - 2x\right)dx - \left(\frac{1}{x} - y\right)dy = 0$$

$$M = \frac{y}{x^2} - 2x, \qquad N = -\frac{1}{x} + y$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2}, \qquad \frac{\partial N}{\partial x} = \frac{1}{x^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int \left(\frac{y}{x^2} - 2x\right)dx + \int y\,dy = c$$

$$-\frac{y}{x} - x^2 + \frac{y^2}{2} = c$$

EXERCISE 4.2

Solve the following differential equations:

1. $(x^2+y^2+x)\,dx+xy\,dy=0$

$\left[\textbf{Ans.: } 3x^4+4x^3+6x^2y^2=c\right]$

2. $(y-2x^3)\,dx-(x-x^2y)\,dy=0$

$\left[\textbf{Ans.: } xy^2-2y-2x^3=cx\right]$

3. $(2xy^4e^y+2xy^3+y)\,dx+(x^2y^4e^y-x^2y^2-3x)\,dy=0$

$\left[\textbf{Ans.: } x^2e^y+\dfrac{x^2}{y}+\dfrac{x}{y^3}=c\right]$

4. $\left(2x\sinh\dfrac{y}{x}+3y\cosh\dfrac{y}{x}\right)dx-\left(3x\cosh\dfrac{y}{x}\right)dy=0$

$\left[\textbf{Ans.: } 3\sinh\dfrac{y}{x}=cx^{\frac{2}{3}}\right]$

5. $(e^xx^4-2mxy^2)\,dx+2mx^2y\,dy=0$

$\left[\textbf{Ans.: } x^2e^x+my^2=cx^2\right]$

6. $\left(y+\dfrac{y^3}{3}+\dfrac{x^2}{2}\right)dx+\dfrac{1}{4}(x+xy^2)\,dy=0$

$\left[\textbf{Ans.: } x^6+3x^4y+x^4y^3=c\right]$

7. $(x\sec^2 y-x^2\cos y)\,dy=(\tan y-3x^4)\,dx$

$\left[\textbf{Ans.: } \dfrac{\tan y}{x}+x^3-\sin y=c\right]$

8. $(4xy+3y^2-x)\,dx+x(x+2y)\,dy=0$

$\left[\textbf{Ans.: } x^4y+x^3y^2-\dfrac{x^4}{4}=c\right]$

9. $(x^2+y^2+2x)\,dx+2y\,dy=0$

$\left[\textbf{Ans.: } e^x(x^2+y^2)=c\right]$

10. $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

$\left[\text{Ans.: } x^3y^3 + x^2 = cy\right]$

11. $y(xy + e^x)\,dx - e^x\,dy = 0$

$\left[\text{Ans.: } \dfrac{x^2}{2} + \dfrac{e^x}{y} = c\right]$

12. $(3x^2y^3e^y + y^3 + y^2)\,dx + (x^3y^3e^y - xy)\,dy = 0$

$\left[\text{Ans.: } x^3e^y + x + \dfrac{x}{y} = c\right]$

13. $y(x^2y + e^x)\,dx - e^x\,dy = 0$

$\left[\text{Ans.: } \dfrac{x^3}{3} + \dfrac{e^x}{y} = c\right]$

14. $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$

$\left[\text{Ans.: } 3x^2y^4 + 6xy^2 + 2y^6 = c\right]$

15. $(2x^2y + e^x)y\,dx - (e^x + y^3)\,dy = 0$

$\left[\text{Ans.: } 4x^3y - 3y^3 + 6e^x = cy\right]$

16. $y\log y\,dx + (x - \log y)\,dy = 0$

$\left[\text{Ans.: } 2x\log y = c + (\log y)^2\right]$

17. $(x - y^2)\,dx + 2xy\,dy = 0$

$\left[\text{Ans.: } \dfrac{y^2}{x} + \log x = c\right]$

18. $2xy\,dx + (y^2 - x^2)\,dy = 0$

$\left[\text{Ans.: } x^2 + y^2 = cy\right]$

19. $(1 + xy)y\,dx + (1 - xy)x\,dy = 0$

$\left[\text{Ans.: } \log\left(\dfrac{x}{y}\right) = c + \dfrac{1}{xy}\right]$

20. $(1 + xy + x^2y^2 + x^3y^3)y\,dx + (1 - xy - x^2y^2 + x^3y^3)x\,dy = 0$

$\left[\text{Ans.: } xy - \dfrac{1}{xy} - \log y^2 = c\right]$

21. $\frac{dy}{dx} = -\frac{x^2y^3 + 2y}{2x - 2x^3y^2}$

$$\left[\text{Ans.: } \frac{1}{3}\log\frac{x}{y^2} - \frac{1}{3x^2y^2} = c\right]$$

22. $y(\sin xy + xy\cos xy)dx + x(xy\cos xy - \sin xy)dy = 0$

$$\left[\text{Ans.: } \frac{x\sin(xy)}{y} = c\right]$$

23. $y(x+y)dx - x(y-x)dy = 0$

$$\left[\text{Ans.: } \log\sqrt{xy} - \frac{y}{2x} = c\right]$$

24. $x^2y\,dx - (x^3 + y^3)dy = 0$

$$\left[\text{Ans.: } y = ce^{\frac{x^3}{3y^3}}\right]$$

25. $3y\,dx + 2x\,dy = 0,\ y(1) = 1$

$$\left[\text{Ans.: } yx^{\frac{3}{2}} = 1\right]$$

26. $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$

$$\left[\text{Ans.: } -\frac{2}{3}x^{-\frac{3}{2}}y^{\frac{3}{2}} + 4x^{\frac{1}{2}}y^{\frac{1}{2}} = c\right]$$

27. $(2x^2y^2 + y)dx - (x^3y - 3x)dy = 0$

$$\left[\text{Ans.: } \frac{7}{5}x^{\frac{10}{7}}y^{-\frac{5}{7}} - \frac{7}{4}x^{-\frac{4}{7}}y^{\frac{12}{7}} = c\right]$$

28. If y^n is an integrating factor of

$(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

find n and solve the equation.

$$\left[\text{Ans.: } n = -4,\ x^2y^3e^y + x^2y^2 + x = cy^3\right]$$

4.3.6 Linear Differential Equations

If each term in a differential equation including the derivative is linear in terms of dependent variable then the equation is called linear.

A differential equation of the form

$$\frac{dy}{dx} + Py = Q \qquad ...(4.11)$$

where P and Q are functions of x, is called a linear differential equation and is linear in y. To solve Eq. (4.11), obtain the integrating factor (IF) as

$$\text{IF} = e^{\int P dx}$$

Multiplying Eq. (4.11) by the IF,

$$e^{\int P dx}\frac{dy}{dx} + Pe^{\int P dx}y = Qe^{\int P dx}$$

$$\frac{d}{dx}\left[e^{\int P dx}y\right] = Qe^{\int P dx}$$

Integrating w.r.t x,

$$e^{\int P dx}y = \int Qe^{\int P dx}dx + c$$

or

$$(\text{IF})\,y = \int (\text{IF})Q + c \qquad \text{...(4.12)}$$

Equation (4.12) is the solution of the differential equation (4.12).

Example 1

[Summer 2018]

Solve $\dfrac{dy}{dx} + y\sin x = e^{\cos x}$.

Solution

The equation is linear in y.

$$P = \sin x, \qquad Q = e^{\cos x}$$

$$\text{IF} = e^{\int \sin x\, dx} = e^{-\cos x}$$

Hence, the general solution is

$$e^{-\cos x}y = \int e^{-\cos x}\cdot e^{\cos x}\,dx + c$$

$$= \int e^0\,dx + c$$

$$= \int dx + c$$

$$= x + c$$

$$y = (x+c)e^{\cos x}$$

Example 2

Solve $\dfrac{dy}{dx} + \dfrac{3y}{x} = \dfrac{\sin x}{x^3}$.

Solution

The equation is linear in y.

$$P = \frac{3}{x}, \quad Q = \frac{\sin x}{x^3}$$

$$IF = e^{\int \frac{3}{x} dx} = e^{3 \log x} = e^{\log x^3} = x^3$$

Hence, the general solution is

$$x^3 y = \int x^3 \frac{\sin x}{x^3} dx + c$$

$$= \int \sin x \, dx + c$$

$$= -\cos x + c$$

$$y = -\frac{\cos x}{x^3} + \frac{c}{x^3}$$

Example 3

Solve $\frac{dy}{dx} + 2y \tan x = \sin x.$ **[Winter 2014]**

Solution

The equation is linear in y.

$$P = 2 \tan x, \qquad Q = \sin x$$

$$IF = e^{\int 2 \tan x \, dx} = e^{2 \log \sec x} = e^{\log \sec^2 x} = \sec^2 x$$

Hence, the general solution is

$$(\sec^2 x) y = \int \sec^2 x \sin x \, dx + c$$

$$y \sec^2 x = \int \sec x \frac{\sin x}{\cos x} dx + c$$

$$= \int \sec x \tan x \, dx + c$$

$$= \sec x + c$$

Example 4

Solve $\frac{dy}{dx} + y \cot x = 2 \cos x.$ **[Summer 2016]**

Solution

The equation is linear in y.

$$P = \cot x, \qquad Q = 2 \cos x$$

$$\text{IF} = e^{\int \cot x\,dx} = e^{\log \sin x} = \sin x$$

Hence, the general solution is

$$(\sin x)y = \int \sin x(2\cos x)\,dx + c$$

$$y\sin x = \int \sin 2x\,dx + c$$

$$= -\frac{1}{2}\cos 2x + c$$

Example 5

Solve $\frac{dy}{dx} + (\tan x)y = \sin 2x \quad y(0) = 0.$ [Summer 2017]

Solution

The equation is linear in y.

$$P = \tan x, \quad Q = \sin x$$

$$\text{IF} = e^{\int \tan dx = e^{\log \sec x}} = \sec x$$

Hence, the general solution is

$$(\sec x)y = \int \sec x(\sin 2x)\,dx + c$$

$$y\sec x = \int \frac{1}{\cos x}(2\sin x\cos x)\,dx + c$$

$$= 2\int \sin x\,dx + c$$

$$= -2\cos x + c$$

$$y\cdot\frac{1}{\cos x} = -2\cos x + c$$

$$y = -2\cos^2 x + c\cos x \quad \text{...(1)}$$

Putting $y(0) = 0$ in Eq. (1),

$$y(0) = -2 + c$$

$$0 = -2 + c$$

$$c = 2$$

Hence, $y = -2\cos^2 x + 2\cos x = 2\cos(1 - \cos x)$

Example 6

Solve $(x+1)\dfrac{dy}{dx} - y = e^{3x}(x+1)^2.$ **[Winter 2016; Summer 2014]**

Solution

Rewriting the equation,

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1)$$

The equation is linear in y.

$$P = -\frac{1}{x+1}, \qquad Q = e^{3x}(x+1)$$

$$IF = e^{\int -\frac{1}{x+1}dx} = e^{-\log(x+1)} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$$

Hence, the general solution is

$$\left(\frac{1}{x+1}\right)y = \int\left(\frac{1}{x+1}\right)e^{3x}(x+1)dx + c$$

$$= \int e^{3x}dx + c$$

$$= \frac{e^{3x}}{3} + c$$

$$y = (x+1)\left(\frac{e^{3x}}{3} + c\right)$$

Example 7

Solve $\dfrac{dy}{dx} + \dfrac{1}{x^2}y = 6e^{\frac{1}{x}}.$ **[Winter 2012]**

Solution

The equation is linear in y.

$$P = \frac{1}{x^2}, \qquad Q = 6e^{\frac{1}{x}}$$

$$IF = e^{\int \frac{1}{x^2}dx} = e^{-\frac{1}{x}}$$

Hence, the general solution is

$$e^{-\frac{1}{x}}y = \int e^{-\frac{1}{x}}(6e^{\frac{1}{x}})dx + c$$

$$= 6\int dx + c$$

$$= 6x + c$$

$$y = (6x + c)e^{\frac{1}{x}}$$

Example 8

Solve $\frac{dy}{dx} + \frac{4x}{1+x^2}y = \frac{1}{(x^2+1)^3}$. **[Winter 2013]**

Solution

The equation is linear in y.

$$P = \frac{4x}{1+x^2}, \quad Q = \frac{1}{(x^2+1)^3}$$

$$IF = e^{\int \frac{4x}{1+x^2}dx} = e^{2\log(1+x^2)} = e^{\log(1+x^2)^2} = (1+x^2)^2$$

Hence, the general solution is

$$(1+x^2)^2 y = \int (1+x^2)^2 \cdot \frac{1}{(x^2+1)^3} dx + c$$

$$= \int \frac{1}{x^2+1} dx + c$$

$$= \tan^{-1} x + c$$

Example 9

Solve $(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$.

Solution

Rewriting the equation,

$$\frac{dy}{dx} + \left(\frac{2x}{1-x^2}\right)y = \frac{x}{\sqrt{1-x^2}}$$

The equation is linear in y.

$$P = \frac{2x}{1-x^2}, \quad Q = \frac{x}{\sqrt{1-x^2}}$$

$$\text{IF} = e^{\int \frac{2x}{1-x^2}\mathrm{d}x} = e^{-\log(1-x^2)} = e^{\log(1-x^2)^{-1}} = (1-x^2)^{-1} = \frac{1}{1-x^2}$$

Hence, the general solution is

$$\left(\frac{1}{1-x^2}\right)y = \int \left(\frac{1}{1-x^2}\right)\left(\frac{x}{\sqrt{1-x^2}}\right)\mathrm{d}x + c$$

$$= \int \frac{x}{(1-x^2)^{\frac{3}{2}}}\mathrm{d}x + c$$

$$= -\frac{1}{2}\int (1-x^2)^{-\frac{3}{2}}(-2x)\,\mathrm{d}x + c$$

$$= -\frac{1}{2}\cdot\frac{(1-x^2)^{-\frac{1}{2}}}{-\frac{1}{2}} + c \qquad \left[\because \int [f(x)]^n f'(x)\,\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$\frac{y}{1-x^2} = (1-x^2)^{-\frac{1}{2}} + c$$

$$y = \sqrt{1-x^2} + c(1-x^2)$$

Example 10

Solve $x\log x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 2\log x.$

Solution

Rewriting the equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{1}{x\log x}\right)y = \frac{2}{x}$$

The equation is linear in y.

$$P = \frac{1}{x\log x}, \quad Q = \frac{2}{x}$$

$$\text{IF} = e^{\int \frac{1}{x\log x}\mathrm{d}x} = e^{\log(\log x)} = \log x$$

Hence, the general solution is

$$(\log x)y = \int (\log x)\cdot\frac{2}{x}\,dx + c$$

$$= 2\frac{(\log x)^2}{2} + c \qquad \left[\because \int f(x)\cdot f'(x)\,dx = \frac{[f(x)]^2}{2}\right]$$

$$= (\log x)^2 + c$$

$$y\log x = (\log x)^2 + c$$

Example 11

Solve $(1+x+xy^2)\,dy + (y+y^3)\,dx = 0.$

Solution

Rewriting the equation,

$$(1+x+xy^2) + (y+y^3)\frac{dx}{dy} = 0$$

$$\frac{dx}{dy} + \frac{(1+y^2)x}{y+y^3} + \frac{1}{y+y^3} = 0$$

$$\frac{dx}{dy} + \left(\frac{1}{y}\right)x = -\frac{1}{y(1+y^2)}$$

The equation is linear in x.

$$P = \frac{1}{y}, \quad Q = -\frac{1}{y(1+y^2)}$$

$$\text{IF} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$$

Hence, the general solution is

$$yx = \int y\left[-\frac{1}{y(1+y^2)}\right]dy + c$$

$$= -\int \frac{1}{1+y^2}\,dy + c$$

$$= -\tan^{-1} y + c$$

$$xy = c - \tan^{-1} y$$

Example 12

Solve $y\log y\,dx+(x-\log y)\,dy=0.$

Solution

Rewriting the equation,

$$y\log y\frac{dx}{dy}+x-\log y=0$$

$$\frac{dx}{dy}+\left(\frac{1}{y\log y}\right)x=\frac{1}{y}$$

The equation is linear in x.

$$P=\frac{1}{y\log y},\quad Q=\frac{1}{y}$$

$$\text{IF}=e^{\int\frac{1}{y\log y}dy}=e^{\log(\log y)}\qquad\left[\because \int\frac{f'(y)}{f(y)}dy=\log f(y)+c\right]$$

$$=\log y$$

Hence, the general solution is

$$(\log y)x=\int(\log y)\frac{1}{y}dy+c$$

$$x\log y=\frac{(\log y)^2}{2}+c$$

Example 13

Solve $(1+\sin y)\,dx=(2y\cos y-x\sec y-x\tan y)\,dy.$

Solution

Rewriting the equation,

$$(1+\sin y)\frac{dx}{dy}=2y\cos y-(\sec y+\tan y)x$$

$$(1+\sin y)\frac{dx}{dy}+\left(\frac{1+\sin y}{\cos y}\right)x=2y\cos y$$

$$\frac{dx}{dy}+\left(\frac{1}{\cos y}\right)x=\frac{2y\cos y}{1+\sin y}$$

The equation is linear in x.

$$P=\frac{1}{\cos y},\quad Q=\frac{2y\cos y}{1+\sin y}$$

$$\text{IF}=e^{\int\frac{1}{\cos y}dy}=e^{\int\sec y\,dy}=e^{\log(\sec y+\tan y)}=\sec y+\tan y$$

Hence, the general solution is

$$(\sec y+\tan y)x=\int(\sec y+\tan y)\left(\frac{2y\cos y}{1+\sin y}\right)dy+c$$

$$=2\int\left(\frac{1+\sin y}{\cos y}\right)\left(\frac{y\cos y}{1+\sin y}\right)dy+c$$

$$=2\int y\,dy+c$$

$$(\sec y+\tan y)x=y^2+c$$

Example 14

Solve $(1+y^2)\,dx=(\tan^{-1}y-x)\,dy.$ **[Summer 2013]**

Solution

Rewriting the equation,

$$(1+y^2)\frac{dx}{dy}=\tan^{-1}y-x$$

$$\frac{dx}{dy}+\left(\frac{1}{1+y^2}\right)x=\frac{\tan^{-1}y}{1+y^2}$$

The equation is linear in x.

$$P=\frac{1}{1+y^2},\quad Q=\frac{\tan^{-1}y}{1+y^2}$$

$$\text{IF}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$$

Hence, the general solution is

$$(e^{\tan^{-1}y})x=\int e^{\tan^{-1}y}\left(\frac{\tan^{-1}y}{1+y^2}\right)dy+c$$

Let $\tan^{-1} y = t$

$$\frac{1}{1+y^2}\,dy = dt$$

$$(e^{\tan^{-1} y})x = \int e^t t\,dt + c$$

$$= te^t - e^t + c$$

$$= e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$$

$$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$$

Example 15

Solve $dr + (2r\cot\theta + \sin 2\theta)\,d\theta = 0.$

Solution

Rewriting the equation,

$$\frac{dr}{d\theta} + (2\cot\theta)r = -\sin 2\theta$$

The equation is linear in r.

$$P = 2\cot\theta, \quad Q = -\sin 2\theta$$

$$\text{IF} = e^{\int 2\cot\theta\,d\theta} = e^{2\log\sin\theta} = e^{\log\sin^2\theta} = \sin^2\theta$$

Hence, the general solution is

$$\sin^2\theta \cdot r = \int \sin^2\theta(-\sin 2\theta)\,d\theta + c$$

$$= -2\int \sin^3\theta\cos\theta\,d\theta + c$$

$$= -2\frac{\sin^4\theta}{4} + c \quad \left[\because \int [f(\theta)]^n f'(\theta)\,d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$r\sin^2\theta = -\frac{\sin^4\theta}{2} + c$$

Example 16

Solve $\cosh x\dfrac{dy}{dx} = 2\cosh^2 x\sinh x - y\sinh x.$

Solution

$$\frac{dy}{dx}+(\tanh x)y = 2\cosh x \sinh x$$

The equation is linear in y.

$$P = \tanh x, \quad Q = 2\cosh x \sinh x$$

$$IF = e^{\int \tanh x\, dx} = e^{\int \frac{\sinh x}{\cosh x} dx} = e^{\log \cosh x} = \cosh x$$

Hence, the general solution is

$$(\cosh x)y = \int \cosh x (2\cosh x \sinh x)\, dx + c$$

$$= 2\int \cosh^2 x \cdot \sinh x\, dx + c$$

$$= 2\frac{\cosh^3 x}{3} + c \quad \left[\because \int [f(\theta)]^n f'(\theta)\, d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$y\cosh x = \frac{2}{3}\cosh^3 x + c$$

Example 17

Solve $x(x-1)\dfrac{dy}{dx} - (x-2)y = x^3(2x-1)$.

Solution

$$\frac{dy}{dx} - \frac{(x-2)}{x(x-1)}y = \frac{x^2(2x-1)}{(x-1)}$$

The equation is linear in y.

$$P = -\frac{x-2}{x(x-1)}, \qquad Q = \frac{x^2(2x-1)}{x-1}$$

$$= -\left(\frac{2}{x} - \frac{1}{x-1}\right)$$

$$IF = e^{\int\left(-\frac{2}{x}+\frac{1}{x-1}\right)dx} = e^{-2\log x + \log(x-1)} = e^{\log\left(\frac{x-1}{x^2}\right)} = \frac{x-1}{x^2}$$

Hence, the general solution is

$$\left(\frac{x-1}{x^2}\right)\cdot y = \int \left(\frac{x-1}{x^2}\right)\cdot x^2 \left(\frac{2x-1}{x-1}\right) dx + c$$

$$= x^2 - x + c$$

$$y = \frac{x^3(x-1)}{x-1} + \frac{cx^2}{x-1}$$

$$y = x^3 + \frac{cx^2}{x-1}$$

Example 18

Solve $(x^2-1)\sin x\dfrac{dy}{dx} + [2x\sin x + (x^2-1)\cos x]y = (x^2-1)\cos x.$

Solution

$$\frac{dy}{dx} + \left(\frac{2x}{x^2-1} + \cot x\right)y = \cot x$$

The equation is linear in y.

$$P = \frac{2x}{x^2-1} + \cot x, \quad Q = \cot x$$

$$IF = e^{\int\left(\frac{2x}{x^2-1}+\cot x\right)dx} = e^{\log(x^2-1)+\log\sin x} = e^{\log\left[(x^2-1)\sin x\right]} = (x^2-1)\sin x$$

Hence, the general solution is

$$(x^2-1)\sin x \cdot y = \int (x^2-1)\sin x \cdot \cot x\, dx + c$$

$$= \int (x^2-1)\cos x\, dx + c$$

$$= (x^2-1)\sin x - 2x(-\cos x) + 2(-\sin x) + c$$

$$y(x^2-1)\sin x = (x^2-3)\sin x + 2x\cos x + c$$

Example 19

If $\dfrac{dy}{dx} + y\tan x = \sin 2x,\ y(0) = 0,$ *show that the maximum value of* y *is* $\dfrac{1}{2}$.

Solution

The equation is linear in y.

$$P = \tan x, \quad Q = \sin 2x$$

$$IF = e^{\int \tan x\, dx} = e^{\log\sec x} = \sec x$$

Hence, the general solution is

$$(\sec x)y = \int \sec x \cdot \sin 2x \, dx + c$$

$$= \int \sec x \cdot 2\sin x \cos x \, dx + c$$

$$= 2\int \sin x \, dx + c$$

$$y \sec x = -2\cos x + c$$

$$y = -2\cos^2 x + c\cos x \quad \text{...(1)}$$

Given $y(0) = 0$

Putting $x = 0, y = 0$ in Eq. (1),

$$0 = -2\cos 0 + c\cos 0 = -2 + c$$

$$c = 2$$

Hence, the general solution is

$$y = -2\cos^2 x + 2\cos x \quad \text{...(2)}$$

For maximum or minimum value,

$$\frac{dy}{dx} = 0$$

$$-4\cos x(-\sin x) - 2\sin x = 0$$

$$2\sin x(2\cos x - 1) = 0$$

$\sin x = 0, x = 0$ and $2\cos x - 1 = 0$, $\cos x = \frac{1}{2}$, $x = \frac{\pi}{3}$

$x = 0$ and $x = \frac{\pi}{3}$ are the points of extreme values.

Now,
$$\frac{dy}{dx} = 2\sin 2x - 2\sin x$$

$$\frac{d^2y}{dx^2} = 4\cos 2x - 2\cos x$$

When $x = 0$, $\frac{d^2y}{dx^2} = 2 > 0$, y is minimum at $x = 0$.

When $x = \frac{\pi}{3}$, $\frac{d^2y}{dx^2} = 4\cos\frac{2\pi}{3} - 2\cos\frac{\pi}{3} = 4\left(-\frac{1}{2}\right) - 2\left(\frac{1}{2}\right) = -3 < 0$, y is maximum at $x = \frac{\pi}{3}$.

Putting $x = \frac{\pi}{3}$ in Eq. (2), we get maximum value of y.

$$y_{max} = -2\cos^2\frac{\pi}{3} + 2\cos\frac{\pi}{3} = -\frac{1}{2} + 1 = \frac{1}{2}$$

EXERCISE 4.3

Solve the following differential equations:

1. $x^2\frac{dy}{dx} = 3x^2 - 2xy + 1$

$$\left[\textbf{Ans.: } y = \frac{c}{x^2} + x + \frac{1}{x}\right]$$

2. $(2y - 3x)dx + x\,dy = 0$

$$\left[\textbf{Ans.: } x^2y = x^3 + c\right]$$

3. $(x+1)\frac{dy}{dx} - 2y = (x+1)^4$

$$\left[\textbf{Ans.: } y = \left(\frac{x^2}{2} + x + c\right)(x+1)^2\right]$$

4. $\frac{dy}{dx} + y\cot x = \cos x$

$$\left[\textbf{Ans.: } y\sin x = \frac{\sin^2 x}{2} + c\right]$$

5. $\frac{1}{x}\frac{dy}{dx} + 2y = e^{-x^2}$

$$\left[\textbf{Ans.: } ye^{x^2} = \frac{x^2}{2} + c\right]$$

6. $(y+1)dx + [x - (y+2)e^y]dy = 0$

$$\left[\textbf{Ans.: } (y+1)(x - e^y) = c\right]$$

7. $dx + x\,dy = e^{-y}\sec^2 y\,dy$

$$\left[\textbf{Ans.: } xe^y - \tan y + c\right]$$

8. $(1+x)\frac{dy}{dx} - y = e^x(x+1)^2$

$$\left[\textbf{Ans.: } y = (1+x)(e^x + c)\right]$$

9. $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1$ $\left[\text{Ans.: } ye^{2\sqrt{x}}=2\sqrt{x}+c\right]$

10. $x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1$ $\left[\text{Ans.: } xy=\sin x+c\cos x\right]$

11. $\cos^2 x\frac{dy}{dx}+y=\tan x$ $\left[\text{Ans.: } y=\tan x-1+ce^{-\tan x}\right]$

12. $(2x+y^4)\frac{dy}{dx}=y$ $\left[\text{Ans.: } \frac{2x}{y^2}=y^2+c\right]$

13. $\sqrt{a^2+x^2}\frac{dy}{dx}+y=\sqrt{a^2+x^2}-x$ $\left[\text{Ans.: } \left(x+\sqrt{x^2+a^2}\right)y=a^2x+c\right]$

14. $\frac{dy}{dx}=\frac{1}{x+e^y}$ $\left[\text{Ans.: } xe^{-y}=c+y\right]$

15. $\frac{dy}{dx}-\left(\frac{3}{x}\right)y=x^3, y(1)=4$ $\left[\text{Ans.: } y=x^3(x+3)\right]$

16. $(1+x^2)\frac{dy}{dx}-2xy=2x(1+x^2),\ y(0)=1$ $\left[\text{Ans.: } y=(1+x^2)[1+\log(1+x^2)]\right]$

17. $x\frac{dy}{dx}-3y=x^4(e^x+\cos x)-2x^2, y(\pi)=\pi^3e^\pi+2\pi^2$ $\left[\text{Ans.: } y=2x^2+(e^x+\sin x)x^3\right]$

18. If $\frac{dy}{dx}+2y\tan x=\sin x,\ y\left(\frac{\pi}{3}\right)=0$, show that maximum value of y is $\frac{1}{8}$.

19. $\frac{dy}{dx}+\frac{y}{x}=\log x,\ y(1)=1$

$$\left[\text{Ans.: } y=\frac{x\log x}{2}-\frac{x}{4}+\frac{5}{4x}\right]$$

20. $\frac{dy}{dx}+2xy=xe^{-x^2}$

$$\left[\text{Ans.: } ye^{x^2}=\frac{x^2}{2}+c\right]$$

4.3.7 Nonlinear Differential Equations Reducible to Linear Form

Type 1 Bernoulli's Equation

The equation of the form

$$\frac{dy}{dx}+Py=Qy^n \quad ...(4.13)$$

where P and Q are functions of x or constants is a nonlinear equation, known as Bernoulli's equation. This equation can be made linear using the following method:

Dividing Eq. (4.13) by y^n,

$$\frac{1}{y^n}\frac{dy}{dx}+\frac{P}{y^{n-1}}=Q \quad ...(4.14)$$

Let $\frac{1}{y^{n-1}}=v$

$$\frac{(1-n)}{y^n}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx}=\frac{1}{(1-n)}\cdot\frac{dv}{dx}$$

Substituting in Eq. (4.14),

$$\frac{1}{1-n}\cdot\frac{dv}{dx}+Pv=Q$$

$$\frac{dv}{dx}+(1-n)Pv=Q$$

The equation is linear in v and can be solved using the method of linear differential equations. Finally, substituting $v=\frac{1}{y^{n-1}}$, we get the solution of Eq. (4.13).

Example 1

Solve $\frac{dy}{dx}+\frac{2y}{x}=y^2x^2$.

Solution

The equation is in Bernoulli's form.
Dividing the equation by y^2,

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}\cdot\frac{2}{x}=x^2 \quad ...(1)$$

Let $\frac{1}{y}=v$, $\quad -\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$

Substituting in Eq. (1),

$$-\frac{dv}{dx}+\left(\frac{2}{x}\right)v=x^2$$

$$\frac{dv}{dx}-\left(\frac{2}{x}\right)v=-x^2 \quad ...(2)$$

The equation is linear in v.

$$P=-\frac{2}{x}, Q=-x^2$$

$$IF=e^{\int -\frac{2}{x}dx}=e^{-2\log x}=e^{\log x^{-2}}=x^{-2}=\frac{1}{x^2}$$

The general solution of Eq. (2) is

$$\frac{1}{x^2}v=\int\frac{1}{x^2}(-x^2)dx+c$$

$$=\int -dx+c$$

$$=-x+c$$

$$v=-x^3+cx^2$$

Hence, $\quad \frac{1}{y}=-x^3+cx^2$

Example 2

Solve $\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2}$. **[Winter 2017]**

Solution

The equation is in Bernoulli's form.

Dividing the equation by y^2,

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}\cdot\frac{1}{x}=\frac{1}{x^2} \qquad ...(1)$$

Let $\frac{1}{y}=v$,

$$-\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$$

Substituting in Eq. (1),

$$-\frac{dv}{dx}+\left(\frac{1}{x}\right)v=\frac{1}{x^2}$$

$$\frac{dv}{dx}-\left(\frac{1}{x}\right)v=-\frac{1}{x^2} \qquad ...(2)$$

The equation is linear in v.

$$P=-\frac{1}{x},\ Q=-\frac{1}{x^2}$$

$$\text{IF}=e^{\int -\frac{1}{x}dx}=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$$

The general solution of Eq. (2) is

$$\frac{1}{x}v=\int\frac{1}{x}\left(-\frac{1}{x^2}\right)dx+c$$

$$=-\int x^{-3}dx+c$$

$$=-\frac{x^{-2}}{-2}+c$$

$$=\frac{1}{2x^2}+c$$

$$v=\frac{1}{2x}+cx$$

Hence,
$$\frac{1}{y}=\frac{1}{2x}+cx$$

Example 3

Solve $\frac{dy}{dx}+y=y^2(\cos x-\sin x)$.

Solution

The equation is in Bernoulli's form.

Dividing the equation by y^2,

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}=\cos x-\sin x \quad ...(1)$$

Let $\frac{1}{y}=v,\quad -\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$

Substituting in Eq. (1),

$$-\frac{dv}{dx}+v=\cos x-\sin x$$

$$\frac{dv}{dx}-v=-\cos x+\sin x \quad ...(2)$$

The equation is linear in v.

$$P=-1,\ Q=-\cos x+\sin x$$

$$IF=e^{\int -dx}=e^{-x}$$

The general solution of Eq. (2) is

$$e^{-x}\cdot v=\int e^{-x}(-\cos x+\sin x)\,dx+c$$

$$=-\int e^{-x}\cos x\,dx+\int e^{-x}\sin x\,dx+c$$

$$=-\left[\frac{e^{-x}}{2}(-\cos x+\sin x)\right]+\left[\frac{e^{-x}}{2}(-\sin x-\cos x)\right]+c$$

$$e^{-x}v=-e^{-x}\sin x+c$$

$$v=-\sin x+ce^{x}$$

Hence, $\frac{1}{y}=-\sin x+ce^{x}$

Example 4

Solve $xy(1+xy^2)\frac{dy}{dx}=1.$

Solution

Rewriting the equation, $\frac{dx}{dy}=xy+x^2y^3$

$$\frac{dx}{dy}-xy=x^2y^3$$

The equation is in Bernoulli's form, where x is a dependent variable.

Dividing the equation by x^2,

$$\frac{1}{x^2}\frac{dx}{dy}-\left(\frac{1}{x}\right)y=y^3 \qquad ...(1)$$

Let $-\frac{1}{x}=v, \qquad \frac{1}{x^2}\frac{dx}{dy}=\frac{dv}{dy}$

Substituting in Eq. (1),

$$\frac{dv}{dy}+vy=y^3 \qquad ...(2)$$

The equation is linear in v.

$$P=y, \qquad Q=y^3$$

$$IF=e^{\int y dy}=e^{\frac{y^2}{2}}$$

The general solution of Eq. (2) is

$$e^{\frac{y^2}{2}}\cdot v=\int e^{\frac{y^2}{2}}y^3 dy+c$$

Putting $\frac{y^2}{2}=t, \quad y dy=dt$

$$e^{\frac{y^2}{2}}\cdot v=\int e^t\cdot 2t\,dt+c$$

$$=2(e^t t-e^t)+c$$

$$=2e^t(t-1)+c$$

$$=2e^{\frac{y^2}{2}}\left(\frac{y^2}{2}-1\right)+c$$

$$v=y^2-2+ce^{-\frac{y^2}{2}}$$

Hence, $$-\frac{1}{x}=y^2-2+ce^{\frac{-y^2}{2}}$$

Example 5

Solve $y^4dx=\left(x^{-\frac{3}{4}}-y^3x\right)dy.$

Solution

Rewriting the equation,

$$\frac{dx}{dy} = \frac{x^{-\frac{3}{4}}}{y^4} - \frac{x}{y}$$

$$\frac{dx}{dy} + \frac{x}{y} = \frac{x^{-\frac{3}{4}}}{y^4}$$

The equation is in Bernoulli's form, where x is a dependent variable.

Dividing the equation by $x^{-\frac{3}{4}}$,

$$x^{\frac{3}{4}}\frac{dx}{dy} + x^{\frac{7}{4}}\left(\frac{1}{y}\right) = \frac{1}{y^4}$$

Let $\quad x^{\frac{7}{4}} = v, \quad \frac{7}{4}x^{\frac{3}{4}}\frac{dx}{dy} = \frac{dv}{dy}$

Substituting in Eq. (1),

$$\frac{4}{7}\frac{dv}{dy} + \left(\frac{1}{y}\right)v = \frac{1}{y^4}$$

$$\frac{dv}{dy} + \left(\frac{7}{4y}\right)v = \frac{7}{4y^4}$$

The equation is linear in v.

$$P = \frac{7}{4y}, \qquad Q = \frac{7}{4y^4}$$

$$\text{IF} = e^{\int \frac{7}{4y}dy} = e^{\frac{7}{4}\log y} = e^{\log y^{\frac{7}{4}}} = y^{\frac{7}{4}}$$

The general solution of Eq. (2) is

$$y^{\frac{7}{4}}v = \int y^{\frac{7}{4}} \cdot \frac{7}{4y^4}dy + c$$

$$= \frac{7}{4}\int y^{\frac{-9}{4}}dy + c$$

$$= \frac{7}{4}\left(\frac{4y^{-\frac{5}{4}}}{-5}\right) + c$$

$$y^{\frac{7}{4}}v = -\frac{7}{5}y^{-\frac{5}{4}} + c$$

$$y^3 v = -\frac{7}{5} + cy^{\frac{5}{4}}$$

Hence,

$$y^3 x^{\frac{7}{4}} = -\frac{7}{5} + cy^{\frac{5}{4}}$$

Example 6

Solve $\dfrac{dr}{d\theta} = r\tan\theta - \dfrac{r^2}{\cos\theta}$.

Solution

Rewriting the equation, $\dfrac{dr}{d\theta} - r\tan\theta = -\dfrac{r^2}{\cos\theta}$

The equation is in Bernoulli's form, where r is a dependent variable.
Dividing the equation by r^2,

$$\frac{1}{r^2}\frac{dr}{d\theta} - \frac{\tan\theta}{r} = -\frac{1}{\cos\theta} \quad \text{...(1)}$$

Let $-\dfrac{1}{r} = v$, $\qquad \dfrac{1}{r^2}\dfrac{dr}{d\theta} = \dfrac{dv}{d\theta}$

Substituting in Eq. (1),

$$\frac{dv}{d\theta} + v\tan\theta = -\frac{1}{\cos\theta} \quad \text{...(2)}$$

The equation is linear in v.

$$P = \tan\theta, \qquad Q = -\frac{1}{\cos\theta}$$

$$IF = e^{\int \tan\theta\, d\theta} = e^{\log\sec\theta} = \sec\theta$$

The general solution of Eq. (2) is

$$\sec\theta \cdot v = \int \sec\theta\left(-\frac{1}{\cos\theta}\right)d\theta + c$$

$$= \int -\sec^2\theta\, d\theta + c$$

$$= -\tan\theta + c$$

Hence,

$$\sec\theta\left(-\frac{1}{r}\right) = -\tan\theta + c$$

$$\frac{\sec\theta}{r} = \tan\theta - c$$

Type 2

The equation of the form $f'(y)\frac{dy}{dx} + Pf(y) = Q$ ...(4.15)

where P and Q are functions of x or constants can be reduced to the linear form by putting $f(y) = v$, $f'(y)\frac{dy}{dx} = \frac{dv}{dx}$ in Eq. (4.15)

$$\frac{dv}{dx} + Pv = Q \qquad ...(4.16)$$

Equation (4.16) is linear in v and can be solved using the method of linear differential equations. Finally, substituting $v = f(y)$, we get the solution of Eq. (4.15).

Example 1

Solve $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$. **[Summer 2015]**

Solution

Dividing the equation by e^y,

$$e^{-y}\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x^2} \qquad ...(1)$$

Let $e^{-y} = v$, $\quad e^{-y}\frac{dy}{dx} = \frac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x^2} \qquad ...(2)$$

The equation is linear in v.

$$P = \frac{1}{x}, \qquad Q = \frac{1}{x^2}$$

$$\text{IF} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

The general solution of Eq. (2) is

$$xv = \int x \cdot \frac{1}{x^2} + c$$

$$= \int \frac{1}{x} + c$$

$$= \log x + c$$

$$v = \frac{1}{x}(\log x + c)$$

Hence,
$$e^{-y} = \frac{1}{x}(\log x + c)$$

Example 2

Solve $\frac{dy}{dx} + \frac{y}{x} = x^3 y^3$. **[Winter 2015]**

Solution

Dividing the equation by y^3,

$$y^{-3}\frac{dy}{dx} + \frac{y^{-2}}{x} = x^3 \quad ...(1)$$

Let $y^{-2} = v$

$$-2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$$

$$y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$$

Substituting in Eq. (1),

$$-\frac{1}{2}\frac{dv}{dx} + \frac{v}{x} = x^3$$

$$\frac{dv}{dx} - \frac{2v}{x} = -2x^3 \quad ...(2)$$

The equation is linear in v.

$$P = -\frac{2}{x}, \qquad Q = -2x^3$$

$$\text{IF} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

The general solution of Eq. (2) is

$$\frac{1}{x^2}v = \int \frac{1}{x^2}(-2x^3)\,dx + c$$

$$= -2\int x\,dx + c$$

$$= -2\frac{x^2}{2} + c$$

$$= -x^2 + c$$

Hence, $$\frac{1}{x^2y^2} = -x^2 + c$$

Example 3

Solve $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y.$

Solution

Dividing the equation by $\cos^2 y$,

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{2\sin y\cos y}{\cos^2 y}x = x^3$$

$$\sec^2 y\frac{dy}{dx} + 2\tan y\cdot x = x^3 \quad \text{...(1)}$$

Let $\tan y = v$, $\quad \sec^2 y\dfrac{dy}{dx} = \dfrac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx} + 2vx = x^3 \quad \text{...(2)}$$

The equation is linear in v.

$$P = 2x, \quad Q = x^3$$

$$\text{IF} = e^{\int 2x\,dx} = e^{x^2}$$

The general solution of Eq. (2) is

$$e^{x^2}v = \int e^{x^2}\cdot x^3\,dx + c$$

Putting $x^2 = t$, $2x\,dx = dt$, $x\,dx = \frac{dt}{2}$

$$e^{x^2} v = \int e^t t \frac{dt}{2} + c$$

$$= \frac{1}{2}\left(te^t - e^t\right) + c$$

$$= \frac{1}{2} e^{x^2}(x^2 - 1) + c$$

$$v = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$$

Hence,

$$\tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$$

Example 4

Solve $x\frac{dy}{dx} + y\log y = xye^x$.

Solution

Dividing the equation by xy,

$$\frac{1}{y}\frac{dy}{dx} + \frac{\log y}{x} = e^x \quad \text{...(1)}$$

Let $\log y = v$, $\frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx} + \frac{v}{x} = e^x \quad \text{...(2)}$$

The equation is linear in v.

$$P = \frac{1}{x}, \qquad Q = e^x$$

$$\text{IF} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

The general solution of Eq. (2) is

$$xv = \int xe^x dx + c$$

$$= xe^x - e^x + c$$

$$= e^x(x - 1) + c$$

Hence,
$$x \log y = e^x (x-1) + c.$$

Example 5

Solve $\dfrac{dy}{dx} + \tan x \tan y = \cos x \sec y.$

Solution

Dividing the equation by sec y,

$$\frac{1}{\sec y}\frac{dy}{dx} + \tan x \sin y = \cos x$$

$$\cos y \frac{dy}{dx} + \tan x \sin y = \cos x \quad \text{...(1)}$$

Let $\sin y = v$, $\cos y \dfrac{dy}{dx} = \dfrac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx} + \tan x \cdot v = \cos x \quad \text{...(2)}$$

The equation is linear in v.

$$P = \tan x, \qquad Q = \cos x$$

$$IF = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

The general solution of Eq. (2) is

$$\sec x \cdot v = \int \sec x \cdot \cos x\, dx + c$$
$$= \int dx + c$$
$$= x + c$$

Hence,
$$\sec x \cdot \sin y = x + c$$

Example 6

Solve $\dfrac{dy}{dx} = e^{x-y}(e^x - e^y).$

Solution

Dividing the equation by e^{-y},

$$e^y \frac{dy}{dx} = e^{2x} - e^x e^y$$

$$e^{y}\frac{dy}{dx}+e^{x}e^{y}=e^{2x} \qquad \text{...(1)}$$

Let $e^{y}=v$, $$e^{y}\frac{dy}{dx}=\frac{dv}{dx}$$

Substituting in Eq. (1),

$$\frac{dv}{dx}+e^{x}v=e^{2x} \qquad \text{...(2)}$$

The equation is linear in v.

$$P=e^{x}, \quad Q=e^{2x}$$

$$IF=e^{\int e^{x}dx}=e^{e^{x}}$$

The general solution of Eq. (2) is

$$e^{e^{x}}\cdot v=\int e^{e^{x}}\cdot e^{2x}\,dx+c$$

Let $e^{x}=t$, $e^{x}dx=dt$

$$e^{e^{x}}\cdot v=\int e^{t}t\,dt+c$$

$$=e^{t}\cdot t-e^{t}+c$$

$$=e^{t}(t-1)+c$$

$$=e^{e^{x}}(e^{x}-1)+c$$

$$v=e^{x}-1+ce^{-e^{x}}$$

Hence, $$e^{y}=e^{x}-1+ce^{-e^{x}}$$

Example 7

Solve $\dfrac{dy}{dx}=\dfrac{y^{3}}{e^{2x}+y^{2}}$.

Solution

Rewriting the equation, $\dfrac{dx}{dy}=\dfrac{e^{2x}}{y^{3}}+\dfrac{1}{y}$

$$e^{-2x}\frac{dx}{dy}-\frac{e^{-2x}}{y}=\frac{1}{y^{3}} \qquad \text{...(1)}$$

Let $e^{-2x}=v$, $\quad -2e^{-2x}\dfrac{dx}{dy}=\dfrac{dv}{dy}$, $\quad e^{-2x}\dfrac{dx}{dy}=-\dfrac{1}{2}\dfrac{dv}{dy}$

Substituting in Eq. (1),

$$-\frac{1}{2}\frac{dv}{dy}-\frac{v}{y}=\frac{1}{y^3}$$

$$\frac{dv}{dy}+\frac{2}{y}\cdot v=\frac{-2}{y^3} \qquad \text{...(2)}$$

The equation is linear in v.

$$P=\frac{2}{y}, \qquad Q=-\frac{2}{y^3}$$

$$IF=e^{\int\frac{2}{y}dy}=e^{2\log y}=e^{\log y^2}=y^2$$

The general solution of Eq. (2) is

$$y^2\cdot v=\int y^2\left(-\frac{2}{y^3}\right)dy+c$$

$$=-2\int\frac{1}{y}dy+c$$

$$=-2\log y+c$$

Hence,
$$y^2e^{-2x}=-2\log y+c$$

Example 8

Solve $\dfrac{dz}{dx}+\dfrac{z}{x}\log z=\dfrac{z}{x^2}(\log z)^2.$

Solution

Rewriting the equation,

$$\frac{1}{z(\log z)^2}\frac{dz}{dx}+\frac{1}{\log z}\cdot\frac{1}{x}=\frac{1}{x^2} \qquad \text{...(1)}$$

Let $\dfrac{-1}{\log z}=v, \qquad \dfrac{1}{(\log z)^2}\cdot\dfrac{1}{z}\dfrac{dz}{dx}=\dfrac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx}-\frac{v}{x}=\frac{1}{x^2} \qquad \text{...(2)}$$

The equation is linear in v.

$$P = -\frac{1}{x}, \qquad Q = \frac{1}{x^2}$$

$$\text{IF} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

The general solution of Eq. (2) is

$$\frac{1}{x}\cdot v = \int \frac{1}{x}\cdot\frac{1}{x^2}dx + c$$

$$= \int x^{-3}dx + c$$

$$= \frac{x^{-2}}{-2} + c$$

Hence,

$$\frac{1}{x}\left(-\frac{1}{\log z}\right) = -\frac{1}{2x^2} + c$$

$$\frac{1}{x\log z} = \frac{1}{2x^2} - c$$

Example 9

Solve $(\sec x \tan x \tan y - e^x)\,dx + \sec x \sec^2 y\,dy = 0$.

Solution

Rewriting the equation,

$$\sec x \sec^2 y\frac{dy}{dx} + \sec x \tan x \tan y - e^x = 0$$

$$\sec^2 y\frac{dy}{dx} + \tan x \tan y = \frac{e^x}{\sec x} \qquad \text{...(1)}$$

Let $\tan y = v$, $\quad \sec^2 y\frac{dy}{dx} = \frac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx} + (\tan x)v = e^x \cos x \qquad \text{...(2)}$$

The equation is linear in v.

$$P = \tan x, \qquad Q = e^x \cos x$$

$$\text{IF} = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x$$

The general solution of Eq. (2) is

$$(\sec x)v = \int \sec x\, e^x \cos x\, dx + c$$

$$= \int e^x dx + c$$

$$= e^x + c$$

Hence, $\sec x \tan y = e^x + c$

Example 10

Solve $\dfrac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1.$

Solution

$$\frac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1$$

Let $x + y = z$, $\quad 1 + \dfrac{dy}{dx} = \dfrac{dz}{dx}, \dfrac{dy}{dx} = \dfrac{dz}{dx} - 1$

Substituting in Eq. (1),

$$\frac{dz}{dx} - 1 + xz = x^3z^3 - 1$$

$$\frac{dz}{dx} + xz = x^3z^3$$

Dividing the Eq. (2) by z^3,

$$\frac{1}{z^3}\frac{dz}{dx} + \frac{x}{z^2} = x^3$$

Let $\quad \dfrac{1}{z^2} = v, \ -\dfrac{2}{z^3}\dfrac{dz}{dx} = \dfrac{dv}{dx}, \ \dfrac{1}{z^3}\dfrac{dz}{dx} = -\dfrac{1}{2}\dfrac{dv}{dx}$

Substituting in Eq. (3),

$$-\frac{1}{2}\frac{dv}{dx} + xv = x^3$$

$$\frac{dv}{dx} - 2xv = -2x^3$$

The equation is linear in v.

$$P = -2x, \qquad Q = -2x^3$$

$$\text{IF} = e^{\int -2x\,dx} = e^{-x^2}$$

The general solution of Eq. (4) is

$$e^{-x^2} \cdot v = \int e^{-x^2}(-2x^3)dx + c$$

Let $x^2 = t$, $2x\,dx = dt$

$$e^{-x^2} \cdot v = -\int te^{-t}dt + c$$
$$= te^{-t} + e^{-t} + c$$
$$= (x^2+1)e^{-x^2} + c$$
$$v = (x^2+1) + ce^{x^2}$$

Substituting value of v,

$$\frac{1}{z^2} = (x^2+1) + ce^{x^2}$$

Hence,
$$\frac{1}{(x+y)^2} = (x^2+1) + ce^{x^2}$$

EXERCISE 4.4

Solve the following differential equations:

1. $\frac{dy}{dx} = x^3y^3 - xy$

$$\left[\textbf{Ans.: } \frac{1}{y^2} = x^2 + 1 + ce^{x^2}\right]$$

2. $x^2y - x^3\frac{dy}{dx} = y^4 \cos x$

$$\left[\textbf{Ans.: } x^3 = y^3(3\sin x - c)\right]$$

3. $x(3x+2y^2)dx + 2y(1+x^2)dy = 0$

$$\left[\textbf{Ans.: } y^2(1+x^2) = -x^3 + c\right]$$

4. $y\,dx + x(1-3x^2y^2)dy = 0$

$$\left[\textbf{Ans.: } y^6 = ce^{-\frac{1}{x^2y^2}}\right]$$

5. $x\,dy - [y + xy^3(1+\log x)]dx = 0$

$$\left[\textbf{Ans.: } x^2 = -\frac{2}{3}x^3y^2\left(\frac{2}{3} + \log x\right) + cy^2\right]$$

6. $\frac{dy}{dx}+y=y^2e^x$

$\left[\text{Ans.: } -\frac{e^{-x}}{y}=x+c\right]$

7. $x\,dy+y\,dx=x^3y^6\,dx$

$\left[\text{Ans.: } \frac{2}{y^5}=5x^3+cx^5\right]$

8. $x\frac{dy}{dx}+y=y^3x^{n+1}$

$\left[\text{Ans.: } \frac{n-1}{y^2}=cx^2-2x^{n+1}\right]$

9. $xy(1+x^2y^2)\frac{dy}{dx}=1$

$\left[\text{Ans.: } \frac{1}{x^2}=ce^{-y^2}-y^2+1\right]$

10. $x^2y^3\,dx+(x^3y-2)\,dy=0$

$\left[\text{Ans.: } x^3=\frac{2}{y}+\frac{2}{3}+ce^{\frac{3}{y}}\right]$

11. $y\frac{dx}{dy}=x-yx^2\cos y$

$\left[\text{Ans.: } \frac{y}{x}=y\sin y+\cos y+c\right]$

12. $\frac{dy}{dx}=\frac{e^y}{x^2}-\frac{1}{x}$

$\left[\text{Ans.: } 2xe^{-y}=1+2cx^2\right]$

13. $y\frac{dy}{dx}+\frac{4}{3}x-\frac{y^2}{3x}=0$

$\left[\text{Ans.: } y^2x^{-\frac{2}{3}}+2x^{\frac{4}{3}}=c\right]$

14. $\frac{dy}{dx}+(2x\tan^{-1}y-x^3)(1+y^2)=0$

$\left[\text{Ans.: } 2\tan^{-1}y=(x^2-1)+ce^{-x^2}\right]$

15. $\tan y\frac{dy}{dx}+\tan x=\cos y\cos^2 x$

$\left[\text{Ans.: } \sec y\sec x=\sin x+c\right]$

16. $(y+e^y-e^{-x})dx+(1+e^y)dy=0$

$\left[\text{Ans.: } y+e^y=(x+c)e^{-x}\right]$

17. $x^2\cos y\frac{dy}{dx}=2x\sin y-1$

$\left[\text{Ans.: } 3x\sin y=cx^3+1\right]$

18. $4x^2y\frac{dy}{dx}=3x(3y^2+2)+2(3y^2+2)^3$

$\left[\text{Ans.: } 4x^9=(3y^2+2)^2(-3x^8+c)\right]$

19. $\frac{dy}{dx}+\frac{1}{x}\tan y=\frac{1}{x^2}\tan y\sin y$

$\left[\text{Ans.: } \operatorname{cosec} y=1+cx\right]$

20. $x\frac{dy}{dx}+3y=x^4e^{\frac{1}{x^2}}y^3$

$\left[\text{Ans.: } \frac{1}{y^2}=\left(e^{\frac{1}{x^2}}+c\right)x^6\right]$

21. $x^2\frac{dy}{dx}=\sin^2 y-(\sin y\cos y)x$

$\left[\text{Ans.: } \cot y=\frac{1}{2x}+cx\right]$

22. $\dfrac{dr}{d\theta} = \dfrac{r\sin\theta - r^2}{\cos\theta}$

$$\left[\text{Ans.: } \frac{1}{r} = c\cos\theta + \sin\theta\right]$$

23. $\cos x \dfrac{dy}{dx} + 4y\sin x = 4\sqrt{y}\sec x$

$$\left[\text{Ans.: } \sqrt{y}\sec^2 x = 2\left(\tan x + \frac{\tan^3 x}{3}\right) + c\right]$$

24. $\sin y \dfrac{dy}{dx} = \cos x(2\cos y - \sin^2 x)$

$$\left[\text{Ans.: } 4\cos y = 2\sin^2 x - 2\sin x + 1 - 4ce^{-2\sin x}\right]$$

25. $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x$

$$\left[\text{Ans.: } e^{x+y} = \frac{e^{2x}}{2} + c\right]$$

4.4 ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND HIGHER DEGREE

If the degree of $\dfrac{dy}{dx}$ in a differential equation of first order is higher than 1, it is convenient to denote $\dfrac{dy}{dx}$ by p. Hence, a differential equation of first order and higher degree can be written as

$$f(x, y, p) = 0$$

There are three cases of such equations, viz.,

(i) Equation solvable for p
(ii) Equation solvable for y
(iii) Equation solvable for x

Case I Equations solvable for p

Let $p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_n = 0$ ...(4.17)

be an ordinary differential equation of first order and n^{th} degree, where $p = \dfrac{dy}{dx}$ and $P_1, P_2, \ldots, P_n$, are functions of x and y.

If Eq. (4.17) is solvable for p, its LHS can be resolved into n rational factors of the first degree. Equation (4.17) can be written as

$$\left[p-f_1(x,y)\right]\left[p-f_2(x,y)\right]....\left[p-f_n(x,y)\right]=0$$

Equating each factor of the LHS to zero,

$$p=f_1(x,y), \qquad p=f_2(x,y),....,\ p=f_n(x,y)$$

Let the general solution of these equations be respectively,

$$F_1(x,y,c)=0, \qquad F_2(x,y,c)=0,......,F_n(x,y,c)=0$$

Since the given equation is of the first order, its general solution will have only one arbitrary constant c.

These n solutions constitute the general solution of Eq. (4.17).

The general solution of Eq. (4.17) is

$$F_1(x,y,c)F_2(x,y,c)....F_n(x,y,c)=0$$

Example 1

Solve $x^2=1+p^2$.

Solution

$$p^2=x^2-1$$

$$p=\pm\sqrt{x^2-1}$$

$$\frac{dy}{dx}=\pm\sqrt{x^2-1}$$

$$dy=\pm\sqrt{x^2-1}\,dx$$

Integrating both the sides,

$$\int dy=\pm\int\sqrt{x^2-1}\,dx$$

$$y=\pm\frac{x}{2}\sqrt{x^2-1}\mp\frac{1}{2}\log\left(x+\sqrt{x^2+1}\right)+c$$

which is the general solution.

Example 2

Solve $x^2p^2+3xyp+2y^2=0$.

Solution

$$x^2p^2+3xyp+2y^2=0$$

$$(xp+y)(xp+2y)=0$$

$$(xp+y)=0, \qquad (xp+2y)=0$$

$$x\frac{dy}{dx}+y=0, \qquad x\frac{dy}{dx}+2y=0$$

$$\frac{dy}{y}+\frac{dx}{x}=0, \qquad \frac{dy}{y}+\frac{2}{x}dx=0$$

Integrating both the sides,

$$\int\frac{dy}{y}+\int\frac{dx}{x}=0, \qquad \int\frac{dy}{y}+\int\frac{2}{x}dx=0$$

$$\log y+\log x=\log c, \qquad \log y+2\log x=\log c$$

$$xy=c, \qquad x^2y=c$$

$$(xy-c)=0, \qquad (x^2y-c)=0$$

Hence, the general solution is

$$(xy-c)(x^2y-c)=0$$

Example 3

Solve $xyp^2+p(3x^2-2y^2)-6xy=0$.

Solution

The given equation is quadratic in p.

$$p=\frac{-(3x^2-2y^2)\pm\sqrt{(3x^2-2y^2)^2+24x^2y^2}}{2xy}$$

$$=\frac{(2y^2-3x^2)\pm(3x^2+2y^2)}{2xy}$$

$$=\frac{2y}{x} \quad \text{or} \quad -\frac{3x}{y}$$

$$p=\frac{2y}{x}, \qquad p=-\frac{3x}{y}$$

$$\frac{dy}{dx}=\frac{2y}{x}, \qquad \frac{dy}{dx}=-\frac{3x}{y}$$

$$\frac{dy}{y}-\frac{2}{x}dx=0, \qquad ydy+3xdx=0$$

Integrating both the sides,

$$\int \frac{dy}{y} - \int \frac{2}{x} dx = 0, \qquad \int y dy + \int 3x dx = 0$$

$$\log y - 2\log x = \log c, \qquad \frac{y^2}{2} + \frac{3x^2}{2} = c_1$$

$$\log y - \log x^2 = \log c, \qquad y^2 + 3x^2 = 2c_1$$

$$\log\left(\frac{y}{x^2}\right) = \log c, \qquad y^2 + 3x^2 = c$$

$$\frac{y}{x^2} = c,$$

$$y = cx^2$$

Hence, the general solution is

$$(y - cx^2)(y^2 + 3x^2 - c) = 0$$

Example 4

Solve $x^2p^2 + xyp - 6y^2 = 0$.

Solution

$$x^2p^2 + xyp - 6y^2 = 0$$

$$(px + 3y)(px - 2y) = 0$$

$$px + 3y = 0, \qquad px - 2y = 0$$

$$x\frac{dy}{dx} + 3y = 0, \qquad x\frac{dy}{dx} - 2y = 0$$

$$\frac{dy}{y} + \frac{3dx}{x} = 0, \qquad \frac{dy}{y} - \frac{2dx}{x} = 0$$

Integrating both the sides,

$$\int \frac{dy}{y} + \int \frac{3dx}{x} = 0, \qquad \int \frac{dy}{y} - \int \frac{2dx}{x} = 0$$

$$\log y + 3\log x = \log c, \qquad \log y - 2\log x = \log c$$

$$\log y + \log x^3 = \log c, \qquad \log y - \log x^2 = \log c$$

$$\log yx^3 = \log c, \qquad \log \frac{y}{x^2} = \log c$$

$$yx^3 = c, \qquad \frac{y}{x^2} = c$$

Hence, the general solution is

$$(yx^3 - c)(y - cx^2) = 0$$

Example 5

Solve $p^2 + 2py\cot x - y^2 = 0$.

Solution

The given equation is quadratic in p.

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$= -y\cot x \pm y\operatorname{cosec} x$$

$$p = y(-\cot x + \operatorname{cosec} x), \qquad p = y(-\cot x - \operatorname{cosec} x)$$

$$\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x), \qquad \frac{dy}{dx} = y(-\cot x - \operatorname{cosec} x)$$

$$\frac{dy}{y} + (\cot x - \operatorname{cosec} x)dx = 0, \qquad \frac{dy}{y} + (\cot x + \operatorname{cosec} x)dx = 0$$

Integrating both the sides,

$$\int\frac{dy}{y} + \int(\cot x + \operatorname{cosec} x)dx = 0, \qquad \int\frac{dy}{y} + \int(\cot x - \operatorname{cosec} x)dx = 0$$

$$\log y + \log\sin x + \log\tan\frac{x}{2} = \log c, \qquad \log y + \log\sin x - \log\tan\frac{x}{2} = \log c$$

$$\log\left(y\sin x\tan\frac{x}{2}\right) = \log c, \qquad \log\frac{y\sin x}{\tan\frac{x}{2}} = \log c$$

$$y\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right) = c, \qquad \frac{y\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)} = c$$

$$y\left(2\sin^2\frac{x}{2}\right) = c, \qquad y\left(2\cos^2\frac{x}{2}\right) = c$$

$$y(1-\cos x) = c, \qquad y(1+\cos x) = c$$

$$y(1-\cos x) - c = 0, \qquad y(1+\cos x) - c = 0$$

Hence, the general solution is

$$[y(1-\cos x) - c][y(1+\cos x) - c] = 0$$

Example 6

Solve $p^2 - 2p\cosh x + 1 = 0$.

Solution

$$p^2 - 2p\cosh x + 1 = 0$$

$$p^2 - p\left(e^x + e^{-x}\right) + 1 = 0$$

$$p\left(p - e^x\right) - e^{-x}\left(p - e^x\right) = 0$$

$$\left(p - e^x\right)\left(p - e^{-x}\right) = 0$$

$$p - e^x = 0, \qquad p - e^{-x} = 0$$

$$\frac{dy}{dx} - e^x = 0, \qquad \frac{dy}{dx} - e^{-x} = 0$$

$$dy - e^x dx = 0, \qquad dy - e^{-x} dx = 0$$

Integrating both the sides,

$$\int dy - \int e^x dx = 0, \qquad \int dy - \int e^{-x} dx = 0$$

$$y - e^x = c, \qquad y + e^{-x} = c$$

$$y - e^x - c = 0, \qquad y + e^{-x} - c = 0$$

Hence, the general solution is

$$\left(y - e^x - c\right)\left(y + e^{-x} - c\right) = 0$$

Example 7

Solve $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$.

Solution

$$p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$$

$$p^2(p + 2x) - y^2p(p + 2x) = 0$$

$$(p + 2x)(p^2 - y^2p^2) = 0$$

$$(p + 2x)p(p - y^2) = 0$$

$$p + 2x = 0, \qquad p = 0, \qquad p - y^2 = 0$$

$$\frac{dy}{dx} + 2x = 0, \qquad \frac{dy}{dx} = 0, \qquad \frac{dy}{dx} - y^2 = 0$$

$$dy + 2xdx = 0 \qquad dy = 0 \qquad dy - y^2dx = 0$$

Integrating both the sides,

$$\int dy + \int 2x dx = 0 \qquad \int dy = 0 \qquad \int dy - \int y^2 dx = 0$$

$$y + x^2 = c, \qquad y = c, \qquad \int \frac{dy}{y^2} - \int dx = 0$$

$$-\frac{1}{y} - x = c$$

$$-\frac{1}{y} = x + c$$

Hence, the general solution is

$$(y + x^2 - c)(y - c)(xy + cy + 1) = 0$$

Example 8

Solve $2p^3 - (2x + 4\sin x - \cos x)p^2 - (x\cos x - 4x\sin x + \sin 2x)p + x\sin 2x = 0.$

Solution

$$2p^3 - (2x + 4\sin x - \cos x)p^2 - (x\cos x - 4x\sin x + \sin 2x)p + x\sin 2x = 0$$

$$(p - x)\left[2p^2 - (4\sin x - \cos x)p - \sin 2x\right] = 0$$

$$(p - x)\left[2p(p - 2\sin x) + \cos x(p - 2\sin x)\right] = 0$$

$$(p - x)(p - 2\sin x)(2p + \cos x) = 0$$

$$p - x = 0, \qquad p - 2\sin x = 0, \qquad 2p + \cos x = 0$$

$$\frac{dy}{dx} - x = 0, \qquad \frac{dy}{dx} - 2\sin x = 0, \qquad 2\frac{dy}{dx} + \cos x = 0$$

$$dy - x dx = 0, \qquad dy - 2\sin x dx = 0, \qquad 2dy + \cos x dx = 0$$

Integrating both the sides,

$$\int dy - \int x dx = 0, \qquad \int dy - \int 2\sin x dx = 0 \qquad \int 2dy + \cos x dx = 0$$

$$y - \frac{x^2}{2} = c, \qquad y + 2\cos x = c \qquad 2y + \sin x = c$$

$$2y - x^2 = c$$

Hence, the general solution is

$$(2y - x^2 - c)(y + 2\cos x - c)(2y + \sin x - c) = 0$$

EXERCISE 4.5

Solve the following differential equations:

1. $p^2-9p+18=0$

$$\left[\textbf{Ans.: } (y-3x+c)(y-6x+c)=0\right]$$

2. $(p-xy)(p-x^2)(p-y^2)=0$

$$\left[\textbf{Ans.: } \left(\log y-\frac{1}{2}x^2-c\right)\left(y-\frac{1}{3}x^3-c\right)\left(x+\frac{1}{y}+c\right)=0\right]$$

3. $(p+y+x)(xp+x+y)(p+2x)=0.$

$$\left[\textbf{Ans.: } (1-x-y+ce^{-x})(2xy+x^2-c)(y+x^2-c)=0\right]$$

4. $p^2-p(e^x+e^{-x})+1=0$

$$\left[\textbf{Ans.: } (y-e^{2x}-c)(y-e^{-2x}-c)=0\right]$$

5. $p^2+(x-e^x)p-xe^x=0$

$$\left[\textbf{Ans.: } (y-e^x+c)\left(y+\frac{1}{2}x^2+c\right)=0\right]$$

6. $p(p-y)=x(x+y)$

$$\left[\textbf{Ans.: } (x+y+1-ce^{-x})(2y+x^2-c)=0\right]$$

7. $p^3+3xp^2-y^3p^2-3xy^3p=0$

$$\left[\textbf{Ans.: } (y-c)\left(y+\frac{3}{2}x^2-c\right)(2xy^2+1-2cy^2)=0\right]$$

8. $xyp^2-(x+y)p+1=0$

$$\left[\textbf{Ans.: } \left(x+\frac{1}{2}y^2-c\right)(y+\log x-c)=0\right]$$

9. $p^3(x+2y)+3p^2(x+y)+(y+2x)p=0$

$$\left[\textbf{Ans.: } \left(x+\frac{1}{2}y^2-c\right)(y+\log x-c)=0\right]$$

Case II *Equations solvable for y*

If the given ordinary differential equation is solvable for y, it can be put in the form

$$y = f(x, p) \quad \text{...(4.18)}$$

Differentiating Eq. (4.18) w.r.t. x,

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right) \quad \text{...(4.19)}$$

Equation (4.19) is a differential equation in p and x.

Let the general solution of Eq. (4.19) be

$$F(x, p, c) = 0 \quad \text{...(4.20)}$$

where c is an arbitrary constant. The elimination of p from Eqs (4.18) and (4.20) gives the general solution.

Note:

(i) If the elimination of p is not possible, Eqs (4.18) and (4.20) are solved for x and y in terms of p.

The two parametric equations

$$x = F_1(p, c)$$

and $$y = F_2(p, c)$$

taken together constitute the general solution of Eq. (4.18), where p is the parameter.

(ii) If Eqs (4.18) and (4.20) are easily not solvable for x and y and then Eqs (4.18) and (4.20) taken together represent the general solution of Eq. (4.20).

(iii) If any factor of Eq. (4.19) does not contain the term $\frac{dp}{dx}$, the elimination of p from that factor equation and Eq. (4.18) gives a solution which does not contain any arbitrary constants. Such a solution is called a *singular solution.*

In this section, only general solutions are obtained.

Example 1

Solve $y = 2px - xp^2$.

Solution

$$y = 2px - xp^2 \quad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. x,

$$p = 2p + 2x\frac{dp}{dx} - p^2 - 2xp\frac{dp}{dx} \qquad \left[\because \frac{dy}{dx} = p\right]$$

$$p + 2x\frac{dp}{dx}(1 - p) - p^2 = 0$$

$$p(1-p)+2x\frac{dp}{dx}(1-p)=0$$

$$\left(p+2x\frac{dp}{dx}\right)(1-p)=0$$

Neglecting the second factor which does not contain $\frac{dp}{dx}$,

$$p+2x\frac{dp}{dx}=0$$

$$2\frac{dp}{p}+\frac{dx}{x}=0$$

Integrating both the sides,

$$\int 2\frac{dp}{p}+\int\frac{dx}{x}=0$$

$$2\log p+\log x=\log c$$

$$p^2x=c$$

$$p^2=\frac{c}{x} \qquad \text{...(2)}$$

From Eq. (1),

$$y+xp^2=2px$$

$$\left(y+xp^2\right)^2=4p^2x^2 \qquad \text{...(3)}$$

Eliminating p from Eqs (2) and (3),

$$(y+c)^2=4\left(\frac{c}{x}\right)x^2=4cx$$

which is the general solution.

Example 2

Solve $y+px=x^4p^2$.

Solution

$$y=-px+x^4p^2 \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. x,

$$p=-p-x\frac{dp}{dx}+4x^3p^2+2x^4p\frac{dp}{dx} \qquad \left[\because \frac{dy}{dx}=p\right]$$

$$2p + x\frac{dp}{dx} - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0$$

$$\left(2p + x\frac{dp}{dx}\right)(1 - 2px^3) = 0$$

Neglecting the second factor which does not contain $\frac{dp}{dx}$,

$$2p + x\frac{dp}{dx} = 0$$

$$\frac{dp}{p} + 2\frac{dx}{x} = 0$$

Integrating both the sides,

$$\int \frac{dp}{p} + \int 2\frac{dx}{x} = 0$$

$$\log p + 2\log x = \log c$$

$$\log px^2 = \log c$$

$$px^2 = c$$

$$p = \frac{c}{x^2}$$

Substituting in Eq. (1),

$$y = -\frac{c}{x} + x^4\left(\frac{c^2}{x^4}\right)$$

$$= -\frac{c}{x} + c^2$$

which is the general solution.

Example 3

Solve $y = (x-a)p - p^2$.

Solution

$$y = (x-a)p - p^2 \quad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. x,

$$p = p + (x-a)\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$0 = (x-a)\frac{dp}{dx} - 2p\cdot\frac{dp}{dx}$$

$$0 = \frac{dp}{dx}(x - a - 2p)$$

Neglecting the second factor which does not contain $\frac{dp}{dx}$,

$$\frac{dp}{dx} = 0$$

Integrating both the sides,

$$\int \frac{dp}{dx} = 0$$

$$p = c$$

Substituting in Eq. (1),

$$y = (x-a)c - c^2$$

which is the general solution.

Example 4

Solve $3x^4p^2 - xp - y = 0$.

Solution

$$3x^4p^2 - xp - y = 0$$

$$y = 3x^4p^2 - xp \quad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. x,

$$p = 12x^3p^2 + 6x^4p\frac{dp}{dx} - p - x\frac{dp}{dx}$$

$$(2p - 12x^3p^2) + (x - 6x^4p)\frac{dp}{dx} = 0$$

$$2p(1-6x^3p) + x\frac{dp}{dx}(1-6x^3p) = 0$$

$$(1-6x^3p)\left(2p + x\frac{dp}{dx}\right) = 0$$

Neglecting the first factor which does not contain $\frac{dp}{dx}$,

$$\left(2p + x\frac{dp}{dx}\right) = 0$$

$$\frac{2dx}{x} + \frac{dp}{p} = 0$$

Integrating both the sides,

$$\int \frac{2\,dx}{x} + \int \frac{dp}{p} = 0$$

$$2\log x + \log p = \log c$$

$$\log x^2 + \log p = \log c$$

$$\log px^2 = \log c$$

$$px^2 = c$$

$$p = \frac{c}{x^2}$$

Substituting in Eq. (1),

$$y = 3x^4\left(\frac{c}{x^2}\right)^2 - x\left(\frac{c}{x^2}\right)$$

$$y = 3c^2 - \frac{c}{x}$$

which is the general solution.

Example 5

Solve $y = 2px + p^n$.

Solution

$$y = 2px + p^n \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. x,

$$p = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} \qquad \left[\because \frac{dy}{dx} = p\right]$$

$$p + \left(2x + np^{n-1}\right)\frac{dp}{dx} = 0$$

$$p\frac{dx}{dp} + 2x + np^{n-1} = 0$$

$$\frac{dx}{dp} + \frac{2}{p}x = -np^{n-2} \qquad \text{...(2)}$$

which is a linear equation in x and p.

$$\text{IF} = e^{\int \frac{2}{p}dp} = e^{2\log p} = p^2$$

The general solution of Eq. (2) is

$$p^2 \cdot x = \int p^2\left(-np^{n-2}\right)dp + c$$

$$xp^2 = -n\int p^n \,dp + c = -\frac{np^{n+1}}{n+1} + c$$

$$x = -\frac{np^{n-1}}{n+1} + \frac{c}{p^2} \quad ...(3)$$

Substituting the value of x in Eq. (1),

$$y = -\frac{2n}{n+1}p^n + \frac{2c}{p} + p^n = \frac{2c}{p} - \frac{n-1}{n+1}p^n \quad ...(4)$$

Equations (3) and (4) taken together, with the parameter p, constitute the general solution.

Example 6

Solve $y = x + a\tan^{-1} p$.

Solution

$$y = x + a\tan^{-1} p \quad ...(1)$$

Differentiating Eq. (1) w.r.t. x,

$$p = 1 + a\frac{1}{1+p^2}\frac{dp}{dx} \qquad \left[\because \frac{dy}{dx} = p\right]$$

$$\frac{a}{1+p^2}\frac{dp}{dx} = p - 1$$

$$dx = \frac{a}{(p-1)(p^2+1)}dp = \frac{a}{2}\left(\frac{1}{p-1} - \frac{p+1}{p^2+1}\right)dp$$

Integrating both the sides,

$$\int dx = \int \frac{a}{2}\left(\frac{1}{p-1} - \frac{p+1}{p^2+1}\right)dp$$

$$x = \frac{a}{2}\left[\log(p-1) - \frac{1}{2}\log(p^2+1) - \tan^{-1} p\right] + c = \frac{a}{2}\left[\log\frac{p-1}{\sqrt{p^2+1}} - \tan^{-1} p\right] + c \quad ...(2)$$

Substituting the value of x in Eq. (1),

$$y = \frac{a}{2}\left[\log\frac{p-1}{\sqrt{p^2+1}} + \tan^{-1} p\right] + c \quad ...(3)$$

Equations (2) and (3) taken together constitute the general solution.

Example 7

Solve $p\tan p - y + \log\cos p = 0$.

Solution

$$y = p\tan p + \log\cos p \quad \ldots(1)$$

Differentiating Eq. (1) w.r.t. x,

$$p = \tan p\frac{dp}{dx} + p\sec^2 p\frac{dp}{dx} + \frac{1}{\cos p}(-\sin p)\frac{dp}{dx}$$

$$= \tan p\frac{dx}{dx} + p\sec^2 p\frac{dp}{dx} - \tan p\frac{dp}{dx}$$

$$p\left(1 - \sec^2 p\frac{dp}{dx}\right) = 0$$

Neglecting the first factor which does not contain $\frac{dp}{dx}$,

$$1 - \sec^2 p\frac{dp}{dx} = 0$$

$$dx - \sec^2 p dp = 0$$

Integrating both the sides,

$$\int dx = \int \sec^2 p dp$$

$$x = \tan p + c \quad \ldots(2)$$

Equations (2) and (3) taken together, with the parameter p, constitute the general solution.

EXERCISE 4.6

Solve the following differential equations:

1. $y = -px + x^4p^2$

$$\left[\textbf{Ans.: } xy = -c + c^2x\right]$$

2. $y = \frac{x}{p} - ap$

$$\left[\begin{aligned}\textbf{Ans.: } x &= \frac{p}{\sqrt{1-p^2}}\left(c + a\sin^{-1}p\right)\\ y &= -ap + \frac{1}{\sqrt{1-p^2}}\left(c + a\sin^{-1}p\right)\end{aligned}\right]$$

3. $y = \sin p - p\cos p$

$$\left[\textbf{Ans.: } x = c - \cos\left\{\frac{\sqrt{1-(c-x)^2} - y}{c-x}\right\}\right].$$

4. $y = 3x + a\log p$

$$\left[\textbf{Ans.: } y = 3x - a\log\left(\frac{1}{3} - ce^{\frac{3x}{a}}\right)\right]$$

5. $y = 2px + p^2$

$$\left[\textbf{Ans.: } 4\left(y^2 - 3cx\right)\left(x^2 + y\right) = \left(xy + 3c\right)^2\right]$$

6. $y = (1+p)x + p^2$

$$\left[\begin{aligned}\textbf{Ans.: } x &= 2(1-p) + ce^{-p}\\ y &= 2 - p^2 + c(1+p)e^{-p}\end{aligned}\right]$$

7. $y = 2xp - p^3$

$$\left[\textbf{Ans.: } x = \frac{3}{4}p^2 + \frac{c}{p^2}, y = \frac{1}{2}p^3 + \frac{2c}{p}\right]$$

8. $xp^2 - 2yp + ax = 0$

$$\left[\textbf{Ans.: } 2y = cx^2 + \frac{a}{c}\right]$$

9. $yp^2 - 2xp + y = 0$

$$\left[\textbf{Ans.: } y^2 = 2cx - c^2\right]$$

10. $y = (1+p)x + ap^2$

$$\left[\begin{aligned}\textbf{Ans.: } x &= ce^{-p} - 2a(p-1)\\ y &= (1+p)ce^{-p} - 2a(p-1) + ap^2\end{aligned}\right]$$

Case III *Equations solvable for x*

If the given ordinary differential equation is solvable for x, it can be put in the form

$$x = f(y, p) \quad \text{...(4.21)}$$

Differentiating Eq. (4.21) w.r.t. y,

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right) \quad \text{...(4.22)}$$

Equation (4.22) is a differential equation in p and y.

Let the general solution of Eq. (4.22) be

$$F(y, p, c) = 0 \quad \text{...(4.23)}$$

where c is an arbitrary constant. The elimination of p from Eqs (4.21) and (4.22) gives the general solution.

Note:

(i) If the elimination of p is not possible, Eqs (4.21) and (4.23) are solved for x and y in terms of p.
The two parametric equations

$$x = F_1(p, c)$$

and

$$y = F_2(p, c)$$

taken together constitute the general solution of Eq. (4.21), where p is the parameter.

(ii) If Eqs (4.21) and (4.23) are not easily solvable for x and y then Eqs (4.21) and (4.23) taken together represent the general solution of Eq. (4.21).

(iii) If any factor of Eq. (4.22) does not contain the term $\frac{dp}{dy}$, the elimination of p from that factor equation and Eq. (4.21) gives a solution which does not contain any arbitrary constants. Such a solution is called a *singular solution*.
In this section, only general solutions are obtained.

Example 1

Solve $p^2 - xp + y = 0$.

Solution

$$p^2 - xp + y = 0 \quad \text{...(1)}$$

$$x = \frac{p^2 + y}{p} \quad \text{...(2)}$$

Differentiating Eq. (2) w.r.t. y,

$$\frac{dx}{dy} = \frac{dp}{dy} + \frac{\left(p - y\frac{dp}{dy}\right)}{p^2}$$

$$\frac{1}{p} = \left(\frac{p^2 - y}{p^2}\right)\frac{dp}{dy} + \frac{1}{p}$$

$$(p^2 - y)\frac{dp}{dy} = 0$$

Neglecting the first factor which does not contain $\frac{dp}{dy}$,

$$\frac{dp}{dy}=0$$

Integrating both the sides,

$$\int \frac{dp}{dy}=0$$

$$p=c$$

From Eq. (1),

$$c^2-xc+y=0$$

$$y=cx-x^2$$

which is the general solution.

Example 2

Solve $xp^2-yp-y=0$.

Solution

$$xp^2-yp-y=0 \qquad \text{...(1)}$$

$$x=\frac{y(1+p)}{p^2} \qquad \text{...(2)}$$

Differentiating Eq. (2) w.r.t. y,

$$\frac{dx}{dy}=\frac{1+p}{p^2}+y\left[\frac{p^2-(1+p)2p}{p^4}\right]\frac{dp}{dy}$$

$$\frac{1}{p}=\frac{1}{p^2}+\frac{1}{p}+\frac{y(-p-2)}{p^3}\frac{dp}{dy}$$

$$\frac{dp}{dy}=\frac{p}{2+p}\cdot\frac{1}{y}$$

$$\frac{2+p}{p}dp=\frac{dy}{y}$$

Integrating both the sides,

$$\int\frac{2+p}{p}dp=\int\frac{dy}{y}$$

$$2\log p+p=\log y+\log c$$

$$\log p^2+p\log e=\log y-\log c$$

$$\log p^2+\log e^p+\log c=\log y$$

$$\log(cp^2e^p)=\log y$$

$$y = cp^2 e^p \qquad \text{...(3)}$$

Equations (2) and (3) together constitute the general solution.

Example 3

Solve $yp^2 - 2xp + y = 0.$

Solution

$$yp^2 - 2xp + y = 0 \qquad \text{...(1)}$$

$$2x = yp + \frac{y}{p} \qquad \text{...(2)}$$

Differentiating Eq. (2) w.r.t. y,

$$\frac{2}{p} = p + y\frac{dp}{dy} + \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}$$

$$p + y\frac{dp}{dy} - \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} = 0$$

$$y\frac{dp}{dy}\left(1 - \frac{1}{p^2}\right) + p\left(1 - \frac{1}{p^2}\right) = 0$$

$$\left(p + y\frac{dp}{dy}\right)\left(1 - \frac{1}{p^2}\right) = 0$$

Neglecting the second factor which does not contain $\frac{dp}{dy}$,

$$p + y\frac{dp}{dy} = 0$$

$$\frac{dp}{p} + \frac{dy}{y} = 0$$

Integrating both the sides,

$$\int\frac{dp}{p} + \int\frac{dy}{y} = 0$$

$$\log p + \log y = \log c$$

$$py = c$$

$$p = \frac{c}{y}$$

Substituting the value of p in Eq. (1),

$$y\frac{c^2}{y^2} - 2x\frac{c}{y} + y = 0$$

$$y^2 = 2cx - c^2$$

which is the general solution.

Example 4

Solve $y = 2px + y^2 p^3$.

Solution

$$y = 2px + y^2 p^3 \qquad \text{...(1)}$$

$$x = \frac{y}{2p} - \frac{y^2 p^2}{2} \qquad \text{...(2)}$$

Differentiating Eq. (2) w.r.t. y,

$$\frac{dx}{dy} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - yp^2 - y^2 p \frac{dp}{dy}$$

$$\frac{1}{p} - \frac{1}{2p} + yp^2 = -y\left(\frac{1}{2p^2} + py\right)\frac{dp}{dy}$$

$$\left(1 + 2yp^3\right)p = -y\left(1 + 2yp^3\right)\frac{dp}{dy}$$

$$\left(1 + 2yp^3\right)p + \left(1 + 2yp^3\right)y\frac{dp}{dy} = 0$$

$$\left(p + y\frac{dp}{dy}\right)\left(1 + 2yp^3\right) = 0$$

Neglecting the second factor which does not contain $\frac{dp}{dy}$,

$$p + y\frac{dp}{dy} = 0$$

$$\frac{dp}{p} + \frac{dy}{y} = 0$$

Integrating both the sides,

$$\int \frac{dp}{p} + \int \frac{dy}{y} = 0$$

$$\log p + \log y = \log c$$

$$py = c$$

$$p = \frac{c}{y}$$

Substituting the value of p in Eq. (1),

$$y = 2\frac{c}{y}x + y^2 \frac{c^3}{y^3}$$

$$y^2 = 2cx + c^3$$

which is the general solution.

Example 5

Solve $p = \tan\left(x - \dfrac{p}{1+p^2}\right)$.

Solution

$$p = \tan\left(x - \frac{p}{1+p^2}\right) \qquad \text{...(1)}$$

$$x = \tan^{-1} p + \frac{p}{1+p^2} \qquad \text{...(2)}$$

Differentiating Eq. (2) w.r.t. y,

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2)-2p^2}{(1+p)^2}\cdot\frac{dp}{dy}$$

$$\frac{1}{p} = \frac{2(1+p^2)-p^2}{(1+p^2)^2}\frac{dp}{dy}$$

$$dy = \frac{2p}{(1+p^2)^2}dp$$

Integrating both the sides,

$$\int dy - \int \frac{2p}{(1+p)^2}dp = 0$$

$$y - \frac{1}{1+p^2} = c \qquad \text{...(3)}$$

Equations (2) and (3) together constitute the general solution.

Example 6

Solve $y^2 \log y = xyp + p^2$.

Solution

$$y^2 \log y = xyp + p^2 \qquad \text{...(1)}$$

$$x = \frac{y \log y}{p} - \frac{p}{y} \qquad \text{...(2)}$$

Differentiating Eq. (2) w.r.t. y,

$$\frac{dx}{dy} = \frac{p(1+\log y) - y\log y\dfrac{dp}{dy}}{p^2} - \frac{y\dfrac{dp}{dy} - p}{y^2}$$

$$\frac{1}{p} = \frac{1}{p} + \frac{1}{p}\log y - \frac{y}{p^2}\log y\frac{dp}{dy} - \frac{1}{y}\frac{dp}{dy} + \frac{p}{y^2}$$

$$\frac{1}{y}\frac{dp}{dy}\left(1 + \frac{y^2}{p^2}\log y\right) = \frac{p}{y^2}\left(1 + \frac{y^2}{p^2}\log y\right)$$

$$\frac{dp}{dy}\left(1 + \frac{y^2}{p^2}\log y\right) - \frac{p}{y}\left(1 + \frac{y^2}{p^2}\log y\right) = 0$$

$$\left(\frac{dp}{dy} - \frac{p}{y}\right)\left(1 + \frac{y^2}{p^2}\log y\right) = 0$$

Neglecting the second factor which does not contain $\frac{dp}{dy}$,

$$\frac{dp}{dy} - \frac{p}{y} = 0$$

$$\frac{dp}{p} - \frac{dy}{y} = 0$$

Integrating both the sides,

$$\int\frac{dp}{p} - \int\frac{dy}{y} = 0$$

$$\log p - \log y = \log c$$

$$\frac{p}{y} = c$$

$$p = cy$$

Substituting the value of p in Eq. (1),

$$y^2\log y = xy(cy) + c^2y^2$$

$$\log y = cx + c^2$$

which is the general solution.

EXERCISE 4.7

Solve the following differential equations:

1. $x = y + a\log p$

$$\left[\textbf{Ans.: } x = c + a\log\frac{p}{p-1}, y = c - a\log(p-1)\right]$$

2. $x = y + p^2$

$$\left[\begin{aligned}\textbf{Ans.: } x &= c - \{2p + 2\log(p-1)\}\\ y &= c - \{p^2 + 2p + 2\log(p-1)\}\end{aligned}\right]$$

3. $p = \tan\left(x - \frac{p}{1+p^2}\right)$

$$\left[\textbf{Ans.: } y = -\frac{1}{1+p^2} + c\right]$$

4. $x = p^3 - p + 2$

$$\left[\textbf{Ans.: } x = p^3 - p + 2, y = \frac{3}{4}p^4 - \frac{p^2}{2} + c\right]$$

5. $y - 2xp + apy^2 = 0$

$$\left[\textbf{Ans.: } 2cx = y^2 + ac^2\right]$$

6. $p^3 - p(y+3) + x = 0$

$$\left[\begin{aligned}\textbf{Ans.: } x &= cp(1-p^2)^{-\frac{1}{2}} + 2p\\ y &= c(1-p^2)^{-\frac{1}{2}}\end{aligned}\right]$$

7. $y - xp = a(y^2 + p)$

$$\left[\textbf{Ans.: } c(x+a)(ay-1) + y = 0\right]$$

8. $xp^3 = a + bp$

$$\left[\textbf{Ans.: } x = \frac{a}{p^3} + \frac{b}{p^2}, y = \frac{3a}{2p^2} + \frac{2b}{p} + c\right]$$

9. $ayp^2 + (2x-b)p - y = 0$

$$\left[\textbf{Ans.: } ac^2 + (2x-b)c - y^2 = 0\right]$$

10. $p^2y + 2px - y = 0$

$$\left[\textbf{Ans.: } 2cx = y^2 - c^2\right]$$

4.4.1 Clairaut's Equation

An equation of the form

$$y = px + f(p) \qquad \text{...(4.24)}$$

is known as Clairaut's equation.

Differentiating Eq. (4.24) w.r.t. x,

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$[x + f'(p)]\frac{dp}{dx} = 0$$

$$x + f'(p) = 0 \quad \text{or} \quad \frac{dp}{dx} = 0$$

The solution of the equation $\frac{dp}{dx} = 0$ is

$$p = c \qquad \text{...(4.25)}$$

Eliminating p from Eqs (4.24) and (4.25),

$$y = cx + f(c) \qquad \text{...(4.26)}$$

which is the general solution of Eq. (4.24). Hence, the solution of Clairaut's equation is obtained on replacing p by c.

Note: Elimination of p from the equation $x + f'(p) = 0$ gives a solution which does not contain any arbitrary constants. Such a solution is called a singular solution of Eq. (4.24) which gives the envelope of the family of straight lines represented by Eq. (4.26).

In this section, only general solutions are obtained.

4.4.2 Lagrange's Equation

An equation of the form

$$y = xf_1(p) + f_2(p) \qquad \text{...(4.27)}$$

is known as Lagrange's equation.

This is a generalization of Clairaut's equation. Putting $f_1(p) = p$, Clairaut's equation is obtained.

Differentiating Eq. (4.27) w.r.t. x,

$$p = f_1(p) + \left[x f_1'(p) + f_2'(p)\right]\frac{dp}{dx}$$

$$= \left[p - f_1(p)\right]\frac{dx}{dp} - x f_1'(p) = f_2'(p)$$

which is a linear differential equation in x, and, hence, it can be solved.

Example 1

Solve $(y - px)(p-1) = p$.

Solution

$$y - px = \frac{p}{p-1}$$

$$y = px + \frac{p}{p-1}$$

which is a differential equation in Clairaut's form.

Hence, the general solution is

$$y = cx + \frac{c}{c-1}$$

Example 2

Solve $p = \log(px - y)$.

Solution

$$p = \log(px - y)$$

$$e^p = px - y$$

$$y = px - e^p$$

which is a differential equation in Clairaut's form.

Hence, the general solution is

$$y = cx - e^c$$

Example 3

Solve $\sin y \cos^2 x = \cos^2 y \cdot p^2 + \sin x \cos x \cos y \cdot p$.

Solution

Let

$$\sin x = u \quad \text{and} \quad \sin y = v$$

$$\cos x\,dx = du \quad \text{and} \quad \cos y\,dy = dv$$

$$\frac{\cos y}{\cos x}\frac{dy}{dx} = \frac{dv}{du}$$

Dividing the given equation by $\cos^2 x$,

$$\sin y = \frac{\cos^2 y}{\cos^2 x}p^2 + \sin x \frac{\cos y}{\cos x}p$$

$$v = \left(\frac{dv}{du}\right)^2 + u\left(\frac{dv}{du}\right)$$

$$= u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2$$

which is a differential equation in Clairaut's form.

Hence, the general solution is

$$v = cu + c^2$$

$$\sin y = c \sin x + c^2$$

Example 4

Solve $(x^2 + y^2)(1+p)^2 = 2(x+y)(1+p)(x+yp) - (x+yp)^2$.

Solution

$$x^2 + y^2 = \frac{2(x+y)(x+yp)}{(1+p)} - \left(\frac{x+yp}{1+p}\right)^2 \qquad \ldots(1)$$

Let

$$x + y = u \quad \text{and} \quad x^2 + y^2 = v$$

$$1 + \frac{dy}{dx} = \frac{du}{dx} \quad \text{and} \quad 2x + 2y\frac{dy}{dx} = \frac{dv}{dx}$$

$$1 + p = \frac{du}{dx} \quad \text{and} \quad 2x + 2yp = \frac{dv}{dx}$$

$$\frac{2(x+yp)}{1+p} = \frac{dv}{du}$$

Substituting the values of u, v and p in the Eq. (1),

$$v = u\frac{dv}{du} - \frac{1}{4}\left(\frac{dv}{du}\right)^2$$

which is a differential equation in Clairaut's form.

Hence, the general solution is

$$v = uc - \frac{1}{4}c^2$$

$$x^2 + y^2 = c(x+y) - \frac{1}{4}c^2$$

Example 5

Solve $e^{3x}(p-1) + p^3 e^{2y} = 0$.

Solution

Let $\quad e^x = u \quad$ and $\quad e^y = v$

$$e^x dx = du \quad \text{and} \quad e^y dy = dv$$

$$\frac{e^y}{e^x} \frac{dy}{dx} = \frac{dv}{du}$$

$$p = \frac{dy}{dx} = \frac{e^x}{e^y} \cdot \frac{dv}{du} = \frac{u}{v} \cdot \frac{dv}{du} = \frac{u}{v} P, \text{ where } P = \frac{dv}{du}$$

Substituting the value of u, v, and p in the given equation,

$$u^3 \left(\frac{u}{v} P - 1 \right) + \left(\frac{u}{v} P \right)^3 v^2 = 0$$

$$\frac{u^3}{v} \left(uP - v + P^3 \right) = 0$$

$$uP - v + P^3 = 0$$

$$v = uP + P^3$$

which is a differential equation in Clairaut's form. Hence, the general solution is

$$v = uc + c^3$$

$$e^y = ce^x + c^3$$

Example 6

Solve $(px - y)(py + x) = a^2 p$.

Solution

Let $\quad x^2 = u \quad$ and $\quad y^2 = v$

$$2x dx = du \quad \text{and} \quad 2y dy = dv$$

$$\frac{y}{x}\frac{dy}{dx}=\frac{dv}{du}$$

$$p=\frac{x}{y}\frac{dv}{du}=\frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}=\frac{\sqrt{u}}{\sqrt{v}}P, \text{ where } P=\frac{dv}{du}$$

Substituting the values of u, v, and p in the given equation,

$$\left(\frac{\sqrt{u}}{\sqrt{v}}P\sqrt{u}-\sqrt{v}\right)\left(\frac{\sqrt{u}}{\sqrt{v}}P\sqrt{v}+\sqrt{u}\right)=a^2\frac{\sqrt{u}}{\sqrt{v}}P$$

$$\frac{1}{\sqrt{v}}(uP-v)\sqrt{u}(P+1)=a^2\frac{\sqrt{u}}{\sqrt{v}}P$$

$$(uP-v)(P+1)=a^2P$$

$$uP-v=\frac{a^2P}{P+1}$$

$$v=uP-\frac{a^2P}{P+1}$$

which is a differential equation in Clairaut's form.

Hence, the general solution is

$$v=uc-\frac{a^2c}{c+1}$$

$$y^2=cx^2-\frac{a^2c}{c+1}$$

Example 7

Solve $x^2p^2+y(2x+y)p+y^2=0$.

Solution

Let

$$y=u \quad\text{and}\quad xy=v$$

$$dy=du \quad\text{and}\quad xdy+ydx=dv$$

$$\frac{xdy+ydx}{dy}=\frac{dv}{du}$$

$$x+y\frac{dx}{dy}=\frac{dv}{du}$$

$$\frac{v}{u}+u\frac{1}{p}=P, \quad\text{where } P\equiv\frac{dv}{du}$$

$$p = \frac{u^2}{uP - v}$$

Substituting the value of u, v, and p in the given equation,

$$\frac{v^2}{u^2}\frac{u^4}{(uP-v)^2} + u\left(2\frac{v}{u} + u\right)\frac{u^2}{uP-v} + u^2 = 0$$

$$v^2u^2 + (2v+u^2)(uP-v)u^2 + u^2(uP-v)^2 = 0$$

$$v^2u^2 + (2v+u^2)u^3P - (2v+u^2)u^2v + u^2(u^2P^2 - 2uvP + v^2) = 0$$

$$-vu^4 + (2vu^3 + u^5 - 2u^3v)P + u^4P^2 = 0$$

$$-vu^4 + u^5P + u^4P^2 = 0$$

$$(-v + uP + P^2)u^4 = 0$$

$$v = uP + P^2$$

which is a differential equation in Clairaut's form.

Hence, the general solution is

$$v = uc + c^2$$

$$xy = yc + c^2$$

EXERCISE 4.8

Solve the following differential equations:

1. $y = px + p - p^2$

   $\left[\textbf{Ans.: } y = cx + c - c^2\right]$

2. $y = px + (1+p^2)^{\frac{1}{2}}$

   $\left[\textbf{Ans.: } y = cx + (1+c^2)^{\frac{1}{2}}\right]$

3. $p = \tan(px - y)$

   $\left[\textbf{Ans.: } c = \tan(cx - y)\right]$

4. $\dfrac{(y-px)^2}{1+p^2} = a^2$

   $\left[\textbf{Ans.: } (y-cx)^2 = a^2(1+c^2)\right]$

5. $xp^3 - (y+3)p^2 + 4 = 0$

   $\left[\textbf{Ans.: } y = cx + \dfrac{4}{c^2} - 3\right]$

6. $p = e^{(y-px)}$

$\left[\textbf{Ans.: } c = e^{(y-cx)}\right]$

7. $(xp-y)^2 = p^2 - 1$

$\left[\textbf{Ans.: } (cx-y)^2 = c^2 - 1\right]$

8. $(y-px)^2(1+p^2) = a^2p^2$

$\left[\textbf{Ans.: } (y-cx)^2(1+c^2) = a^2c^2\right]$

9. $(x-a)p^2 + (x-y)p - y = 0$

$\left[\textbf{Ans.: } y = cx - \dfrac{ac^2}{c+1}\right]$

10. $y = px + \sin^{-1} p$

$\left[\textbf{Ans.: } y = cx + \sin^{-1} c\right]$

11. $xyp^2 - (x^2 + y^2 - 1)p + xy = 1$

$\left[\textbf{Ans.: } c^2x^2 - c(x^2 + y^2 + 1) + y^2 = 0\right]$

12. $y = px + \dfrac{p}{x}$

$\left[\textbf{Ans.: } y = cx^3 + c\right]$

13. $xy(y-px) = x + py$

$\left[\textbf{Ans.: } y^2 = cx^2 + (1+c)\right]$

14. $xp^2 - 2yp + x + 2y = 0$

$\left[\textbf{Ans.: } 2c^2x^2 - 2c(y-x) + 1 = 0\right]$

15. $y = 2xp + \tan^{-1}(xp^2)$

$\left[\textbf{Ans.: } y = c\sqrt{x} + \tan^{-1}\left(\dfrac{c^2}{4}\right)\right]$

16. $(px^2 + y^2)(px + y) = (p+1)^2$

$\left[\textbf{Ans.: } c^2(x+y) - cxy - 1 = 0\right]$

17. $(2x^2+1)p^2 + (x^2 + y^2 + 2xy + 2)p + 2y^2 + 1 = 0$

$\left[\textbf{Ans.: } xy - 1 = c(x+y) + c^2\right]$

18. $y^2 \log y = xyp + p^2$

$\left[\textbf{Ans.: } \log y = c(c+x)\right]$

Points to Remember

First-Order Differential Equation

Sr. No.	Differential Equation	Type	Integrating Factor	Solution
1.	$M(x, y)dx + N(x, y)dy = 0$	Exact, i.e., $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$	–	(i) $\int M(x, y)dx + \int$(terms of N not containing x)$dy = c$ (ii) $\int$(terms of M not containing y)$dx + \int N(x, y)dy = c$
2.	$M(x, y)dx + N(x, y)dy = 0$	Non-exact and $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$	$IF = e^{\int f(x)dx}$	(i) $\int M_1(x, y)dx + \int$(terms of N_1 not containing x)$dy = c$ (ii) $\int$(terms of M_1 not containing y)$dx + \int N_1(x, y)dy = c$
3.	$M(x, y)dx + N(x, y)dy = 0$	Non-exact and $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$	$IF = e^{\int f(y)dy}$	(i) $\int M_1(x, y)dx + \int$(terms of N_1 not containing x)$dy = c$ (ii) $\int$(terms of M_1 not containing y)$dx + \int N_1(x, y)dy = c$
4.	$f_1(xy)y\,dx + f_2(xy)x\,dy = 0,$	Non-exact	$IF = \frac{1}{Mx - Ny}$	(i) $\int M_1(x, y)dx + \int$(terms of N_1 not containing x)$dy = c$ (ii) $\int$(terms of M_1 not containing y)$dx + \int N_1(x, y)dy = c$
5.	$M(x, y)dx + N(x, y)dy = 0$	Non-exact and homogeneous	$IF = \frac{1}{Mx + Ny}$	(i) $\int M_1(x, y)dx + \int$(terms of N_1 not containing x)$dy = c$ (ii) $\int$(terms of M_1 not containing y)$dx + \int N_1(x, y)dy = c$

Sr. No.	Differential Equation	Type	Integrating Factor	Solution
6.	$x^{m_1}y^{n_1}(a_1 y\,dx + b_1 x\,dy) + x^{m_2}y^{n_2}(a_2 y\,dx + b_2 x\,dy) = 0$	Non-exact	$IF = x^h y^k$ where $\frac{m_1+h+1}{a_1} = \frac{n_1+k+1}{b_1}$ and $\frac{m_2+h+1}{a_2} = \frac{n_2+k+1}{b_2}$	(i) $\int M_1(x,y)\,dx + \int$ (terms of N_1 not containing x) $dy = c$ (ii) $\int$ (terms of M_1 not containing y) $dx + \int N_1(x,y)\,dy = c$
7.	$\frac{dy}{dx} + Py = Q$, where P and Q are functions of x	Linear in y	$IF = e^{\int P\,dx}$	$ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c$
8.	$\frac{dy}{dx} + Py = Qy^n$	Nonlinear	$IF = e^{\int P_1\,dx}$ where $P_1 = (1-n)v$ and $v = y^{1-n}$	$ve^{\int P_1\,dx} = \int Q_1 e^{\int P_1\,dx}\,dx + c$ where $Q_1 = (1-n)Q$
9.	$f'(y)\frac{dy}{dx} + Pf(y) = Q$	Nonlinear	$IF = e^{\int P\,dx}$	$ve^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c$ where $f(y) = v$

Note: In the cases 1 to 6 after multiplication by IF, differential equation reduces to $M_1(x, y)\,dx + N_1(x, y)\,dy = 0$

Ordinary Differential Equations of First Order and Higher Degree

If the degree of $\frac{dy}{dx}$ in a differential equation of first order is higher than 1, it is convenient to denote $\frac{dy}{dx}$ by p. Hence, a differential equation of first order and higher degree can be written as

$$f(x, y, p) = 0$$

Case I Equations solvable for p

Let $p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_n = 0$ ...(1)

be an ordinary differential equation of first order and n^{th} degree, where $p = \frac{dy}{dx}$ and $P_1, P_2, \ldots, P_n$, are functions of x and y.

The general solution of Eq. (1) is

$$F_1(x, y, c) F_2(x, y, c) \ldots . F_n(x, y, c) = 0$$

Case II *Equations solvable for y*

If the given ordinary differential equation is solvable for y, it can be put in the form

$$y = f(x, p) \qquad \text{...(2)}$$

Differentiating Eq. (2) w.r.t. x,

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right) \qquad \text{...(3)}$$

Equation (3) is a differential equation in p and x.

The general solution of Eq. (3) is

$$F(x, p, c) = 0 \qquad \text{...(4)}$$

where c is an arbitrary constant. The elimination of p from Eqs (2) and (4) gives the general solution.

Case III *Equations solvable for x*

If the given ordinary differential equation is solvable for x, it can be put in the form

$$x = f(y, p) \qquad \text{...(5)}$$

Differentiating Eq. (5) w.r.t. y,

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right) \qquad \text{...(6)}$$

Equation (6) is a differential equation in p and y.

The general solution of Eq. (6) is

$$F(y, p, c) = 0 \quad \text{...(7)}$$

where c is an arbitrary constant. The elimination of p from Eqs (5) and (6) gives the general solution.

Clairaut's Equation

An equation of the form

$$y = px + f(p) \quad \text{...(8)}$$

is known as Clairaut's equation.

The general solution of Eq. (8) is

$$y = cx + f(c) \quad \text{...(9)}$$

Hence, the solution of Clairaut's equation is obtained on replacing p by c.

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. Integrating factor of the differential equation $\frac{dx}{dy} + \frac{3x}{y} = \frac{1}{y^2}$ is **[Summer 2016]**

   (a) y^2 (b) y (c) y^3 (d) $2y^3$

2. The general solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = \tan 2x$ is **[Summer 2016]**

   (a) $\sin(yx) = c$ (b) $\sin\left(\frac{y}{x}\right) = c$

   (c) $\sin y = c$ (d) $\sin x = c$

3. The type, order and degree of the differential equation $\left(\frac{dx}{dy}\right)^2 + 5y^{\frac{1}{3}} = x$ are **[Summer 2016]**

   (a) Linear, First, Two (b) Nonlinear, First, Two

   (c) Linear, Second, First (d) Nonlinear, Second, First

4. The solution of $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$ is **[Winter 2016]**

   (a) $3e^{2y} = 2(e^{3x} + x^3) + 6c$ (b) $e^{2y} = e^{3x} + x^3 + c$

   (c) $3e^{2y} = (e^{3x} + x^3) + 6c$ (d) $e^{2y} = 2(e^{3x} + x^3) + 6c$

5. Integrating factor of the differential equation $\frac{dy}{dx}+\frac{y}{1+x^2}=x^2$ is

(a) $e^{\frac{1}{1+y^2}}$ (b) $e^{\frac{x^2}{y^2}}$ (c) y^2 (d) $e^{\log y}$

6. The Bernoulli's differential equation $\frac{dy}{dx}-y\tan x=y^4\sec x$ reduces to linear differential equation

(a) $\frac{du}{dx}+(3\tan x)u=-3\sec x$ where $y^{-3}=u$

(b) $\frac{du}{dx}(\tan x)u=3\sec x$ where $y^{-3}=u$

(c) $\frac{du}{dx}+(\tan x)u=-\sec x$ where $y^{-3}=u$

(d) None of these

7. The value of α so that $e^{\alpha y^2}$ is an integrating factor of the linear differential equation $\frac{dx}{dy}+xy=e^{-\frac{y^2}{2}}$ is

(a) -1 (b) $-\frac{1}{2}$ (c) 1 (d) $\frac{1}{2}$

8. The general solution of $\frac{dy}{dx}+(\cot x)y=\sin 2x$ with integrating factor $\sin x$ is

(a) $y\sin x=\frac{2}{3}\sin^2 x+c$ (b) $y\sin x=\sin^3 x+c$

(c) $y\sin x=\frac{2}{3}\sin^3 x+c$ (d) None of these

9. Which of the following differential equation is not exact?

[Winter 2015; Summer 2017]

(a) $(y^2-x^2)dx+2xy\,dy=0$ (b) $(x^3+3xy^2)dx+(3x^2y+y^3)dy=0$

(c) $\frac{dy}{dx}=\frac{y}{x}$ (d) $ye^x dx+(2y+e^x)dy=0$

10. The integrating factor of the linear differential equation $y'-\left(\frac{1}{x}\right)y=x^2$ is

[Summer 2017]

(a) $\frac{1}{x^2}$ (b) x (c) $\frac{1}{x}$ (d) x^2

11. The order of the differential equation whose general solution is given by $y = (c_1 + c_2)\cos(x + c_3) - c_4 e^{x+c_5}$, where c_1, c_2, c_3, c_4, c_5 are arbitrary constants is

(a) 5 (b) 4 (c) 3 (d) 2

12. The value of n so that e^{ny^2} is an integrating factor of the differential equation $\left(e^{\frac{y^2}{2}} - xy\right)dy - dx = 0$ is

(a) -1 (b) 1 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

13. The solution of the differential equation $\frac{dy}{dx} = x - 1$ satisfying $y(1) = 1$ is

(a) $y^2 = x^2 - 2x + 2$ (b) $y^2 = 2x^2 - x - 1$
(c) $y = x^2 - 2x + 2$ (d) None of these

14. Which of the following is a solution to the differential equation $\frac{dx}{dt} + 3x = 0$?

(a) $x = 3e^{-t}$ (b) $x = 2e^{-3t}$ (c) $x = -\frac{3}{2}t^2$ (d) $x = 3t^2$

15. The solution of the differential equation $\frac{dy}{dx} + 2xy = e^{-x^2}$ with $y(0) = 1$ is

(a) $(1+x)e^{x^2}$ (b) $(1+x)e^{-x^2}$ (c) $(1-x)e^{x^2}$ (d) $(1-x)e^{-x^2}$

16. The solution of the differential equation $\frac{dy}{dx} + y^2 = 0$ is

(a) $y = \frac{1}{x + c}$ (b) $y = -\frac{x^3}{3} + c$
(c) ce^x (d) unsolvable as equation is nonlinear

17. On conversion $\frac{dy}{dx} = \frac{xy^2 - y}{x}$ into exact equation, the differential equation becomes

(a) $\frac{xy-1}{x^2y}dx - \frac{1}{xy^2}dy = 0$ (b) $\frac{x-1}{xy}dx - \frac{1}{xy}dy = 0$

(c) $\frac{1}{x}dx - \frac{1}{y}dy = 0$ (d) none of these

Answers

1. (c)	2. (b)	3. (b)	4. (a)	5. (a)	6. (a)	7. (d)	8. (c)
9. (c)	10. (c)	11. (c)	12. (c)	13. (a)	14. (b)	15. (a)	16. (a)
17. (a)							

CHAPTER 5

Ordinary Differential Equations of Higher Orders

Chapter Outline

5.1 INTRODUCTION

Higher-order ordinary differential equations are expressions that involve derivatives other than the first order and their properties are different to those of first-order ordinary differential equations. In first-order ordinary differential equations the general solution contains a single constant of integration, which can be determined from an initial condition. For n^{th} order ordinary differential equations the highest order derivative will be $\frac{d^n y}{dx^n}$ and to find the solution, ordinary differential equations are integrated

n times, which yields n constants of integration. Therefore, n unknown constants are found in the general solution of an n^{th} order ordinary differential equations. In order to determine these constants, n constraints are required. In a first-order system only one constraint is applied at a single value of the independent variable. In higher-order systems the (multiple) constraints do not have to be applied at the same value of the independent variable. In fact, precisely where the constraints are applied can dramatically affect the behaviour of the system.

5.2 HOMOGENEOUS LINEAR ORDINARY DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

An ordinary differential equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = 0 \qquad \ldots(5.1)$$

where a_0, a_1, a_2, ..., a_n are constants, is known as a homogeneous linear ordinary differential equation of order n with constant coefficients. This equation is known as linear since the degree of the dependent variable y and all its differential coefficients is one.

Equation (5.1) can also be written as

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n)y = 0$$

$$f(D)y = 0$$

where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n$.

Here, $D \equiv \frac{d}{dx}$ is known as the *differential operator*.

The operator D obeys the laws of algebra.

General Solution of a Homogeneous Linear Ordinary Differential Equation

The homogeneous equation

$$f(D)y = 0 \qquad \ldots(5.2)$$

can be solved by replacing D by m in $f(D)$ and solving the auxiliary equation (AE)

$$f(m) = 0 \qquad \ldots(5.3)$$

The general solution of Eq. (5.2) depends upon the nature of the roots of the auxiliary Eq. (5.3).

If $m_1, m_2, m_3, \ldots, m_n$ are n roots of the auxiliary equation, the following cases arise:

Case I Real and distinct roots: If roots $m_1, m_2, m_3, \ldots, m_n$ are real and distinct then the solution of Eq. (5.1) is given as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

Case II Real and repeated roots: If two roots m_1, m_2 are real and equal, and the remaining $(n-2)$ roots $m_3, m_4, \ldots, m_n$ are all real and distinct then the solution of Eq. (5.1) is given as

$$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Note: If, however, r roots $m_1, m_2, m_3, \ldots, m_r$ are equal and remaining $(n-r)$ roots $m_{r+1}, m_{r+2}, \ldots, m_n$ are all real and distinct then the solution of Eq. (5.1) is given as

$$y = (c_1 + c_2 x + c_3 x^2 + \ldots + c_r x^{r-1}) e^{m_1 x} + c_{r+1} e^{m_{r+1} x} + \ldots + c_n e^{m_n x}$$

Case III Complex roots: If two roots m_1, m_2 are complex say, $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ (conjugate pair) and remaining $(n-2)$ roots $m_3, m_4, \ldots, m_n$ are real and distinct then the solution of Eq. (5.1) is given as

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Here, α is the real part and β is the imaginary part of the conjugate pair of complex roots.

Note: If, however, two pairs of complex roots m_1, m_2 and m_3, m_4 are equal, say, $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$ and remaining $(n-4)$ roots $m_5, m_6, \ldots, m_n$ are real and distinct then the solution of Eq. (5.1) is given as

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \ldots + c_n e^{m_n x}$$

Remark

(i) In all the above cases, $c_1, c_2, \ldots, c_n$ are arbitrary constants.

(ii) In the general solution of a homogeneous equation, the number of arbitrary constants is always equal to the order of that homogeneous equation.

Example 1

Solve $(D^2 + 2D - 1)y = 0$.

Solution

The auxiliary equation is

$$m^2 + 2m - 1 = 0$$

$$m = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2} \qquad \text{(real and distinct)}$$

Hence, the general solution is

$$y = c_1 e^{(-1+\sqrt{2})x} + c_2 e^{(-1-\sqrt{2})x}$$

Example 2

Solve $2D^2y + Dy - 6y = 0.$

Solution

The equation can be written as

$$(2D^2 + D - 6)y = 0$$

The auxiliary equation is

$$2m^2 + m - 6 = 0$$

$$(2m - 3)(m + 2) = 0$$

$$m = -2, \frac{3}{2} \qquad \textbf{(real and distinct)}$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{\frac{3}{2}x}$$

Example 3

Solve $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 9x = 0.$

Solution

$$(D^2 + 6D + 9)x = 0$$

The auxiliary equation is

$$m^2 + 6m + 9 = 0$$

$$(m + 3)^2 = 0$$

$$m = -3, -3 \qquad \textbf{(real and repeated)}$$

Hence, the general solution is

$$x = (c_1 + c_2 t)e^{-3x}$$

Example 4

Solve $y'' + 4y' + 4y = 0, y(0) = 1, y'(0) = 1.$ **[Summer 2016]**

Solution

$$(D^2 + 4D + 4)y = 0$$

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The auxiliary equation is

$$m^2 + 4m + 4 = 0$$

$$(m + 2)^2 = 0$$

$$m = -2, -2 \text{ (real and repeated)}$$

Hence, the general solution is

$$y = (c_1 + c_2x)\, e^{-2x} \quad \text{...(1)}$$

Differentiating Eq. (1),

$$y' = -2(c_1 + c_2x)\, e^{-2x} + e^{-2x}c_2 \quad \text{...(2)}$$

Putting $x = 0$ in Eqs (1) and (2),

$$y(0) = c_1$$

$$c_1 = 1 \quad \text{...(3)}$$

$$y'(0) = -2c_1 + c_2$$

$$1 = -2c_1 + c_2$$

$$1 = -2 + c_2$$

$$c_2 = 3 \quad \text{...(4)}$$

Hence, the particular solution is

$$y = (1 + 3x)e^{-2x}$$

Example 5

Solve the initial-value problem $y'' - 4y' + 4y = 0$, $y(0) = 3$, $y'(0) = 1$.

[Winter 2014]

Solution

$$(D^2 - 4D + 4)y = 0$$

The auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$(m - 2)^2 = 0$$

$$m = 2, 2 \qquad \text{(real and repeated)}$$

Hence, the general solution is

$$y = (c_1 + c_2x)e^{2x} \quad \text{...(1)}$$

Differentiating Eq. (1),

$$y' = 2c_1e^{2x} + 2c_2e^{2x}x + c_2e^{2x} \quad \text{...(2)}$$

Putting $x = 0$ in Eqs (1) and (2),

$$y(0) = c_1$$

$$3 = c_1$$
$$y'(0) = 2c_1 + c_2$$
$$1 = 2c_1 + c_2$$
$$1 = 2(3) + c_2$$
$$c_2 = -5$$

Hence, the particular solution is

$$y = (3 - 5x)e^{2x}$$

Example 6

Solve the initial-value problem $y'' - 9y = 0$, $y(0) = 2$, $y'(0) = -1$.

[Winter 2(

Solution

$$(D^2 - 9)y = 0$$

The auxiliary equation is

$$m^2 - 9 = 0$$
$$m = \pm 3 \qquad \text{(real and distinct)}$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-3x}$$

Differentiating Eq. (1),

$$y' = 3c_1 e^{3x} - 3c_2 e^{-3x}$$

Putting $x = 0$ in Eqs (1) and (2),

$$y(0) = c_1 + c_2$$
$$2 = c_1 + c_2$$
$$y'(0) = 3c_1 - 3c_2$$
$$-1 = 3c_1 - 3c_2$$

Solving Eqs (3) and (4),

$$c_1 = \frac{5}{6}, \qquad c_2 = \frac{7}{6}$$

Hence, the particular solution is

$$y = \frac{5}{6}e^{3x} + \frac{7}{6}e^{-3x}$$

Example 7

Solve $y''' - 6y'' + 11y' - 6y = 0.$ **[Summer 2018]**

Solution

$$(D^3 - 6D^2 + 11D - 6)y = 0$$

The auxiliary equation is

$$m^3 - 6m^2 + 11m - 6 = 0$$
$$(m-1)(m^2 - 5m + 6) = 0$$
$$(m-1)(m-2)(m-3) = 0$$
$$m = 1, 2, 3 \text{ (real and distinct)}$$

Hence, the general solution is

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}$$

Example 8

Solve $(D^3 - 3D^2 - D + 3)y = 0.$

Solution

The auxiliary equation is

$$m^3 - 3m^2 - m + 3 = 0$$
$$(m-3)(m^2 - 1) = 0$$
$$m = 3, -1, 1 \quad \text{(real and distinct)}$$

Hence, the general solution is

$$y = c_1e^{3x} + c_2e^{-x} + c_3e^x$$

Example 9

Solve $(D^3 - 5D^2 + 8D - 4)y = 0.$

Solution

The auxiliary equation is

$$m^3 - 5m^2 + 8m - 4 = 0$$
$$(m-1)(m^2 - 4m + 4) = 0$$
$$(m-1)(m-2)^2 = 0$$
$$m = 1 \text{ (real and distinct)}, \quad m = 2, 2 \text{ (real and repeated)}$$

Hence, the general solution is

$$y = c_1e^x + (c_2 + c_3x)e^{2x}$$

Example 10

Solve $(D^3 + 1)\,y = 0.$

$$D = \frac{-(-1) \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

Solution

The auxiliary equation is

$$m^3 + 1 = 0$$

$$(m+1)(m^2 - m + 1) = 0$$

$$m + 1 = 0,\ m^2 - m + 1 = 0$$

$$m = -1 \quad \text{(real and distinct)}, \qquad m = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \quad \text{(complex)}$$

Hence, the general solution is

$$y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$$

Example 11

Solve $(D^4 - 2D^3 + D^2)y = 0.$

Solution

The auxiliary equation is

$$m^4 - 2m^3 + m^2 = 0$$

$$m^2(m^2 - 2m + 1) = 0$$

$$m^2(m-1)^2 = 0$$

$$m = 0, 0, 1, 1 \qquad \text{(real and repeated)}$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{0x} + (c_3 + c_4 x)e^{x}$$

$$= c_1 + c_2 x + (c_3 + c_4 x)e^{x}$$

Example 12

Solve $(D^4 - 6D^3 + 12D^2 - 8D)y = 0.$

Solution

The auxiliary equation is

$$m^4 - 6m^3 + 12m^2 - 8m = 0$$
$$m(m^3 - 6m^2 + 12m - 8) = 0$$
$$m(m-2)(m^2 - 4m + 4) = 0$$
$$m(m-2)(m-2)^2 = 0$$

$m = 0$ (real and distinct), $m = 2,2,2$ (real and repeated)

Hence, the general solution is

$$y = c_1 e^{0x} + (c_2 + c_3 x + c_4 x^2) e^{2x}$$
$$= c_1 + (c_2 + c_3 x + c_4 x^2) e^{2x}$$

Example 13

Solve $(D^4 - 1)y = 0$.

Solution

The auxiliary equation is

$$m^4 - 1 = 0$$
$$m^4 = 1$$
$$m^2 = 1, \quad m^2 = -1$$

$m = \pm 1$ ((real and distinct), $m = \pm i$ (complex)

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{-x} + e^{0x}(c_3 \cos x + c_4 \sin x)$$
$$= c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

Example 14

Solve $(D^4 + 4D^2)y = 0$.

Solution

The auxiliary equation is

$$m^4 + 4m^2 = 0$$
$$m^2(m^2 + 4) = 0$$

$m = 0, 0$ (real and distinct), $m = \pm 2i$ (complex)

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{0x} + c_3 \cos 2x + c_4 \sin 2x$$
$$= c_1 + c_2 x + c_3 \cos 2x + c_4 \sin 2x$$

Example 15

Solve $(D^4 + 4)y = 0$.

Solution

The auxiliary equation is

$$m^4 + 4 = 0$$
$$m^4 + 4 + 4m^2 - 4m^2 = 0$$
$$(m^2 + 2)^2 - (2m)^2 = 0$$
$$(m^2 + 2 + 2m)(m^2 + 2 - 2m) = 0$$
$$(m^2 + 2m + 2)(m^2 - 2m + 2) = 0$$
$$m = -1 \pm i \text{ and } m = 1 \pm i \quad \text{(complex)}$$

Hence, the general solution is

$$y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^{x}(c_3 \cos x + c_4 \sin x)$$

Example 16

Solve $(D^4 + 8D^2 + 16)y = 0$.

Solution

The auxiliary equation is

$$m^4 + 8m^2 + 16 = 0$$
$$(m^2 + 4)^2 = 0$$
$$m = \pm 2i, \ \pm 2i \quad \text{(complex)}$$

Hence, the general solution is

$$y = e^{0x}[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x]$$
$$= (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$$

5.3 HOMOGENEOUS LINEAR ORDINARY DIFFERENTIAL EQUATIONS: METHOD OF REDUCTION OF ORDER

This method is used to obtain the second solution of a homogeneous linear ordinary differential equation of second order if one solution is known. Since a second linearly independent solution is obtained by solving a first-order ordinary differential equation, it is known as the *method of reduction of order.*

Consider the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0 \qquad \text{...(5.4)}$$

Let y_1 be the known solution of Eq. (5.4).

Putting $y = y_1$ in Eq. (5.4),

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0 \qquad \text{...(5.5)}$$

Let $y = y_2 = uy_1$ be the second solution of Eq. (5.4).

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$$

$$= u''y_1 + 2u'y_1' + uy_1''$$

Substituting in Eq. (5.4),

$$(u''y_1 + 2u'y_1' + uy_1'') + P(u'y_1 + uy_1') + Qy_1 = 0$$

$$u''y_1 + 2u'y_1' + Pu'y_1 + u(y_1'' + Py_1' + Qy_1) = 0$$

$$u''y_1 + u'(2y_1' + Py_1) = 0 \qquad \text{[Using Eq. (5.5)]}$$

$$u'' + u'\left(\frac{2y_1'}{y_1} + P\right) = 0 \qquad \text{...(5.6)}$$

Putting $u' = U$ in Eq. (5.6),

$$U' + \left(\frac{2y_1'}{y_1} + P\right)U = 0$$

$$\frac{dU}{dx} = -\left(\frac{2y_1'}{y_1} + P\right)U$$

$$\frac{dU}{U} = -\left(\frac{2y_1'}{y_1} + P\right)dx$$

Integrating both the sides,

$$\int \frac{dU}{U} = -2\int \frac{y_1'}{y_1}dx - \int P dx$$

$$\ln U = -2\ln y_1 - \int P dx$$

$$= -\ln y_1^2 - \int P dx$$

$$\ln U + \ln y_1^2 = -\int P dx$$

$$\ln Uy_1^2 = -\int P dx$$

$$Uy_1^2 = e^{-\int P\,dx}$$

$$U = \frac{1}{y_1^2} e^{-\int P\,dx}$$

$$u' = \frac{1}{y_1^2} e^{-\int P\,dx}$$

$$\frac{du}{dx} = \frac{1}{y_1^2} e^{-\int P\,dx}$$

Integrating both the sides,

$$u = \int \frac{1}{y_1^2} e^{-\int P\,dx} dx$$

Hence,

$$y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int P\,dx} dx$$

Since $U > 0$,

$$u = \int U\,dx \text{ cannot be constant.}$$

$$\frac{y_2}{y_1} = u \neq \text{constant}$$

Hence, y_1 and y_2 are linearly independent solutions.

Example 1

If $y_1 = x$ is one solution of $x^2y'' + xy' - y = 0$, find the second solution.

Solution

Rewriting the equation,

$$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0 \quad \text{...(1)}$$

Comparing Eq. (1) with the standard equation,

$$y'' + P(x)y' + Q(x)y = 0$$

$$P = \frac{1}{x}$$

Let $y_2 = uy_1$ be the second solution of Eq. (1).

where

$$u = \int \frac{1}{y_1^2} e^{-\int P\,dx} dx, \; y_1 = x$$

$$e^{-\int P\,dx} = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

$$u = \int \frac{1}{x^2} \cdot \frac{1}{x} \mathrm{d}x$$
$$= \int x^{-3} \mathrm{d}x$$
$$= \frac{x^{-2}}{-2}$$
$$= -\frac{1}{2x^2}$$
$$y_2 = \left(-\frac{1}{2x^2}\right) x$$
$$= -\frac{1}{2x}$$

Example 2

If $y_1 = x^2$ is one solution of $x^2y'' - 4xy' + 6y = 0$, $x > 0$ find the second solution. Also, determine the general solution.

Solution

Rewriting the equation,

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 0 \qquad \text{...(1)}$$

Comparing Eq. (1) with standard equation,

$$y'' + P(x)y' + Q(x)y = 0$$
$$P = -\frac{4}{x}$$

Let $y_2 = uy_1$ be the second solution of Eq. (1).

where $u = \int \frac{1}{y_1^2} e^{-\int P \mathrm{d}x} \mathrm{d}x, \quad y_1 = x^2$

$$e^{-\int P \mathrm{d}x} = e^{-\int -\frac{4}{x} \mathrm{d}x}$$
$$= e^{4 \ln x}$$
$$= e^{\ln x^4}$$
$$= x^4$$
$$u = \int \frac{1}{x^4} \cdot x^4 \mathrm{d}x$$
$$= \int \mathrm{d}x$$
$$= x$$

$$y_2 = x \cdot x^2 = x^3$$

Hence, the general solution is

$$y = c_1 x^2 + c_2 x^3$$

Example 3

If $y_1 = \dfrac{\sin x}{x}$ *is one solution of* $xy'' + 2y' + xy = 0$, *find the second solution. Also, determine the general solution.*

Solution

Rewriting the equation,

$$y'' + \frac{2}{x}y' + y = 0 \qquad \text{...(1)}$$

Comparing Eq. (1) with the standard equation,

$$y'' + P(x)y' + Q(x)y = 0$$

$$P = \frac{2}{x}$$

Let $y_2 = uy_1$ be the second solution of Eq. (1).

where $\quad u = \int \frac{1}{y_1^2} e^{-\int P dx} dx, \qquad y_1 = \frac{\sin x}{x}$

$$\begin{aligned} e^{-\int P dx} &= e^{-\int \frac{2}{x} dx} \\ &= e^{-2\log x} \\ &= e^{\ln x^{-2}} \\ &= x^{-2} \end{aligned}$$

$$\begin{aligned} u &= \int \frac{x^2}{\sin^2 x} \cdot \frac{1}{x^2} dx \\ &= \int \operatorname{cosec}^2 x \, dx \\ &= -\cot x \end{aligned}$$

$$\begin{aligned} y_2 &= (-\cot x)\frac{\sin x}{x} \\ &= -\frac{\cos x}{x} \end{aligned}$$

Hence, the general solution is

$$y = c_1 \frac{\sin x}{x} - c_2 \frac{\cos x}{x}$$

$$= c_1 \frac{\sin x}{x} + c_2' \frac{\cos x}{x}, \qquad c_2' = -c_2$$

EXERCISE 5.1

Solve the following differential equations:

1. $(D^2 + D - 2)y = 0$

$$\left[\text{Ans.}: y = c_1 e^{-2x} + c_2 e^{x}\right]$$

2. $(4D^2 + 8D - 5y) = 0$

$$\left[\text{Ans.}: y = c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{5x}{2}}\right]$$

3. $(D^2 - 4D - 12)y = 0$

$$\left[\text{Ans.}: y = c_1 e^{6x} + c_2 e^{-2x}\right]$$

4. $(D^2 + 2D - 8)y = 0$

$$\left[\text{Ans.}: y = c_1 e^{2x} + c_2 e^{-4x}\right]$$

5. $(D^2 + 4D + 1)y = 0$

$$\left[\text{Ans.}: y = c_1 e^{(-2+\sqrt{3})x} + c_2 e^{(-2-\sqrt{3})x}\right]$$

6. $(4D^2 + 4D + 1)y = 0$

$$\left[\text{Ans.}: y = (c_1 + c_2 x)e^{-\frac{x}{2}}\right]$$

7. $(D^2 + 2\pi D + \pi^2)y = 0$

$$\left[\text{Ans.}: y = (c_1 + c_2 x)e^{-\pi x}\right]$$

8. $(9D^2 - 12D + 4)y = 0$

$$\left[\text{Ans.}: y = (c_1 + c_2 x)e^{\frac{2x}{3}}\right]$$

9. $(25D^2 - 20D + 4)y = 0$

$$\left[\text{Ans.}: y = (c_1 + c_2 x)e^{\frac{2x}{5}}\right]$$

10. $(9D^2 - 30D + 25)y = 0$

$$\left[\text{Ans.}: y = (c_1 + c_2 x)e^{\frac{5x}{3}}\right]$$

11. $(D^2 - 6D + 25)y = 0$ $\left[\text{Ans.}: y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)\right]$

12. $(D^2 + 6D + 11)y = 0$ $\left[\text{Ans.}: y = e^{-3x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)\right]$

13. $[D^2 - 2aD + (a^2 + b^2)y] = 0$ $[\text{Ans.}: y = e^{ax}(c_1 \cos bx + c_2 \sin bx)]$

14. $(D^3 - 9D)y = 0$ $\left[\text{Ans.}: y = c_1 + c_2 e^{3x} + c_3 e^{-3x}\right]$

15. $(D^3 - 3D^2 - D + 3)y = 0$ $\left[\text{Ans.}: y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{3x}\right]$

16. $(D^3 - 6D^2 + 11D - 6)y = 0$ $\left[\text{Ans.}: y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}\right]$

17. $(D^3 - 6D^2 + 12D - 8)y = 0$ $\left[\text{Ans.}: y = (c_1 + c_2 x + c_3 x^2)e^{2x}\right]$

18. $(D^3 + D)y = 0$ $[\text{Ans.}: y = c_1 + c_2 \cos x + c_3 \sin x]$

19. $(D^3 + 5D^2 + 8D + 6)y = 0$ $\left[\text{Ans.}: y = c_1 e^{-3x} + e^{-x}(c_2 \cos x + c_3 \sin x)\right]$

20. $(8D^4 - 6D^3 - 7D^2 + 6D - 1)y = 0$ $\left[\text{Ans.}: y = c_1 e^{\frac{x}{4}} + c_2 e^{\frac{x}{2}} + c_3 e^{x} + c_4 e^{-x}\right]$

21. $(D^4 - 2D^3 + D^2)y = 0$ $\left[\text{Ans.}: y = c_1 + c_2 x + (c_3 + c_4 x)e^{x}\right]$

22. $(D^4 - 3D^3 + 3D^2 - D)y = 0$ $\left[\text{Ans.}: y = c_1 + (c_2 + c_3 x + c_4 x^2)e^{x}\right]$

23. $(D^4 + 8D^2 - 9)y = 0$ $\left[\text{Ans.}: y = c_1 e^{-3x} + c_2 e^{-x} c_3 \cos 3x + c_4 \sin 3x\right]$

24. $(D^4 + D^3 + 14D^2 + 16D - 32)y = 0$

$$\left[\text{Ans.}: y = c_1 e^{x} + c_2 e^{-2x} c_3 \cos 4x + c_4 \sin 4x\right]$$

25. $(D^4 + 2D^3 - 9D^2 - 10D + 50)y = 0$

$$\left[\text{Ans.}: y = e^{2x}\left(c_1 \cos x + c_2 \sin x\right) + e^{-3x}\left(c_3 \cos x + c_4 \sin x\right)\right]$$

26. $(D^4 + 18D^3 + 81)y = 0$

$$\left[\text{Ans.}: y = \left(c_1 + c_2 x\right)\cos 3x + \left(c_3 + c_4 x\right)\sin 3x\right]$$

27. $(D^4 - 4D^3 + 14D^2 - 20D + 25)y = 0$

$$\left[\text{Ans.}: y = e^{x}\left[\left(c_1 + c_2 x\right)\cos 2x + \left(c_3 + c_4 x\right)\sin 2x\right]\right]$$

5.4 NONHOMOGENEOUS LINEAR ORDINARY DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

An ordinary differential equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n+2} y}{dx^{n-2}} + \ldots + a_n y = Q(x) \qquad \ldots(5.7)$$

where a_0, a_1, a_2, ..., a_n are constants and Q is a function of x, is known as a *nonhomogeneous linear ordinary differential equation with constant coefficients.*
Equation (5.7) can also be written as

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n)y = Q(x) \qquad \ldots(5.8)$$

$$f(D)y = Q(x)$$

where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n$

5.4.1 General Solution of a Nonhomogeneous Linear Ordinary Differential Equations

A general solution of Eq. (5.7) is obtained in two parts as

General solution = Complementary function + Particular integral

$$y = \text{CF} + \text{PI}$$

The complementary function (CF) is the general solution of the homogeneous equation obtained by putting $Q(x) = 0$ in Eq. (5.7).

The particular integral (PI) is any particular solution of the nonhomogeneous Eq. (5.7) and contains no arbitrary constants.

Inverse Operator and Particular Integral

$f(D)$ is known as the *differential operator* and $\frac{1}{f(D)}$ is known as the *inverse differential operator.*

$$f(D)\left[\frac{1}{f(D)}Q(x)\right] = Q(x)$$

This shows that $\frac{1}{f(D)}Q(x)$ satisfies the equation $f(D)y = Q(x)$ and since $\frac{1}{f(D)}Q(x)$ does not contain any arbitrary constants, it gives the PI of the equation $f(D)y = Q(x)$.

Hence,
$$PI = \frac{1}{f(D)}Q(x)$$

(i) If $f(D) = D$ then

$$PI = \frac{1}{D}Q(x) = \int Q(x)\,dx$$

(ii) If $f(D) = D - a$ then the equation $f(D)y = Q(x)$ becomes

$$(D - a)y = Q(x)$$

$$\frac{dy}{dx} - ay = Q(x)$$

is a first-order linear differential equation.

$$IF = e^{\int -a dx} = e^{-ax}$$

The solution is

$$ye^{-ax} = \int e^{-ax}Q(x)\,dx + c$$

$$y = e^{ax}\int Q(x)e^{-ax}dx + ce^{ax}$$

Here, ce^{ax} is the complementary function since it contains the arbitrary constant c and $e^{ax}\int Q(x)e^{-ax}dx$ is the particular integral.

Hence,
$$PI = \frac{1}{D-a}Q(x) = e^{ax}\int Q(x)e^{-ax}dx$$

5.4.2 Direct (Short-cut) Method of Obtaining Particular Integrals

This method depends on the nature of Q(x) in Eq. (5.7). The particular integral by this method can be obtained when Q(x) has the following forms:

(i) $Q(x)=e^{ax+b}$

(ii) $Q(x)=\sin(ax+b)$ or $\cos(ax+b)$

(iii) $Q(x)=x^m$ or polynomial in x

(iv) $Q(x)=e^{ax}v(x)$

(v) $Q(x)=xv(x)$

$M_y = \frac{\partial M}{\partial y}$?
$N_x = \frac{\partial N}{\partial x}$

Case I $Q(x)=e^{ax+b}$

$$f(\mathrm{D})y=e^{ax+b}$$

Now, $\mathrm{D}(e^{ax+b})=ae^{ax+b}$, $\mathrm{D}^2(e^{ax+b})=a^2e^{ax+b}, \ldots, \mathrm{D}^n e^{ax+b}=a^n e^{ax+b}$

Consider

$$\begin{aligned} f(\mathrm{D})(e^{ax+b}) &= (a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1}+\ldots+a_n)e^{ax+b} \\ &= (a_0a^n + a_1a^{n-1}+\ldots+a_n)e^{ax+b} \\ &= f(a)e^{ax} \end{aligned}$$

Operating both the sides with $\frac{1}{f(\mathrm{D})}$,

$$\frac{1}{f(\mathrm{D})}\left[f(\mathrm{D})\left(e^{ax+b}\right)\right]=\frac{1}{f(\mathrm{D})}\left[f(a)e^{ax+b}\right]$$

$$e^{ax+b}=f(a)\frac{1}{f(\mathrm{D})}e^{ax+b}$$

$$\frac{1}{f(a)}e^{ax+b}=\frac{1}{f(\mathrm{D})}e^{ax+b}, \quad f(a)\neq 0$$

$$\frac{1}{f(\mathrm{D})}e^{ax+b}=\frac{1}{f(a)}e^{ax+b}, \quad f(a)\neq 0$$

Hence, $\mathrm{PI}=\frac{1}{f(a)}e^{ax+b}$ if $f(a)\neq 0$

Note: If $f(a)=0$ then $(\mathrm{D}-a)$ is a factor of $f(\mathrm{D})$ and, hence, the above rule fails.

Let $f(\mathrm{D})=(\mathrm{D}-a)\,\phi(\mathrm{D})$, where $\phi(a)\neq 0$

$$\text{PI} = \frac{1}{f(\text{D})} e^{ax+b}$$

$$= \frac{1}{(\text{D}-a)\phi(\text{D})} e^{ax+b}$$

$$= \frac{1}{\phi(a)} \cdot \frac{1}{(\text{D}-a)} e^{ax+b}$$

$$= \frac{1}{\phi(a)} \cdot e^{ax} \int e^{-ax} e^{ax+b} \text{d}x$$

$$= \frac{1}{\phi(a)} \cdot e^{ax} \cdot xe^{b}$$

$$= x \frac{1}{\phi(a)} e^{ax+b} \qquad \text{...(5.9)}$$

Since

$$f(\text{D}) = (\text{D}-a)\phi(\text{D})$$

$$f'(\text{D}) = (\text{D}-a)\phi'(\text{D}) + \phi(\text{D})$$

$$f'(a) = \phi(a)$$

Substituting in Eq. (5.9),

$$\frac{1}{f(\text{D})} e^{ax+b} = x \cdot \frac{1}{f'(a)} e^{ax+b} \quad \text{where } f'(a) \neq 0$$

If $f'(a) = 0$ then repeating the above process,

$$\frac{1}{f(\text{D})} e^{ax+b} = x \left[x \cdot \frac{1}{f''(a)} e^{ax+b} \right]$$

$$= x^2 \frac{1}{f''(a)} e^{ax+b} \quad \text{where } f''(a) \neq 0$$

In general, if $(\text{D}-a)^r$ is a factor of $f(\text{D})$ then

$$\frac{1}{f(\text{D})} e^{ax} = x^r \frac{1}{f^{(r)}(a)} e^{ax+b}$$

Hence,

$$\text{PI} = x^r \frac{1}{f^{(r)}(a)} e^{ax+b}$$

Example 1

Solve $(4\text{D}^2 - 4\text{D} + 1)y = 4$.

Solution

The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)^2 = 0$$

$$m = \frac{1}{2}, \frac{1}{2} \quad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2 x)e^{\frac{x}{2}}$$

$$\text{PI} = \frac{1}{4\text{D}^2 - 4\text{D} + 1} 4e^{0x}$$

$$= 4 \cdot \frac{1}{4(0) - 4(0) + 1} e^{0x}$$

$$= 4$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{\frac{x}{2}} + 4$$

Example 2

Solve $(\text{D}^2 + 5\text{D} + 6)y = e^x$. **[Summer 2014]**

Solution

The auxiliary equation is

$$m^2 + 5m + 6 = 0$$

$$(m+3)(m+2) = 0$$

$$m = -2, -3 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{-2x} + c_2 e^{-3x}$$

$$\text{PI} = \frac{1}{\text{D}^2 + 5\text{D} + 6} e^x$$

$$= \frac{1}{1^2 + 5(1) + 6} e^x$$

$$= \frac{1}{12} e^x$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{12} e^x$$

Example 3

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$. **[Winter 2013]**

Solution

$$(D^2 - 5D + 6)y = e^{4x}$$

The auxiliary equation is

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1e^{2x} + c_2e^{3x}$$

$$\text{PI} = \frac{1}{D^2 - 5D + 6}e^{4x}$$

$$= \frac{1}{4^2 - 5(4) + 6}e^{4x}$$

$$= \frac{1}{2}e^{4x}$$

Hence, the general solution is

$$y = c_1e^{2x} + c_2e^{3x} + \frac{1}{2}e^{4x}$$

Example 4

Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = e^{6x}$. **[Winter 2012]**

Solution

$$(D^2 + D - 12)y = e^{6x}$$

The auxiliary equation is

$$m^2 + m - 12 = 0$$

$$(m-3)(m+4) = 0$$

$$m = 3, -4 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1e^{3x} + c_2e^{-4x}$$

$$\text{PI} = \frac{1}{D^2 + D - 12}e^{6x}$$

$$= \frac{1}{6^2 + 6 - 12}e^{6x}$$

$$= \frac{1}{30}e^{6x}$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-4x} + \frac{1}{30} e^{6x}$$

Example 5

Solve $y'' - 3y' + 2y = e^{3x}$. **[Summer 2018]**

Solution

$$(D^2 - 3D + 2)y = e^{3x}$$

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, 2 \quad \text{(real and distinct)}$$

$$CF = c_1 e^x + c_2 e^{2x}$$

$$PI = \frac{1}{D^2 - 3D + 2} e^{3x}$$

$$= \frac{1}{3^2 - 3(3) + 2} e^{3x}$$

$$= \frac{1}{2} e^{3x}$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{2x} + \frac{1}{2} e^{3x}$$

Example 6

Solve $(D^2 + 1)y = e^{-x}$.

Solution

The auxiliary equation is

$$m^2 - 1 = 0$$

$$m^2 = -1$$

$$m = \pm i \quad \text{(complex)}$$

$$CF = c_1 \cos x + c_2 \sin x$$

$$PI = \frac{1}{D^2 + 1} e^{-x}$$

$$= \frac{1}{(-1)^2 + 1} e^{-x}$$

$$= \frac{1}{2} e^{-x}$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} e^{-x}$$

Example 7

Solve $(D^2 + 2D + 1)\, y = e^{-x}$.

Solution

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$(m+1)^2 = 0$$

$$m = -1, -1 \quad \text{(real and repeated)}$$

$$CF = (c_1 + c_2 x)e^{-x}$$

$$PI = \frac{1}{D^2 + 2D + 1} e^{-x}$$

$$= x \frac{1}{2D + 2} e^{-x}$$

$$= x^2 \frac{1}{2} e^{-x}$$

$$= \frac{1}{2} x^2 e^{-x}$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-x} + \frac{1}{2} x^2 e^{-x}$$

Example 8

[Summer 2015]

Solve $(D^2 - 2D + 1)y = 10\, e^x$.

Solution

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$(m - 1)^2 = 0$$

$$m = 1, 1 \quad \text{(real and repeated)}$$

$$CF = (c_1 + c_2x)\, e^x$$

$$PI = \frac{1}{D^2 - 2D + 1} 10e^x$$

$$= \frac{1}{(D-1)^2} 10\, e^x$$

$$= \frac{x}{2(D-1)} 10e^x$$

$$= \frac{x^2}{2}(10e^x)$$

$$= 5x^2\, e^x$$

Hence, the general solution is

$$y = (c_1 + c_2x)\, e^x + 5x^2 e^x$$

Example 9

Solve $(4D^2 - 4D + 1)y = e^{\frac{x}{2}}$.

Solution

The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)^2 = 0$$

$$m = \frac{1}{2}, \frac{1}{2} \quad \text{(real and repeated)}$$

$$CF = (c_1 + c_2 x)e^{\frac{x}{2}}$$

$$PI = \frac{1}{4D^2 - 4D + 1} e^{\frac{x}{2}}$$

$$= x \cdot \frac{1}{8D - 4} e^{\frac{x}{2}}$$

$$= x^2 \cdot \frac{1}{8} e^{\frac{x}{2}}$$

$$= \frac{x^2}{8} e^{\frac{x}{2}}$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{\frac{x}{2}} + \frac{x^2}{8} e^{\frac{x}{2}}$$

Example 10

Solve $(D^2 - 4)y = e^{2x} + e^{-4x}$.

Solution

The auxiliary equation is

$$m^2 - 4 = 0$$

$$(m-2)(m+2) = 0$$

$$m = 2, -2 \quad \text{(real and distinct)}$$

$$CF = c_1 e^{2x} + c_2 e^{-2x}$$

$$PI = \frac{1}{D^2 - 4}(e^{2x} + e^{-4x})$$

$$= \frac{1}{D^2-4}e^{2x} + \frac{1}{D^2-4}e^{-4x}$$

$$= x \cdot \frac{1}{2D}e^{2x} + \frac{1}{(-4)^2 - 4}e^{-4x}$$

$$= x \cdot \frac{1}{2(2)}e^{2x} + \frac{1}{12}e^{-4x}$$

$$= \frac{x}{4}e^{2x} + \frac{1}{12}e^{-4x}$$

Hence, the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{x}{4}e^{2x} + \frac{1}{12}e^{-4x}$$

Example 11

Solve $(D^2 + 4D + 5)y = -2\cosh x$.

Solution

The auxiliary equation is

$$m^2 + 4m + 5 = 0$$

$$m = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i \quad \text{(complex)}$$

$$CF = e^{-2x}(c_1 \cos x + c_2 \sin x)$$

$$\text{PI} = \frac{1}{D^2+4D+5}(-2\cosh x)$$

$$= -2\frac{1}{D^2+4D+5}\left(\frac{e^x+e^{-x}}{2}\right)$$

$$= -\frac{1}{D^2+4D+5}e^x - \frac{1}{D^2+4D+5}e^{-x}$$

$$= -\frac{1}{(1)^2+4(1)+5}e^x - \frac{1}{(-1)^2+4(-1)+5}e^{-x}$$

$$= -\frac{1}{10}e^x - \frac{1}{2}e^{-x}$$

Hence, the general solution is

$$y = e^{-2x}(c_1\cos x + c_2\sin x) - \frac{1}{10}e^x - \frac{1}{2}e^{-x}$$

Example 12

Solve $(D^2+6D+9)y = 5^x - \log 2.$

Solution

The auxiliary equation is

$$m^2+6m+9=0$$

$$(m+3)^2=0$$

$$m=-3,-3 \quad \text{(real and repeated)}$$

$$\text{CF} = (c_1+c_2x)e^{-3x}$$

$$\text{PI} = \frac{1}{D^2+6D+9}(5^x - \log 2)$$

$$= \frac{1}{(D+3)^2}(e^{x\log 5}) - \frac{1}{(D+3)^2}(\log 2)e^{0\cdot x}$$

$$= \frac{1}{(\log 5+3)^2}e^{x\log 5} - \log 2\cdot\frac{1}{(0+3)^2}e^{0\cdot x}$$

$$= \frac{5^x}{(\log 5+3)^2} - \frac{\log 2}{9}$$

Hence, the general solution is

$$y = (c_1+c_2x)e^{-3x} + \frac{5^x}{(\log 5+3)^2} - \frac{\log 2}{9}$$

Example 13

Solve $y''' - 3y'' + 3y' - y = 4e^t$. **[Winter 2014]**

Solution

$$(D^3 - 3D^2 + 3D - 1)y = 4e^t$$

The auxiliary equation is

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$(m-1)^3 = 0$$

$$m = 1, 1, 1 \quad \text{(real and repeated)}$$

$$CF = (c_1 + c_2 t + c_3 t^2)e^t$$

$$PI = \frac{1}{D^3 - 3D^2 + 3D - 1} 4e^t$$

$$= \frac{1}{(D-1)^3} 4e^t$$

$$= 4t \frac{1}{3(D-1)^2} e^t$$

$$= \frac{4t^2}{3} \frac{1}{2(D-1)} e^t$$

$$= \frac{4}{3} t^3 \frac{1}{2} e^t$$

$$= \frac{2}{3} t^3 e^t$$

Hence, the general solution is

$$y = (c_1 + c_2 t + c_3 t^2)e^t + \frac{2}{3} t^3 e^t$$

Example 14

Solve $(D^3 - D^2 + 4D - 4)y = e^x$.

Solution

The auxiliary equation is

$$m^3 - m^2 + 4m - 4 = 0$$

$$(m-1)(m^2 + 4) = 0$$

$$m - 1 = 0, \qquad m^2 + 4 = 0$$

$$m = 1 \text{ (real and distinct)}, \qquad m = \pm 2i \text{ (complex)}$$

$$\text{CF} = c_1e^x + (c_2\cos 2x + c_3\sin 2x)e^{0x}$$
$$= c_1e^x + c_2\cos 2x + c_3\sin 2x$$
$$\text{PI} = \frac{1}{D^3 - D^2 + 4D - 4}e^x$$
$$= x\cdot\frac{1}{3D^2 - 2D + 4}e^x$$
$$= x\frac{1}{3-2+4}e^x$$
$$= \frac{x}{5}e^x$$

Hence, the general solution is

$$y = c_1e^x + c_2\cos 2x + c_3\sin 2x + \frac{x}{5}e^x$$

Example 15

Solve $\frac{d^3y}{dx^3} + 8y = \cosh 2x.$ **[Winter 2015]**

Solution

$$(D^3 + 8)y = \left(\frac{e^{2x} + e^{-2x}}{2}\right)$$

The auxiliary equation is

$$m^3 + 8 = 0$$
$$(m+2)(m^2 - 2m + 4) = 0$$

$m = -2$ (real and distinct), $m = \frac{2 \pm \sqrt{4-16}}{2}$

$$= \frac{2 \pm \sqrt{12}\,i}{2} = \frac{2 \pm 2\sqrt{3}\,i}{2}$$
$$= 1 \pm i\sqrt{3} \quad \text{(complex)}$$

$$\text{CF} = c_1e^{-2x} + e^x(c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x)$$
$$\text{PI} = \frac{1}{D^3 + 8}\left(\frac{e^{2x} + e^{-2x}}{2}\right)$$
$$= \frac{1}{2}\left[\frac{1}{D^3+8}e^{2x} + \frac{1}{D^3+8}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(2)^3+8}e^{2x} + \frac{x}{3D^2}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{16}e^{2x} + \frac{x}{3(-2)^2}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{16}e^{2x} + \frac{x}{12}e^{-2x}\right]$$

$$= \frac{1}{32}e^{2x} + \frac{x}{24}e^{-2x}$$

Hence, the general solution is

$$y = c_1 e^{-2x} + e^x(c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x) + \frac{1}{32}e^{2x} + \frac{x}{24}e^{-2x}$$

Example 16

Solve $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2e^x + 3e^{-x} + 2.$

Solution

The auxiliary equation is

$$m^3 - 5m^2 + 8m - 4 = 0$$

$$(m-1)(m^2 - 4m + 4) = 0$$

$$(m-1)(m-2)^2 = 0$$

$$m = 1 \text{ (real and distinct)}, \quad m = 2, 2 \text{ (real and repeated)}$$

$$\text{CF} = c_1e^x + (c_2 + c_3x)e^{2x}$$

$$\text{PI} = \frac{1}{D^3 - 5D^2 + 8D - 4}(e^{2x} + 2e^x + 3e^{-x} + 2e^{0x})$$

$$= \frac{1}{D^3 - 5D^2 + 8D - 4}e^{2x} + \frac{1}{D^3 - 5D^2 + 8D - 4}2e^x + \frac{1}{D^3 - 5D^2 + 8D - 4}3e^{-x} + \frac{1}{D^3 - 5D^2 + 8D - 4}2e^{0x}$$

$$= x\cdot\frac{1}{3D^2 - 10D + 8}e^{2x} + x\cdot\frac{1}{3D^2 - 10D + 8}2e^x + \frac{1}{-1-5-8-4}3e^{-x} + \frac{1}{-4}2e^{0x}$$

$$= x^2\cdot\frac{1}{6D - 10}e^{2x} + x\frac{1}{3-10+8}2e^x - \frac{1}{18}\cdot 3e^{-x} - \frac{1}{2}$$

$$= x^2\frac{1}{12-10}e^{2x} + 2xe^x - \frac{1}{6}e^{-x} - \frac{1}{2}$$

$$=\frac{x^2}{2}e^{2x}+2xe^x-\frac{1}{6}e^{-x}-\frac{1}{2}$$

Hence, the general solution is

$$y=c_1e^x+(c_2+c_3x)e^{2x}+\frac{x^2}{2}e^{2x}+2xe^x-\frac{1}{6}e^{-x}-\frac{1}{2}$$

Example 17

Solve $(D^3-12D+16)y=(e^x+e^{-2x})^2$.

Solution

$$(D^3-12D+16)y=(e^x+e^{-2x})^2$$
$$=e^{2x}+2e^{-x}+e^{-4x}$$

The auxiliary equation is

$m^3-12m+16=0$

$m=-4$ (real and distinct), $m=2, 2$(real and repeated)

$$CF=c_1e^{-4x}+(c_2+c_3x)e^{2x}$$

$$PI=\frac{1}{D^3-12D+16}(e^{2x}+2e^{-x}+e^{-4x})$$

$$=\frac{1}{D^3-12D+16}e^{2x}+2\frac{1}{D^3-12D+16}e^{-x}+\frac{1}{D^3-12D+16}e^{-4x}$$

$$=\frac{1}{(D+4)(D-2)^2}e^{2x}+2\frac{1}{(-1)^3-12(-1)+16}e^{-x}+\frac{1}{(D+4)(D-2)^2}e^{-4x}$$

$$=\frac{1}{(D-2)^2}\left[\frac{1}{D+4}e^{2x}\right]+2\frac{1}{-1+12+16}e^{-x}+\frac{1}{(D+4)}\left[\frac{1}{(D-2)^2}e^{-4x}\right]$$

$$=\frac{1}{(D-2)^2}\frac{e^{2x}}{6}+\frac{2}{27}e^{-x}+\frac{1}{(D+4)}\cdot\frac{1}{(-4-2)^2}e^{-4x}$$

$$=\frac{1}{6}x\frac{1}{2(D-2)}e^{2x}+\frac{2}{27}e^{-x}+\frac{1}{36}\frac{1}{D+4}e^{-4x}$$

$$=\frac{x}{12}x\cdot\frac{1}{1}e^{2x}+\frac{2}{27}e^{-x}+\frac{1}{36}x\cdot\frac{1}{1}e^{-4x}$$

$$=\frac{x^2}{12}e^{2x}+\frac{2}{27}e^{-x}+\frac{x}{36}e^{-4x}$$

Hence, the general solution is

$$y=c_1e^{-4x}+(c_2+c_3x)e^{2x}+\frac{x^2}{12}e^{2x}+\frac{2}{27}e^{-x}+\frac{x}{36}e^{-4x}$$

Example 18

Solve $(D^6 - 64)y = e^x \cosh 2x$.

Solution

The auxiliary equation is

$$m^6 - 64 = 0$$

$$(m^3)^2 - (8)^2 = 0$$

$$(m^3 + 8)(m^3 - 8) = 0$$

$$(m+2)(m^2 - 2m + 4)(m-2)(m^2 + 2m + 4) = 0$$

$$m + 2 = 0, m^2 - 2m + 4 = 0$$

$$m - 2 = 0, m^2 + 2m + 4 = 0$$

$$m = -2, \quad m = 1 \pm i\sqrt{3}, \qquad m = 2, \quad m = -1 \pm i\sqrt{3}$$

Two roots are real and the two pairs of the roots are complex.

$$\text{CF} = c_1 e^{-2x} + c_2 e^{2x} + e^x(c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x) + e^{-x}(c_5 \cos\sqrt{3}x + c_6 \sin\sqrt{3}x)$$

$$\text{PI} = \frac{1}{D^6 - 64} e^x \cosh 2x$$

$$= \frac{1}{D^6 - 64}\left[e^x\left(\frac{e^{2x} + e^{-2x}}{2}\right)\right]$$

$$= \frac{1}{D^6 - 64}\left[\frac{1}{2}(e^{3x} + e^{-x})\right]$$

$$= \frac{1}{2}\left[\frac{1}{3^6 - 64} e^{3x} + \frac{1}{(-1)^6 - 64} e^{-x}\right]$$

$$= \frac{1}{2}\left(\frac{e^{3x}}{665} - \frac{e^{-x}}{63}\right)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{2x} + e^x(c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x)$$
$$+ e^{-x}(c_5 \cos\sqrt{3}x + c_6 \sin\sqrt{3}x) + \frac{1}{2}\left(\frac{e^{3x}}{665} - \frac{e^{-x}}{63}\right)$$

Case II **$Q(x) = \sin(ax + b)$ or $\cos(ax + b)$**

(i) If $Q(x) = \sin(ax + b)$ then Eq. (5.8) reduces to

$$f(D)y = \sin(ax + b)$$

Now

$$D[\sin(ax+b)] = a\cos(ax+b)$$

$$D^2[\sin(ax+b)] = (-a^2)\sin(ax+b)$$

$$D^3[\sin(ax+b)] = -a^3\cos(ax+b)$$

$$D^4[\sin(ax+b)] = a^4\sin(ax+b)$$

$$(D^2)^2[\sin(ax+b)] = (-a^2)^2\sin(ax+b)$$

In general,

$$(D^2)^r[\sin(ax+b)] = (-a^2)^r\sin(ax+b)$$

This shows that

$$\phi(D^2)\sin(ax+b) = \phi(-a^2)\sin(ax+b)$$

Operating both the sides with $\dfrac{1}{\phi(D^2)}$,

$$\frac{1}{\phi(D^2)}\left[\phi(D^2)\sin(ax+b)\right] = \frac{1}{\phi(D^2)}\left[\phi(-a^2)\sin(ax+b)\right]$$

$$\sin(ax+b) = \phi(-a^2)\frac{1}{\phi(D^2)}\sin(ax+b)$$

$$\frac{1}{\phi(-a^2)}\sin(ax+b) = \frac{1}{\phi(D^2)}\sin(ax+b)$$

$$\frac{1}{\phi(D^2)}\sin(ax+b) = \frac{1}{\phi(-a^2)}\sin(ax+b)$$

If $f(D) = \phi(D^2)$ then

$$\begin{aligned}
PI &= \frac{1}{f(D)}\sin(ax+b) \\
&= \frac{1}{\phi(D^2)}\sin(ax+b) \\
&= \frac{1}{\phi(-a^2)}\sin(ax+b), \text{ if } \phi(-a^2) \neq 0
\end{aligned}$$

If $\phi(-a^2) = 0$ then $(D^2 + a^2)$ is a factor of $\phi(D^2)$ and, hence, the above rule fails.

$$\begin{aligned}
PI &= \frac{1}{\phi(D^2)}\sin(ax+b) \\
&= \frac{1}{\phi(D^2)}\left[\text{I.P. of } e^{i(ax+b)}\right] \\
&= \text{IP of } \frac{1}{\phi(D^2)}e^{i(ax+b)}
\end{aligned}$$

$$= \text{IP of } x \cdot \frac{1}{\phi'(D^2)} e^{i(ax+b)} \qquad \left[\because \phi(i^2a^2) = \phi(-a^2) = 0\right]$$

$$= \text{IP of } x \cdot \frac{1}{\phi'(i^2a^2)} e^{i(ax+b)}$$

$$= \text{IP of } x \cdot \frac{1}{\phi'(-a^2)} e^{i(ax+b)}$$

$$= x \cdot \frac{1}{\phi'(-a^2)} \sin(ax+b)$$

If $\phi'(-a^2) = 0$ then

$$\frac{1}{\phi(D^2)} \sin(ax+b) = x^2 \cdot \frac{1}{\phi''(-a^2)} \sin(ax+b), \qquad \text{where } \phi''(-a^2) \neq 0$$

In general, if $\phi^{(r)}(-a^2) = 0$ then

$$\text{PI} = \frac{1}{\phi(D^2)} \sin(ax+b)$$

$$= x^{(r+1)} \frac{1}{\phi^{(r+1)}(-a^2)} \sin(ax+b), \qquad \text{where } \phi^{(r+1)}(-a^2) \neq 0$$

(ii) Similarly, if $Q(x) = \cos(ax+b)$

$$\text{PI} = \frac{1}{\phi(D^2)} \cos(ax+b)$$

$$= \frac{1}{\phi(-a^2)} \cos(ax+b), \quad \phi(-a^2) \neq 0$$

If $\phi(-a^2) = 0$ then

$$\text{PI} = \frac{1}{\phi(D^2)} \cos(ax+b)$$

$$= x \cdot \frac{1}{\phi'(-a^2)} \cos(ax+b)$$

If $\phi'(-a^2) = 0$ then

$$\text{PI} = \frac{1}{\phi(D^2)} \cos(ax+b)$$

$$= x^2 \cdot \frac{1}{\phi''(-a^2)} \cos(ax+b), \qquad \text{where } \phi''(-a^2) \neq 0$$

In general, if $\phi^{(r)}(-a^2) = 0$ then

$$\text{PI} = \frac{1}{\phi(\text{D}^2)} \cos(ax+b)$$

$$= x^{r+1} \frac{1}{\phi^{(r+1)}(-a^2)} \cos(ax+b), \qquad \text{where } \phi^{(r+1)}(-a^2) \neq 0$$

Note: If after replacing D^2 by $-a^2$, $f(\text{D})$ contains terms of D then the denominator is rationalized to obtain the even powers of D.

Example 1

Solve $(\text{D}^2 + 9)y = \cos 4x$. **[Summer 2018]**

Solution

The auxiliary equation is

$$m^2 + 9 = 0$$

$$m = \pm 3i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos 3x + c_2 \sin 3x$$

$$\text{PI} = \frac{1}{\text{D}^2 + 9} \sin 4x$$

$$= \frac{1}{-4^2 + 9} \cos 4x$$

$$= \frac{1}{-7} \cos 4x$$

Hence, the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{7} \cos 4x$$

Example 2

Solve $(\text{D}^2 + 1)y = \sin^2 x$.

Solution

The auxiliary equation is

$$m^2 + 1 = 0$$

$$m = \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos x + c_2 \sin x$$

$$\text{PI} = \frac{1}{\text{D}^2 + 1} \sin^2 x$$

$$= \frac{1}{D^2+1}\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{D^2+1}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{D^2+1}\cos 2x$$

$$= \frac{1}{2}\cdot\frac{1}{D^2+1}e^{0x} - \frac{1}{2}\cdot\frac{1}{-2^2+1}\cos 2x$$

$$= \frac{1}{2}\cdot\frac{1}{0+1}e^{0x} - \frac{1}{2}\cdot\frac{1}{-3}\cos 2x$$

$$= \frac{1}{2} + \frac{1}{6}\cos 2x$$

Hence, the general solution is

$$y = c_1\cos x + c_2\sin x + \frac{1}{2} + \frac{1}{6}\cos 2x$$

Example 3

Solve $(D^2+3D+2)y = \sin 2x$.

Solution

The auxiliary equation is

$$m^2+3m+2=0$$

$$(m+1)(m+2)=0$$

$$m = -1, -2 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1e^{-x} + c_2e^{-2x}$$

$$\text{PI} = \frac{1}{D^2+3D+2}\sin 2x$$

$$= \frac{1}{-4+3D+2}\sin 2x$$

$$= \frac{1}{3D-2}\sin 2x$$

$$= \frac{1}{(3D-2)}\cdot\frac{(3D+2)}{(3D+2)}\sin 2x$$

$$= \frac{(3D+2)}{9D^2-4}\sin 2x$$

$$= \frac{3D+2}{9(-2^2)-4}\sin 2x$$

$$= \frac{3D+2}{-40}\sin 2x$$

$$= -\frac{3}{40}(D\sin 2x) - \frac{1}{20}\sin 2x$$

$$= -\frac{3}{40}\cdot 2\cos 2x - \frac{1}{20}\sin 2x$$

$$= -\frac{1}{20}(3\cos 2x + \sin 2x)$$

Hence, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{20}(3\cos 2x + \sin 2x)$$

Example 4

Solve $(D^2 + 9)y = 2\sin 3x + \cos 3x.$ **[Summer 2016]**

Solution

The auxiliary equation is

$$m^2 + 9 = 0$$

$$m = \pm 3i \quad \text{(complex)}$$

$$CF = c_1 \cos 3x + c_2 \sin 3x$$

$$PI = \frac{1}{D^2 + 9}(2\sin 3x + \cos 3x)$$

$$= 2\frac{1}{D^2 + 9}\sin 3x + \frac{1}{D^2 + 9}\cos 3x$$

$$= 2\frac{x}{2D}\sin 3x + \frac{x}{2D}\cos 3x$$

$$= x\int \sin 3x\,dx + \frac{x}{2}\int \cos 3x\,dx$$

$$= -\frac{x}{3}\cos 3x + \frac{x}{2}\left(\frac{\sin 3x}{3}\right)$$

$$= -\frac{x}{3}\cos 3x + \frac{x}{6}\sin 3x$$

Hence, the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{3}\cos 3x + \frac{x}{6}\sin 3x$$

Example 5

Solve $(D^2 - 4D + 3)y = \sin 3x \cos 2x.$ **[Winter 2013]**

Solution

The auxiliary equation is

$$m^2-4m+3=0$$
$$(m-1)(m-3)=0$$
$$m=1,3 \quad \text{(real and distinct)}$$
$$\text{CF}=c_1e^x+c_2e^{3x}$$
$$\text{PI}=\frac{1}{D^2-4D+3}(\sin 3x\cos 2x)$$
$$=\frac{1}{D^2-4D+3}\frac{1}{2}(\sin 5x+\sin x)$$
$$=\frac{1}{2}\cdot\frac{1}{D^2-4D+3}\sin 5x+\frac{1}{2}\cdot\frac{1}{D^2-4D+3}\sin x$$
$$=\frac{1}{2}\cdot\frac{1}{-5^2-4D+3}\sin 5x+\frac{1}{2}\cdot\frac{1}{-1^2-4D+3}\sin x$$
$$=\frac{1}{2}\cdot\frac{1}{-4D-22}\sin 5x+\frac{1}{2}\cdot\frac{1}{2-4D}\sin x$$
$$=-\frac{1}{4}\cdot\frac{1}{2D+11}\cdot\frac{2D-11}{2D-11}\sin 5x+\frac{1}{4}\cdot\frac{1}{1-2D}\cdot\frac{1+2D}{1+2D}\sin x$$
$$=-\frac{1}{4}\cdot\frac{2D-11}{4D^2-121}\sin 5x+\frac{1}{4}\cdot\frac{1+2D}{1-4D^2}\sin x$$
$$=-\frac{1}{4}\cdot\frac{2D-11}{4\left(-5^2\right)-121}\sin 5x+\frac{1}{4}\cdot\frac{1+2D}{1-4\left(-1^2\right)}\sin x$$
$$=\frac{2}{884}(D\sin 5x)-\frac{11}{884}\sin 5x+\frac{1}{20}\sin x+\frac{2}{20}(D\sin x)$$
$$=\frac{10}{884}\cos 5x-\frac{11}{884}\sin 5x+\frac{1}{20}\sin x+\frac{1}{10}\cos x$$

Hence, the general solution is

$$y=c_1e^x+c_2e^{3x}+\frac{10}{884}\cos 5x-\frac{11}{884}\sin 5x+\frac{1}{20}\sin x+\frac{1}{10}\cos x$$

Example 6

Solve $(D^2+6D+8)\,y=\cos^2 x$.

Solution

The auxiliary equation is

$$m^2+6m+8=0$$
$$(m+4)(m+2)=0$$

$$m = -4, -2 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{-2x} + c_2 e^{-4x}$$

$$\text{PI} = \frac{1}{D^2 + 6D + 8}(\cos^2 x)$$

$$= \frac{1}{D^2 + 6D + 8}\left(\frac{1 + \cos 2x}{2}\right)$$

$$= \frac{1}{D^2 + 6D + 8}\frac{1}{2} + \frac{1}{D^2 + 6D + 8}\frac{\cos 2x}{2}$$

$$= \frac{1}{2}\frac{1}{D^2 + 6D + 8}e^{0x} + \frac{1}{2}\frac{1}{-2^2 + 6D + 8}\cos 2x$$

$$= \frac{1}{2}\frac{1}{0 + 0 + 8}e^{0x} + \frac{1}{2}\frac{1}{6D + 4}\cos 2x$$

$$= \frac{1}{16}e^{0x} + \frac{1}{4}\frac{1}{3D + 2}\cos 2x$$

$$= \frac{1}{16} + \frac{1}{4}\frac{3D - 2}{9D^2 - 4}\cos 2x$$

$$= \frac{1}{16} + \frac{1}{4}\frac{3D(\cos 2x) - 2\cos 2x}{9(-2)^2 - 4}$$

$$= \frac{1}{16} - \frac{1}{160}(-6\sin 2x - 2\cos 2x)$$

$$= \frac{1}{16} + \frac{1}{80}(3\sin 2x + \cos 2x)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-4x} + \frac{1}{16} + \frac{1}{80}(3\sin 2x + \cos 2x)$$

Example 7

Solve $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = \cos 2x \sin x.$ **[Winter 2015]**

Solution

$$(D^2 - 6D + 9)y = \frac{1}{2}(2\cos 2x \sin x)$$

$$(D^2 - 6D + 9)y = \frac{1}{2}(\sin 3x - \sin x)$$

The auxiliary equation is

$$m^2 - 6m + 9 = 0$$

$$(m-3)^2 = 0$$

$$m = 3, 3 \qquad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2 x)\, e^{3x}$$

$$\text{PI} = \frac{1}{D^2 - 6D + 9} \cdot \frac{1}{2}(\sin 3x - \sin x)$$

$$= \frac{1}{2}\frac{1}{D^2 - 6D + 9}\sin 3x - \frac{1}{2}\frac{1}{D^2 - 6D + 9}\sin x$$

$$= \frac{1}{2}\frac{1}{-9 - 6D + 9}\sin 3x - \frac{1}{2}\frac{1}{-1 - 6D + 9}\sin x$$

$$= -\frac{1}{12}\int \sin 3x - \frac{1}{2}\frac{1}{8 - 6D}\sin x$$

$$= \frac{1}{12}\left(\frac{-\cos 3x}{3}\right) - \frac{1}{2}\frac{8 + 6D}{64 - 36D^2}\sin x$$

$$= \frac{1}{36}\cos 3x - \frac{4 + 3D}{64 + 36}\sin x$$

$$= \frac{1}{36}\cos 3x - \frac{1}{100}(4\sin x + 3\cos x)$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{36}\cos 3x - \frac{2}{50}\sin x - \frac{3}{100}\cos x$$

Example 8

Solve $(D^2 - 4D + 4) = e^{2x} + \cos 2x$.

Solution

The auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2 \quad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2 x)e^{2x}$$

$$\text{PI} = \frac{1}{D^2 - 4D + 4}(e^{2x} + \cos 2x)$$

$$= \frac{1}{D^2 - 4D + 4}e^{2x} + \frac{1}{D^2 - 4D + 4}\cos 2x$$

$$= x\frac{1}{2D - 4}e^{2x} + \frac{1}{-2^2 - 4D + 4}\cos 2x$$

$$= x^2 \frac{1}{2} e^{2x} + \frac{1}{-4D} \cos 2x$$

$$= \frac{x^2}{2} e^{2x} - \frac{1}{4} \int \cos 2x \, dx$$

$$= \frac{x^2}{2} e^{2x} - \frac{1}{4} \frac{\sin 2x}{2}$$

$$= \frac{x^2}{2} e^{2x} - \frac{1}{8} \sin 2x$$

Hence, the general solution is

$$y = (c_1 + c_2 x) e^{2x} + \frac{x^2}{2} e^{2x} - \frac{1}{8} \sin 2x$$

Example 9

Solve $(D^2 - 3D + 2)y = 2\cos(2x + 3) + 2e^x$.

Solution

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0$$

$$m = 2, 1 \quad \text{(real and distinct)}$$

$$CF = c_1 e^x + c_1 e^{2x}$$

$$PI = \frac{1}{D^2 - 3D + 2} \left[2\cos(2x+3) + 2e^x \right]$$

$$= 2 \frac{1}{D^2 - 3D + 2} \cos(2x+3) + 2 \frac{1}{D^2 - 3D + 2} e^x$$

$$= 2 \frac{1}{-2^2 - 3D + 2} \cos(2x+3) + 2 \frac{1}{(D-1)(D-2)} e^x$$

$$= 2 \frac{1}{-2 - 3D} \cos(2x+3) + 2 \frac{1}{D-1} \frac{1}{(1-2)} e^x$$

$$= -2 \frac{3D-2}{9D^2 - 4} \cos(2x+3) - 2x \frac{1}{1} e^x$$

$$= \frac{-2\left[3D\cos(2x+3) - 2\cos(2x+3)\right]}{9\left(-2^2\right) - 4} - 2xe^x$$

$$= \frac{-2\left[-3\sin(2x+3)(2) - 2\cos(2x+3)\right]}{-36-4} - 2xe^x$$

$$= \frac{1}{20}\left[-6\sin(2x+3) - 2\cos(2x+3)\right] - 2xe^x$$

$$= -\frac{1}{10}\left[3\sin(2x+3)+\cos(2x+3)\right]-2xe^x$$

Hence, the general solution is

$$y = c_1e^x + c_2e^{2x} - \frac{1}{10}[3\sin(2x+3)+\cos(2x+3)] - 2xe^x$$

Example 10

Solve $(D^3 - 3D^2 + 9D - 27)y = \cos 3x.$ **[Winter 2016]**

Solution

The auxiliary equation is

$$m^3 - 3m^2 + 9m - 27 = 0$$

$$m^2(m-3) + 9(m-3) = 0$$

$$(m-3)(m^2+9) = 0$$

$$m = 3 \text{ (real)}, \quad m = \pm 3i \quad \text{(complex)}$$

$$\text{CF} = c_1 e^{3x} + c_2 \cos 3x + c_3 \sin 3x$$

$$\text{PI} = \frac{1}{D^3 - 3D^2 + 9D - 27}\cos 3x$$

$$= \frac{x}{3D^2 - 6D + 9}\cos 3x$$

$$= \frac{x}{-27 - 6D + 9}\cos 3x$$

$$= -\frac{x}{6D + 18}\cos 3x$$

$$= -\frac{x}{6}\frac{1}{(D+3)}\cos 3x$$

$$= -\frac{x}{6}\frac{D-3}{D^2-9}\cos 3x$$

$$= -\frac{x}{6}\frac{D-3}{-18}\cos 3x$$

$$= \frac{x}{6\cdot 18}[D-3]\cos 3x$$

$$= \frac{x}{6\cdot 18}[-3\sin 3x - 3\cos 3x]$$

$$= -\frac{x}{36}(\sin 3x + \cos 3x)$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 \cos 3x + c_3 \sin 3x - \frac{x}{36}(\sin 3x + \cos 3x)$$

Example 11

Solve $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x.$

Solution

The auxiliary equation is

$$m^3 - 3m^2 + 4m - 2 = 0$$

$$(m-1)(m^2 - 2m + 2) = 0$$

$$m - 1 = 0, \qquad m^2 - 2m + 2 = 0$$

$$m = 1 \text{ (real)}, \qquad m = 1 \pm i \text{ (complex)}$$

$$\text{CF} = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x)$$

$$\text{PI} = \frac{1}{D^3 - 3D^2 + 4D - 2}(e^x + \cos x)$$

$$= x \cdot \frac{1}{3D^2 - 6D + 4} e^x + \frac{1}{D(-1^2) - 3(-1^2) + 4D - 2}\cos x$$

$$= x \frac{1}{3 - 6 + 4} \cdot e^x + \frac{1}{3D + 1}\cos x$$

$$= xe^x + \frac{1}{(3D+1)} \cdot \frac{(3D-1)}{(3D-1)}\cos x$$

$$= xe^x + \frac{3D - 1}{9D^2 - 1}\cos x$$

$$= xe^x + \frac{3D - 1}{9(-1^2) - 1}\cos x$$

$$= xe^x - \frac{1}{10}(3D\cos x - \cos x)$$

$$= xe^x - \frac{1}{10}(-3\sin x - \cos x)$$

$$= xe^x + \frac{1}{10}(3\sin x + \cos x)$$

Hence, the general solution is

$$y = (c_1 + c_2 \cos x + c_3 \sin x)e^x + xe^x + \frac{1}{10}(3\sin x + \cos x)$$

Example 12

Solve $(D^4 + 2a^2D^2 + a^4)y = 8\cos ax.$

Solution

The auxiliary equation is

$$m^4 + 2a^2m^2 + a^4 = 0$$

$$(m^2 + a^2)^2 = 0$$

$$m = \pm ia, \pm ia \text{ (complex and repeated)}$$

$$\text{CF} = (c_1 + c_2x)\cos ax + (c_3 + c_4x)\sin ax$$

$$\text{PI} = \frac{1}{D^4 + 2a^2D^2 + a^4} 8\cos ax$$

$$= x \cdot \frac{1}{4D^3 + 4a^2D} 8\cos ax$$

$$= x^2 \cdot \frac{1}{12D^2 + 4a^2} 8\cos ax$$

$$= x^2 \cdot \frac{1}{12(-a^2) + 4a^2} 8\cos ax$$

$$= -\frac{x^2}{a^2}\cos ax$$

Hence, the general solution is

$$y = (c_1 + c_2x)\cos ax + (c_3 + c_4x)\sin ax - \frac{x^2}{a^2}\cos ax$$

Example 13

Solve $(D-1)^2(D^2+1)y = e^x + \sin^2\frac{x}{2}.$

Solution

The auxiliary equation is

$$(m-1)^2(m^2+1) = 0$$

$$(m-1)^2 = 0, \quad m^2 + 1 = 0$$

$$m = 1, 1 \text{ (real and repeated)}, \quad m = \pm i \text{ (complex)}$$

$$\text{CF} = (c_1 + c_2x)e^x + c_3\cos x + c_4\sin x$$

$$\text{PI} = \frac{1}{(D-1)^2(D^2+1)}\left(e^x + \sin^2\frac{x}{2}\right)$$

$$=\frac{1}{(D-1)^2(D^2+1)}\left(e^x+\frac{1-\cos x}{2}\right)$$

$$=\frac{1}{(D-1)^2(D^2+1)}\left(e^x+\frac{e^{0x}}{2}-\frac{\cos x}{2}\right)$$

$$=\frac{1}{(D-1)^2}\cdot\frac{1}{(1^2+1)}e^x+\frac{1}{(0-1)^2(0+1)}\cdot\frac{e^{0x}}{2}-\frac{1}{(D^2+1)(D^2-2D+1)}\cdot\frac{\cos x}{2}$$

$$=x\cdot\frac{1}{2(D-1)}\cdot\frac{e^x}{2}+\frac{1}{2}-\frac{1}{(D^2+1)(-1^2-2D+1)}\cdot\frac{\cos x}{2}$$

$$=\frac{x^2}{2}\cdot\frac{e^x}{2}+\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{(D^2+1)}\frac{1}{D}\cos x$$

$$=\frac{x^2e^x}{4}+\frac{1}{2}+\frac{1}{4}\frac{1}{(D^2+1)}\int\cos x\,dx$$

$$=\frac{x^2e^x}{4}+\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{D^2+1}\sin x$$

$$=\frac{x^2e^x}{4}+\frac{1}{2}+\frac{1}{4}x\frac{1}{2D}\sin x$$

$$=\frac{x^2e^x}{4}+\frac{1}{2}+\frac{x}{8}\int\sin x\,dx$$

$$=\frac{x^2e^x}{4}+\frac{1}{2}+\frac{x}{8}(-\cos x)$$

Hence, the general solution is

$$y=(c_1+c_2x)e^x+c_3\cos x+c_4\sin x+\frac{x^2e^x}{4}+\frac{1}{2}-\frac{x\cos x}{8}$$

Case III $Q(x)=x^m$

In this case, Eq. (5.8) reduces to $f(D)y=x^m$.

Hence,
$$\text{PI}=\frac{1}{f(D)}x^m$$
$$=[f(D)]^{-1}x^m$$
$$=[1+\phi(D)]^{-1}x^m$$

Expanding in ascending powers of D up to D^m using Binomial Expansion, since $D^nx^m=0$ when $n>m$,

$$\text{PI}=(a_0+a_1D+a_2D^2+\ldots+a_mD^m)x^m$$

Example 1

Solve $(D^2+2D+1)y=x$.

Solution

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$
$$(m+1)^2 = 0$$
$$m = -1, -1 \text{ (real and repeated)}$$
$$\text{CF} = (c_1 + c_2 x)e^{-x}$$
$$\text{PI} = \frac{1}{D^2 + 2D + 1} x$$
$$= \frac{1}{(1+D)^2} x$$
$$= (1+D)^{-2} x$$
$$= (1 - 2D + 3D^2 - \ldots)x$$
$$= x - 2Dx + 3D^2 x - \ldots$$
$$= x - 2 + 0$$
$$= x - 2$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-x} + x - 2$$

Example 2

Solve $y'' + 2y' + 3y = 2x^2$. **[Winter 2014]**

Solution

$$(D^2 + 2D + 3)y = 2x^2$$

The auxiliary equation is

$$m^2 + 2m + 3 = 0$$
$$m = \frac{-2 \pm \sqrt{4-12}}{2} = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm \sqrt{2}i \quad \text{(complex)}$$
$$\text{CF} = e^{-x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$$
$$\text{PI} = \frac{1}{D^2 + 2D + 3}(2x^2)$$
$$= \frac{1}{3\left(1 + \frac{D^2 + 2D}{3}\right)}(2x^2)$$
$$= \frac{2}{3}\left(1 + \frac{D^2 + 2D}{3}\right)^{-1} x^2$$

$$= \frac{2}{3}\left[1 - \left(\frac{D^2 + 2D}{3}\right) + \left(\frac{D^2 + 2D}{3}\right)^2 - \cdots\right] x^2$$

$$= \frac{2}{3}\left[x^2 - \frac{2}{3}Dx^2 - \frac{D^2}{3}x^2 + \frac{4}{9}D^2 x^2 - \cdots\right]$$

$$= \frac{2}{3}\left[x^2 - \frac{2}{3}(2x) - \frac{2}{3} + \frac{4}{9}(2)\right]$$

$$= \frac{2}{3}\left(x^2 - \frac{4}{3}x + \frac{2}{9}\right)$$

Hence, the general solution is

$$y = e^{-x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + \frac{2}{3}\left(x^2 - \frac{4}{3}x + \frac{2}{9}\right)$$

Example 3

Solve $(D^2 + D)y = x^2 + 2x + 4$.

Solution

The auxiliary equation is

$$m^2 + m = 0$$

$$m(m + 1) = 0$$

$m = 0, -1$ (real and distinct)

$$CF = c_1 e^{0x} + c_2 e^{-x}$$

$$= c_1 + c_2 e^{-x}$$

$$PI = \frac{1}{D^2 + D}(x^2 + 2x + 4)$$

$$= \frac{1}{D(D+1)}(x^2 + 2x + 4)$$

$$= \frac{1}{D}(1 + D)^{-1}(x^2 + 2x + 4)$$

$$= \frac{1}{D}(1 - D + D^2 - D^3 + \cdots)(x^2 + 2x + 4)$$

$$= \frac{1}{D}\left[(x^2 + 2x + 4) - D(x^2 + 2x + 4) + D^2(x^2 + 2x + 4) - D^3(x^2 + 2x + 4) + \cdots\right]$$

$$= \frac{1}{D}\left[(x^2 + 2x + 4) - (2x + 2) + 2 - 0\right]$$

$$= \frac{1}{D}(x^2 + 4)$$

$$= \int (x^2 + 4)\,dx$$

$$= \frac{x^3}{3} + 4x$$

Hence, the general solution is

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3} + 4x$$

Example 4

[Winter 2014]

Solve $(D^2 + 16)y = x^4 + e^{3x} + \cos 3x.$

Solution

The auxiliary equation is

$$m^2 + 16 = 0$$

$$m = \pm 4i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos 4x + c_2 \sin 4x$$

$$\text{PI} = \frac{1}{D^2 + 16}(x^4 + e^{3x} + \cos 3x)$$

$$= \frac{1}{D^2+16}x^4 + \frac{1}{D^2+16}e^{3x} + \frac{1}{D^2+16}\cos 3x$$

$$= \frac{1}{16}\left(1 + \frac{D^2}{16}\right)^{-1} x^4 + \frac{1}{9+16}e^{3x} + \frac{\cos 3x}{(-3^2)+16}$$

$$= \frac{1}{16}\left(1 - \frac{D^2}{16} + \frac{(-1)(-2)}{2!}\frac{D^4}{256} - \cdots\right)x^4 + \frac{1}{25}e^{3x} + \frac{1}{7}\cos 3x$$

$$= \frac{1}{16}\left(x^4 - \frac{3}{4}x^2 + \frac{3}{32}\right) + \frac{1}{25}e^{3x} + \frac{1}{7}\cos 3x$$

Hence, the general solution is

$$y = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{16}\left(x^4 - \frac{3}{4}x^2 + \frac{3}{32}\right) + \frac{1}{25}e^{3x} + \frac{1}{7}\cos 3x$$

Example 5

Solve $(D^2 + 2)y = x^3 + x^2 + e^{-2x} + \cos 3x.$

Solution

The auxiliary equation is

$$m^2 + 2 = 0,$$

$$m = \pm i\sqrt{2} \quad \text{(imaginary)}$$

$$CF = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$$

$$PI = \frac{1}{D^2+2}(x^3 + x^2 + e^{-2x} + \cos 3x)$$

$$= \frac{1}{2\left(1+\frac{D^2}{2}\right)}(x^3+x^2) + \frac{1}{D^2+2}e^{-2x} + \frac{1}{D^2+2}\cos 3x$$

$$= \frac{1}{2}\left(1+\frac{D^2}{2}\right)^{-1}(x^3+x^2) + \frac{1}{4+2}e^{-2x} + \frac{1}{-3^2+2}\cos 3x$$

$$= \frac{1}{2}\left(1-\frac{D^2}{2}+\frac{D^4}{4}-\dots\right)(x^3+x^2) + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

$$= \left[\frac{1}{2}(x^3+x^2) - \frac{1}{4}D^2(x^3+x^2) + \frac{D^4}{8}(x^3+x^2) - \dots\right] + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

$$= \left[\frac{1}{2}(x^3+x^2) - \frac{1}{4}(6x+2) + 0\right] + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

Hence, the general solution is

$$y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + \frac{1}{2}(x^3+x^2-3x-1) + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

Example 6

Solve $(D^3 - D)y = x^3$. **[Winter 2016]**

Solution

The auxiliary equation is

$$m^3 - m = 0$$

$$m(m^2-1) = 0$$

$$m = 0, \pm 1 \qquad \text{(real and distinct)}$$

$$CF = c_1 + c_2 e^x + c_3 e^{-x}$$

$$PI = \frac{1}{D^3 - D}x^3$$

$$= -\frac{1}{D}\left[\frac{1}{1-D^2}\right]x^3$$

$$= -\frac{1}{D}[(1-D^2)^{-1}]x^3$$

$$= -\frac{1}{D}[1 + D^2 + \cdots]x^3$$

$$= -\frac{1}{D}[x^3 + D^2(x^3)]$$

$$= -\frac{1}{D}[x^3 + 6x]$$

$$= -\int (x^3 + 6x)\, dx$$

$$= -\frac{x^4}{4} - 3x^2$$

$$= -\frac{1}{4}x^4 - 3x^2$$

Hence, the general solution is

$$y = c_1 + c_2 e^x + c_3 e^{-x} - \frac{1}{4}x^4 - 3x^2$$

Example 7

Solve $(D^3 + 8)y = x^4 + 2x + 1.$

Solution

The auxiliary equation is

$$m^3 + 8 = 0$$

$$m = -2 \text{ (real)}, \quad m = 1 \pm i\sqrt{3} \quad \text{(imaginary)}$$

$$\text{CF} = c_1 e^{-2x} + e^x\left(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right)$$

$$\text{PI} = \frac{1}{D^3 + 8}(x^4 + 2x + 1)$$

$$= \frac{1}{8\left(1 + \frac{D^3}{8}\right)}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 + \frac{D^3}{8}\right)^{-1}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 - \frac{D^3}{8} + \frac{D^6}{64} - \cdots\right)(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left[(x^4 + 2x + 1) - \frac{1}{8}D^3(x^4 + 2x + 1) + \frac{1}{64}D^6(x^4 + 2x + 1) - \cdots\right]$$

$$= \frac{1}{8}(x^4 + 2x + 1 - 3x)$$

$$= \frac{1}{8}(x^4 - x + 1)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + e^x (c_2 \cos\sqrt{3x} + c_3 \sin\sqrt{3}x) + \frac{1}{8}(x^4 - x + 1)$$

Example 8

Solve $(D^3 - D^2 - 6D)y = 1 + x^2$. **[Summer 2013]**

Solution

The auxiliary equation is

$$m^3 - m^2 - 6m = 0$$

$$m(m^2 - m - 6) = 0$$

$$m(m-3)(m+2) = 0$$

$$m = 0,\ 3,\ -2 \text{ (real and distinct)}$$

$$CF = c_1 e^{0x} + c_2 e^{3x} + c_3 e^{-2x}$$

$$= c_1 + c_2 e^{3x} + c_3 e^{-2x}$$

$$PI = \frac{1}{D^3 - D^2 - 6D}(1 + x^2)$$

$$= \frac{1}{-6D\left[1 - \frac{D^2 - D}{6}\right]}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \left(\frac{D^2 - D}{6}\right)\right]^{-1}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 + \left(\frac{D^2 - D}{6}\right) + \left(\frac{D^2 - D}{6}\right)^2 + \ldots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[1 + \frac{D^2 - D}{6} + \frac{D^4 - 2D^3 + D^2}{36} + \ldots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{7D^2}{36} - \frac{D^3}{18} + \ldots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[(1+x^2) - \frac{1}{6}D(1+x^2) + \frac{7}{36}D^2(1+x^2) - \frac{1}{18}D^3(1+x^2) + \ldots\right]$$

$$= -\frac{1}{6D}\left[1+x^2 - \frac{1}{6}(2x) + \frac{7}{36}(2) - 0\right]$$

$$= -\frac{1}{6D}\left[x^2 - \frac{x}{3} + \frac{25}{18}\right]$$

$$= -\frac{1}{6}\int\left(x^2 - \frac{x}{3} + \frac{25}{18}\right)dx$$

$$= -\frac{1}{6}\left(\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right)$$

Hence, the general solution is

$$y = c_1 + c_2e^{3x} + c_3e^{-2x} - \frac{1}{18}\left(x^3 - \frac{x^2}{2} + \frac{25}{6}x\right)$$

Example 9

Solve $(D^3 - 2D + 4)y = x^4 + 3x^2 - 5x + 2$.

Solution

The auxiliary equation is

$$m^3 - 2m + 4 = 0$$

$$(m+2)(m^2 - 2m + 2) = 0$$

$$m + 2 = 0, \qquad m^2 - 2m + 2 = 0$$

$$m = -2 \text{ (real)}, \qquad m = 1 \pm i \text{ (complex)}$$

$$\text{CF} = c_1e^{-2x} + e^x(c_2\cos x + c_3\sin x)$$

$$\text{PI} = \frac{1}{(D^3 - 2D + 4)}(x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left(1 + \frac{D^3 - 2D}{4}\right)^{-1}(x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 - \left(\frac{D^3 - 2D}{4}\right) + \left(\frac{D^3 - 2D}{4}\right)^2 - \left(\frac{D^3 - 2D}{4}\right)^3 + \left(\frac{D^3 - 2D}{4}\right)^4 - \ldots\right](x^4 + 3x^2 - 5x + 2)$$

$$=\frac{1}{4}\left[1-\left(\frac{D^3-2D}{4}\right)+\frac{4D^2}{16}-\frac{4D^4}{16}+\frac{8D^3}{64}\right.$$

$$\left.+\frac{16D^4}{256}+\text{higher powers of D}\right](x^4+3x^2-5x+2)$$

$$=\frac{1}{4}\left[1+\frac{D}{2}+\frac{D^2}{4}-\frac{D^3}{8}-\frac{3D^4}{16}+\text{higher powers of D}\right](x^4+3x^2-5x+2)$$

$$=\frac{1}{4}\left[(x^4+3x^2-5x+2)+\frac{1}{2}D(x^4+3x^2-5x+2)+\frac{1}{4}D^2(x^4+3x^2-5x+2)\right.$$

$$-\frac{1}{8}D^3(x^4+3x^2-5x+2)-\frac{3}{16}D^4(x^4+3x^2-5x+2)$$

$$\left.+\text{higher powers of } D(x^4+3x^2-5x+2)\right]$$

$$=\frac{1}{4}\left[(x^4+3x^2-5x+2)+\frac{1}{2}(4x^3+6x-5)+\frac{1}{4}(12x^2+6)-\frac{1}{8}(24x)-\frac{3}{16}(24)+0\right]$$

$$=\frac{1}{4}\left(x^4+2x^3+6x^2-5x-\frac{7}{2}\right)$$

Hence, the general solution is

$$y=c_1e^{-2x}+e^x(c_2\cos x+c_3\sin x)+\frac{1}{4}\left(x^4+2x^3+6x^2-5x-\frac{7}{2}\right)$$

Example 10

Solve $(D^4-2D^3+D^2)y=x^3$.

Solution

The auxiliary equation is

$$m^4-2m^3+m^2=0$$

$$m^2(m^2-2m+1)=0$$

$$m^2(m-1)^2=0$$

$$m=0,0,1,1 \text{ (real and repeated)}$$

Both the roots are real and repeated twice.

$$CF=(c_1+c_2x)e^{0x}+(c_3+c_4x)e^x$$

$$=c_1+c_2x+(c_3+c_4x)e^x$$

$$PI=\frac{1}{D^4-2D^3+D^2}x^3$$

$$= \frac{1}{D^2(D^2-2D+1)} x^3$$

$$= \frac{1}{D^2(1-D)^2} \cdot x^3$$

$$= \frac{1}{D^2}(1-D)^{-2} x^3$$

$$= \frac{1}{D^2}(1+2D+3D^2+4D^3+5D^4+\ldots)x^3$$

$$= \frac{1}{D^2}(x^3+2Dx^3+3D^2x^3+4D^3x^3+5D^4x^3+\ldots)$$

$$= \frac{1}{D^2}(x^3+2\cdot 3x^2+3\cdot 6x+4\cdot 6+0)$$

$$= \frac{1}{D^2}(x^3+6x^2+18x+24)$$

$$= \frac{1}{D}\left[\int (x^3+6x^2+18x+24)dx\right]$$

$$= \frac{1}{D}\left(\frac{x^4}{4}+6\frac{x^3}{3}+18\frac{x^2}{2}+24x\right)$$

$$= \int\left(\frac{x^4}{4}+2x^3+9x^2+24x\right)dx$$

$$= \frac{x^5}{20}+2\frac{x^4}{4}+9\frac{x^3}{3}+24\frac{x^2}{2}$$

$$= \frac{x^5}{20}+\frac{x^4}{2}+3x^3+12x^2$$

Hence, the general solution is

$$y = c_1 + c_2 x + (c_3 + c_4 x)e^x + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$$

Example 11

Solve $(D^4-16)y = e^{2x} + x^4$ where $D = \frac{d}{dx}$. **[Summer 2017]**

Solution

The auxiliary equation is

$$m^4 - 16 = 0$$

$$(m^2-4)(m^2+4)=0$$

$$(m-2)(m+2)(m^2+4)=0$$

$m=2,-2$ (real and distinct), $m=\pm 2i$ (complex)

$$\text{CF}=c_1 e^{2x}+c_2 e^{-2x}+c_3 \cos 2x+c_4 \sin 2x$$

$$\text{PI}=\frac{1}{D^4-16}(e^{2x}+x^4)$$

$$=\frac{1}{D^4-16}e^{2x}+\frac{1}{D^4-16}x^4$$

$$=\frac{x}{4D^3}e^{2x}+\left(-\frac{1}{16}\right)\frac{1}{1-\frac{D^4}{16}}x^4$$

$$=\frac{x}{4(2)^3}e^{2x}-\frac{1}{16}\left[1-\frac{D^4}{16}\right]^{-1}x^4$$

$$=\frac{x}{32}e^{2x}-\frac{1}{16}\left[1+\frac{D^4}{16}+\cdots\right]x^4$$

$$=\frac{x}{32}e^{2x}-\frac{1}{16}\left[x^4+\frac{1}{16}D^4(x^4)\right]$$

$$=\frac{x}{32}e^{2x}-\frac{1}{16}\left[x^4+\frac{24}{16}\right]$$

$$=\frac{x}{32}e^{2x}-\frac{x^4}{16}-\frac{1}{16}\frac{24}{16}$$

$$=\frac{x}{32}e^{2x}-\frac{x^4}{16}-\frac{3}{32}$$

$$=\frac{1}{32}(xe^{2x}-3-2x^4)$$

Hence, the general solution is

$$y=c_1 e^{2x}+c_2 e^{-2x}+c_3 \cos 2x+c_4 \sin 2x+\frac{1}{32}(xe^{2x}-3-2x^4)$$

Example 12

Solve $(D^4+2D^3-3D^2)y = x^2+3e^{2x}$.

Solution

The auxiliary equation is

$$m^4+2m^3-3m^2=0$$
$$m^2(m^2+2m-3)=0$$
$$m^2(m-1)(m+3)=0$$

$m=0, 0$ (real and repeated), $\quad m=1,-3$ (real and distinct)

$$\begin{aligned}
\text{CF} &= (c_1+c_2x)e^{0x}+c_3e^x+c_4e^{-3x}\\
&= c_1+c_2x+c_3e^x+c_4e^{-3x}
\end{aligned}$$

$$\begin{aligned}
\text{PI} &= \frac{1}{D^4+2D^3-3D^2}\left(x^2+3e^{2x}\right)\\
&= \frac{1}{D^4+2D^3-3D^2}x^2+\frac{1}{D^4+2D^3-3D^2}3e^{2x}\\
&= \frac{1}{-3D^2\left(1-\frac{D^2+2D}{3}\right)}x^2+\frac{1}{16+16-12}3e^{2x}\\
&= -\frac{1}{3D^2}\left(1-\frac{D^2+2D}{3}\right)^{-1}x^2+\frac{3e^{2x}}{20}\\
&= -\frac{1}{3D^2}\left[1+\frac{D^2+2D}{3}+\left(\frac{D^2+2D}{3}\right)^2+\dots\right]x^2+\frac{3e^{2x}}{20}\\
&= -\frac{1}{3D^2}\left(1+\frac{D^2+2D}{3}+\frac{D^4+4D^2+4D^3}{9}+\cdots\right)x^2+\frac{3e^{2x}}{20}\\
&= -\frac{1}{3D^2}\left(x^2+\frac{2}{3}Dx^2+\frac{7}{9}D^2x^2+\frac{4}{9}D^3x^2+\cdots\right)+\frac{3}{20}e^{2x}\\
&= -\frac{1}{3D^2}\left[x^2+\frac{2}{3}(2x)+\frac{7}{9}(2)+0\right]+\frac{3e^{2x}}{20}
\end{aligned}$$

$$= -\frac{1}{3D}\left[\int\left(x^2+\frac{4}{3}x+\frac{14}{9}\right)dx\right]+\frac{3e^{2x}}{20}$$

$$= -\frac{1}{3D}\left(\frac{x^3}{3}+\frac{4}{3}\frac{x^2}{2}+\frac{14}{9}x\right)+\frac{3e^{2x}}{20}$$

$$= -\frac{1}{3}\int\left(\frac{x^3}{3}+\frac{2}{3}x^2+\frac{14}{9}x\right)dx+\frac{3e^{2x}}{20}$$

$$= -\frac{1}{3}\left(\frac{x^4}{12}+\frac{2x^3}{9}+\frac{7x^2}{9}\right)+\frac{3e^{2x}}{20}$$

Hence, the general solution is

$$y = c_1 + c_2x + c_3e^x + c_4e^{-3x} - \frac{x^2}{9}\left(\frac{x^2}{4}+\frac{2x}{3}+\frac{7}{3}\right)+\frac{3e^{2x}}{20}$$

Case IV **$Q = e^{ax}V$, where V is a function of x.**

In this case, Eq. (5.8) reduces to $f(D)y = e^{ax}V$.

Let u be a function of x.

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u$$

$$= e^{ax}(D+a)u$$

$$D^2(e^{ax}u) = D\left[e^{ax}(D+a)u\right]$$

$$= ae^{ax}(D+a)u + e^{ax}(D^2+aD)u$$

$$= e^{ax}(D^2+2aD+a^2)u$$

$$= e^{ax}(D+a)^2u$$

In general,

$$D^r(e^{ax}u) = e^{ax}(D+a)^r u$$

Let $D^r = f(D)$, $\quad (D+a)^r = f(D+a)$

$$f(D)(e^{ax}u) = e^{ax}f(D+a)u$$

Operating both the sides with $\frac{1}{f(D)}$,

$$\frac{1}{f(D)}\left[f(D)(e^{ax}u)\right] = \frac{1}{f(D)}\left[e^{ax}f(D+a)u\right]$$

$$e^{ax}u = \frac{1}{f(D)}\left[e^{ax}f(D+a)u\right]$$

Putting $f(D+a)u = V$, $u = \dfrac{1}{f(D+a)} V$

$$e^{ax} \cdot \frac{1}{f(D+a)} V = \frac{1}{f(D)} (e^{ax} V)$$

Hence,

$$PI = \frac{1}{f(D)} \cdot e^{ax} V$$

$$= e^{ax} \cdot \frac{1}{f(D+a)} V$$

Example 1

Solve $(D+2)^2 y = e^{-2x} \sin x$.

Solution

The auxiliary equation is

$$(m+2)^2 = 0$$

$$m = -2, -2 \quad \text{(real and repeated)}$$

$$CF = (c_1 + c_2 x) e^{-2x}$$

$$PI = \frac{1}{(D+2)^2} e^{-2x} \sin x$$

$$= e^{-2x} \frac{1}{(D-2+2)^2} \sin x$$

$$= e^{-2x} \frac{1}{D^2} \sin x$$

$$= e^{-2x} \frac{1}{-1^2} \sin x$$

$$= -e^{-2x} \sin x$$

Hence, the general solution is

$$y = (c_1 + c_2 x) e^{-2x} - e^{-2x} \sin x$$

$$= (c_1 + c_2 x - \sin x) e^{-2x}$$

Example 2

Solve $(D^2 + 2D + 1) y = \dfrac{e^{-x}}{x^2}$.

Solution

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$(m+1)^2 = 0$$

$$m = -1, -1 \quad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2 x)e^{-x}$$

$$\text{PI} = \frac{1}{D^2 + 2D + 1}\left(\frac{e^{-x}}{x^2}\right)$$

$$= \frac{1}{(D+1)^2}\left(\frac{e^{-x}}{x^2}\right)$$

$$= e^{-x}\frac{1}{(D-1+1)^2}\left(\frac{1}{x^2}\right)$$

$$= e^{-x}\frac{1}{D^2}x^{-2}$$

$$= e^{-x}\frac{1}{D}\int x^{-2}dx$$

$$= e^{-x}\frac{1}{D}\left(\frac{x^{-2+1}}{-2+1}\right)$$

$$= e^{-x}\frac{1}{D}x^{-1}$$

$$= -e^{-x}\int\frac{dx}{x}$$

$$= e^{-x}\log x$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-x} - e^{-x}\log x$$

$$= e^{-x}(c_1 + c_2 x - \log x)$$

Example 3

Solve $(D^2 - 2D - 1)y = e^x \cos x.$

Solution

The auxiliary equation is

$$m^2 - 2m - 1 = 0$$

$$m = 1 \pm \sqrt{2} \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x}$$

$$\text{PI} = \frac{1}{D^2 - 2D - 1}e^x \cos x$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) - 1} \cos x$$

$$= e^x \frac{1}{(D^2 - 2)} \cos x$$

$$= e^x \frac{1}{-1^2 - 2} \cos x$$

$$= -\frac{1}{3} e^x \cos x$$

Hence, the general solution is

$$y = c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x} - \frac{1}{3} e^x \cos x$$

Example 4

Solve $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = e^x \cos x.$ **[Winter 2017]**

Solution

$$(D^3 - 2D + 4)y = e^x \cos x$$

The auxiliary equation is

$$m^3 - 2m + 4 = 0$$

$$m^2(m+2) - 2m(m+2) + 2(m+2) = 0$$

$$(m+2)(m^2 - 2m + 2) = 0$$

$$m = -2 \quad \text{(real)}, \quad m = 1 \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 e^{-2x} + e^x (c_2 \cos x + c_3 \sin x)$$

$$\text{PI} = \frac{1}{D^3 - 2D + 4} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^3 - 2(D+1) + 4} \cos x$$

$$= e^x \frac{1}{D^3 + 3D^2 + D + 3} \cos x$$

$$= e^x \left[x \frac{1}{3D^2 + 6D + 1} \cos x \right] \qquad \left[\begin{aligned} \because D^3 + 3D^2 + D + 3 = 0 \\ \text{at } D^2 = -1^2 = -1 \end{aligned} \right]$$

$$= e^x x \frac{1}{3(-1)^2 + 6D + 1} \cos x$$

$$= e^x x \frac{1}{6D - 2} \cos x$$

$$= e^x x \frac{1}{2(3D-1)} \cdot \frac{(3D+1)}{(3D+1)} \cos x$$

$$= e^x x \frac{3D+1}{2(9D^2-1)} \cos x$$

$$= e^x x \frac{(3D+1)\cos x}{2\left[9(-1^2)-1\right]}$$

$$= -\frac{e^x x}{20}(3D\cos x + \cos x)$$

$$= -\frac{e^x x}{20}(-3\sin x + \cos x)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x) + \frac{e^x x}{20}(3\sin x - \cos x)$$

Example 5

Solve $(D^2 - 2D + 5)y = e^{2x} \sin x$.

Solution

The auxiliary equation is

$$m^2 - 2m + 5 = 0$$

$$m = 1 \pm 2i \quad \text{(complex)}$$

$$CF = e^x(c_1 \cos 2x + c_2 \sin 2x)$$

$$PI = \frac{1}{D^2 - 2D + 5} e^{2x} \sin x$$

$$= e^{2x} \frac{1}{(D+2)^2 - 2(D+2) + 5} \sin x$$

$$= e^{2x} \frac{1}{D^2 + 4 + 4D - 2D - 4 + 5} \sin x$$

$$= e^{2x} \frac{1}{D^2 + 2D + 5} \sin x$$

$$= e^{2x} \frac{1}{-1^2 + 2D + 5} \sin x$$

$$= e^{2x} \frac{1}{2D+4} \sin x$$

$$= \frac{1}{2} e^{2x} \frac{1}{D+2} \sin x$$

$$= \frac{1}{2}e^{2x}\frac{D-2}{D^2-4}\sin x$$
$$= \frac{1}{2}e^{2x}\frac{D-2}{-1^2-4}\sin x$$
$$= -\frac{1}{10}e^{2x}(D\sin x - 2\sin x)$$
$$= -\frac{1}{10}e^{2x}(\cos x - 2\sin x)$$

Hence, the general solution is

$$y = e^x(c_1\cos 2x + c_2\sin 2x) - \frac{1}{10}e^{2x}(\cos x - 2\sin x)$$

Example 6

Solve $(D^2 + 2D + 2)y = e^x \sin x + 7.$

Solution

The auxiliary equation is

$$m^2 + 2m + 2 = 0$$
$$m = -1 \pm \quad \text{(complex)}$$
$$CF = e^{-x}(c_1\cos x + c_2\sin x)$$
$$PI = \frac{1}{D^2+2D+2}(e^x\sin x + 7)$$
$$= \frac{1}{D^2+2D+2}e^x\sin x + \frac{1}{D^2+2D+2}7$$
$$= e^x\frac{1}{(D+1)^2+2(D+1)+2}\sin x + 7\frac{1}{D^2+2D+2}e^{0x}$$
$$= e^x\frac{1}{D^2+2D+1+2D+2+2}\sin x + 7\frac{1}{0+0+2}e^{0x}$$
$$= e^x\frac{1}{D^2+4D+5}\sin x + \frac{7}{2}e^{0x}$$
$$= e^x\frac{1}{(-1)^2+4D+5}\sin x + \frac{7}{2}$$
$$= e^x\frac{1}{4D+4}\sin x + \frac{7}{2}$$
$$= \frac{e^x}{4}\frac{1}{D+1}\sin x + \frac{7}{2}$$

$$= \frac{e^x}{4}\frac{D-1}{D^2-1}\sin x + \frac{7}{2}$$

$$= \frac{e^x}{4}\frac{D-1}{-1^2-1}\sin x + \frac{7}{2}$$

$$= \frac{e^x}{-8}(D\sin x - \sin x) + \frac{7}{2}$$

$$= -\frac{1}{8}e^x(\cos x - \sin x) + \frac{7}{2}$$

Hence, the general solution is

$$y = e^{-x}(c_1\cos x + c_2\sin x) - \frac{1}{8}e^x(\cos x - \sin x) + \frac{7}{2}$$

Example 7

Solve $(D^2 - 2D + 2)y = e^x x^2 + 5 + e^{-2x}$.

Solution

The auxiliary equation is

$$m^2 - 2m + 2 = 0$$

$$m = 1 \pm i \quad \text{(complex)}$$

$$CF = e^x(c_1\cos x + c_2\sin x)$$

$$PI = \frac{1}{D^2-2D+2}(e^x x^2 + 5 + e^{-2x})$$

$$= \frac{1}{D^2-2D+2}e^x x^2 + \frac{1}{D^2-2D+2}5 + \frac{1}{D^2-2D+2}e^{-2x}$$

$$= e^x\frac{1}{(D+1)^2-2(D+1)+2}x^2 + \frac{1}{D^2-2D+2}5e^{0x} + \frac{1}{4-2(-2)+2}e^{-2x}$$

$$= e^x\frac{1}{D^2+2D+1-2D-2+2}x^2 + \frac{1}{0-0+2}5e^{0x} + \frac{1}{10}e^{-2x}$$

$$= e^x\frac{1}{D^2+1}x^2 + \frac{5}{2} + \frac{1}{10}e^{-2x}$$

$$= e^x(1+D^2)^{-1}x^2 + \frac{5}{2} + \frac{1}{10}e^{-2x}$$

$$= e^x(1 - D^2 + D^4 - \ldots)x^2 + \frac{5}{2} + \frac{1}{10}e^{-2x}$$

$$= e^x\left[x^2 - D^2(x^2) + D^4(x^2) - \cdots\right] + \frac{5}{2} + \frac{1}{10}e^{-2x}$$

$$= e^{x}(x^2-2)+\frac{5}{2}+\frac{1}{10}e^{-2x}$$

Hence, the general solution is

$$y = e^{x}(c_1\cos x + c_2\sin x) + e^{x}(x^2-2)+\frac{5}{2}+\frac{1}{10}e^{-2x}$$

Example 8

Solve $(D^2-4D-5)y = xe^{2x}+3\cos 4x.$

Solution

The auxiliary equation is

$$m^2-4m-5=0$$

$$(m-5)(m+1)=0$$

$$m=5,-1 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1e^{5x}+c_2e^{-x}$$

$$\text{PI} = \frac{1}{D^2-4D-5}(xe^{2x}+3\cos 4x)$$

$$= \frac{1}{D^2-4D-5}xe^{2x}+\frac{1}{D^2-4D-5}3\cos 4x$$

$$= e^{2x}\frac{1}{(D+2)^2-4(D+2)-5}x+3\frac{1}{-4^2-4D-5}\cos 4x$$

$$= e^{2x}\frac{1}{D^2+4D+4-4D-8-5}x+3\frac{1}{-4D-21}\cos 4x$$

$$= e^{2x}\frac{1}{D^2-9}x-3\frac{1}{4D+21}\cos 4x$$

$$= \frac{e^{2x}}{-9}\frac{1}{\left(1-\frac{D^2}{9}\right)}x-3\frac{4D-21}{16D^2-441}\cos 4x$$

$$= -\frac{e^{2x}}{9}\left(1-\frac{D^2}{9}\right)^{-1}x-3\frac{4D-21}{16(-4^2)-441}\cos 4x$$

$$= -\frac{e^{2x}}{9}\left[1+\frac{D^2}{9}+\left(\frac{D^2}{9}\right)^2+\ldots\right]x+3\frac{1}{697}(4D\cos 4x-21\cos 4x)$$

$$= -\frac{e^{2x}}{9}\left[x+\frac{1}{9}D^2x+\ldots\right]+\frac{3}{697}[4(-\sin 4x)4-21\cos 4x]$$

$$= -\frac{e^{2x}}{9}(x+0) + \frac{3}{697}(-16\sin 4x - 21\cos 4x)$$

$$= -\frac{1}{9}xe^{2x} - \frac{3}{697}(16\sin 4x + 21\cos 4x)$$

Hence, the general solution is

$$y = c_1e^{5x} + c_2e^{-x} - \frac{1}{9}xe^{2x} - \frac{3}{697}(16\sin 4x + 21\cos 4x)$$

Example 9

Solve $(D^2 + 4D + 3)y = e^{-x}x\sin x + xe^{3x}$.

Solution

The auxiliary equation is

$$m^2 + 4m + 3 = 0$$

$$(m+3)(m+1) = 0$$

$$m = -3, -1 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1e^{-3x} + c_2e^{-x}$$

$$\text{PI} = \frac{1}{D^2+4D+3}(e^{-x}\sin x + xe^{3x})$$

$$= \frac{1}{D^2+4D+3}e^{-x}\sin x + \frac{1}{D^2+4D+3}xe^{3x}$$

$$= e^{-x}\frac{1}{(D-1)^2+4(D-1)+3}\sin x + e^{3x}\frac{1}{(D+3)^2+4(D+3)+3}x$$

$$= e^{-x}\frac{1}{D^2-2D+1+4D-4+3}\sin x + e^{3x}\frac{1}{D^2+6D+9+4D+12+3}x$$

$$= e^{-x}\frac{1}{D^2+2D}\sin x + e^{3x}\frac{1}{D^2+10D+24}x$$

$$= e^{-x}\frac{1}{-1^2+2D}\sin x + \frac{e^{3x}}{24}\frac{1}{\left(1+\frac{10D+D^2}{24}\right)}x$$

$$= e^{-x}\frac{2D+1}{4D^2-1}\sin x + \frac{e^{3x}}{24}\left[1+\frac{10D+D^2}{24}\right]^{-1}x$$

$$= e^{-x}\frac{(2D+1)\sin x}{4(-1^2)-1} + \frac{e^{3x}}{24}\left[1-\frac{10D+D^2}{24}+\cdots\right]x$$

$$= -\frac{e^{-x}}{5}(2D\sin x + \sin x) + \frac{e^{3x}}{24}\left[x - \frac{5}{12}D(x) - \frac{1}{24}D^2(x) + \ldots\right]$$

$$= -\frac{e^{-x}}{5}(2\cos x + \sin x) + \frac{e^{3x}}{24}\left(x - \frac{5}{12} - 0\right)$$

$$= -\frac{1}{5}e^{-x}(2\cos x + \sin x) + \frac{e^{3x}}{24}\left(x - \frac{5}{12}\right)$$

Hence, the general solution is

$$y = c_1 e^{-3x} + c_2 e^{-x} - \frac{1}{5}e^{-x}(2\cos x + \sin x) + \frac{e^{3x}}{24}\left(x - \frac{5}{12}\right)$$

Example 10

Solve $(D^3 + 3D^2 - 4D - 12)y = 12xe^{-2x}$.

Solution

The auxiliary equation is

$$m^3 + 3m^2 - 4m - 12 = 0$$

$$m^2(m+3) - 4(m+3) = 0$$

$$(m+3)(m^2 - 4) = 0$$

$$m = -3, -2, 2 \text{ (real and distinct)}$$

$$\text{CF} = c_1 e^{-3x} + c_2 e^{-2x} + c_3 e^{2x}$$

$$\text{PI} = \frac{1}{(D+3)(D+2)(D-2)} 12xe^{-2x}$$

$$= 12e^{-2x} \frac{1}{(D-2+3)(D-2+2)(D-2-2)} x$$

$$= 12e^{-2x} \frac{1}{(D+1)D(D-4)} x$$

$$= 12e^{-2x} \frac{1}{D(D^2 - 3D - 4)} x$$

$$= 12e^{-2x} \frac{1}{-4D\left(1 + \frac{3D - D^2}{4}\right)} x$$

$$= -3e^{-2x} \frac{1}{D}\left(1 + \frac{3D - D^2}{4}\right)^{-1} x$$

$$= -3e^{-2x}\frac{1}{D}\left(1-\frac{3D-D^2}{4}+\cdots\right)x$$

$$= -3e^{-2x}\frac{1}{D}\left[x-\frac{3}{4}D(x)+\frac{1}{4}D^2(x)+\cdots\right]$$

$$= -3e^{-2x}\frac{1}{D}\left(x-\frac{3}{4}+0\right)$$

$$= -3e^{-2x}\int\left(x-\frac{3}{4}\right)dx$$

$$= -3e^{-2x}\left(\frac{x^2}{2}-\frac{3}{4}x\right)$$

Hence, the general solution is

$$y = c_1e^{-3x}+c_2e^{-2x}+c_3e^{2x}-3e^{-2x}\left(\frac{x^2}{2}-\frac{3}{4}x\right)$$

Example 11

Solve $(D^3+1)y = e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$.

Solution

The auxiliary equation is

$$m^3+1=0$$

$$(m+1)(m^2-m+1)=0$$

$$m=-1 \text{ (real)},\quad m=\frac{1}{2}\pm i\frac{\sqrt{3}}{2} \text{ (complex)}$$

$$\text{CF} = c_1e^{-x}+e^{\frac{x}{2}}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)$$

$$\text{PI} = \frac{1}{D^3+1}e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}\frac{1}{\left[\left(D+\frac{1}{2}\right)^3+1\right]}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}\frac{1}{\left[\left(D^3+\frac{1}{8}+\frac{3}{2}D^2+\frac{3}{4}D\right)+1\right]}\sin\left(\frac{\sqrt{3}}{2}x\right) \quad \left[\because (a+b)=a^3+b^3+3a^2b+3ab^2\right]$$

$$= e^{\frac{x}{2}}\left[x\frac{1}{3D^2+3D+\frac{3}{4}}\sin\left(\frac{\sqrt{3}}{2}x\right)\right] \quad \left[\because \left(D^3+\frac{1}{8}+\frac{3}{2}D^2+\frac{3}{4}D\right)+1=0 \quad \text{at } D^2 = -\left(\frac{\sqrt{3}}{2}\right)^2 = -\frac{3}{4}\right]$$

$$= e^{\frac{x}{2}}x\frac{1}{3\left[-\left(\frac{\sqrt{3}}{2}\right)^2\right]+3D+\frac{3}{4}}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{1}{\left(-\frac{9}{4}+3D+\frac{3}{4}\right)}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{1}{3D-\frac{3}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{2}{3(2D-1)}\frac{(2D+1)}{(2D+1)}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{2(2D+1)}{3(4D^2-1)}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{2(2D+1)}{3\left[4\left\{-\left(\frac{\sqrt{3}}{2}\right)^2\right\}-1\right]}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{2\left[2D\left\{\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}+\sin\left(\frac{\sqrt{3}}{2}x\right)\right]}{3(-4)}$$

$$= -\frac{1}{6}xe^{\frac{x}{2}}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}x\right)+\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

Hence, the general solution is

$$y = c_1e^{-x}+e^{\frac{x}{2}}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)-\frac{1}{6}xe^{\frac{x}{2}}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}x\right)+\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

Case V $Q = xV$, where V is a function of x.

In this case Eq. (5.8) reduces to $f(D)y = xV$.

Let u be a function of x.

$$D(xu) = xDu + u$$

$$D^2(xu) = D(xDu + u) = xD^2u + Du + Du = xD^2u + 2Du$$

$$D^3(xu) = D(xD^2u + 2Du) = xD^3u + D^2u + 2D^2u = xD^3u + 3D^2u$$

In general,

$$D^r(xu) = xD^r u + rD^{r-1}u = xD^r u + \left[\frac{d}{dD}(D^r)\right]u$$

Let $D^r = f(D)$

$$f(D)(xu) = x\,f(D)u + \left[\frac{d}{dD}f(D)\right]u$$

$$= xf(D)u + f'(D)u$$

Putting $f(D)u = V$, $u = \frac{1}{f(D)}V$ in the above equation,

$$f(D)\left[x\frac{1}{f(D)}V\right] = xV + f'(D)\left[\frac{1}{f(D)}V\right]$$

$$xV = f(D)\left[x\frac{1}{f(D)}V\right] - f'(D)\left[\frac{1}{f(D)}V\right]$$

Operating both the sides with $\frac{1}{f(D)}$,

$$\frac{1}{f(D)}xV = \frac{1}{f(D)}\left[f(D)\left(x\frac{1}{f(D)}V\right)\right] - \frac{1}{f(D)}\left[f'(D)\left(\frac{1}{f(D)}\right)V\right]$$

$$= x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V$$

Hence,

$$PI = \frac{1}{f(D)}xV$$

$$= x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V$$

Example 1

Solve $(D^2 - 5D + 6)y = x\cos 2x$.

Solution

The auxiliary equation is

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2,\ 3 \text{ (real and distinct)}$$

$$CF = c_1 e^{2x} + c_2 e^{3x}$$

$$PI = \frac{1}{D^2 - 5D + 6} x\cos 2x$$

$$= x\frac{1}{D^2 - 5D + 6}\cos 2x - \frac{2D-5}{(D^2 - 5D + 6)^2}\cos 2x$$

$$= x\frac{1}{-2^2 - 5D + 6}\cos 2x - \frac{2D-5}{(-2^2 - 5D + 6)^2}\cos 2x$$

$$= x\frac{1}{(2-5D)}\cdot\frac{(2+5D)}{(2+5D)}\cos 2x - \frac{2D-5}{(4-20D+25D^2)}\cos 2x$$

$$= x\frac{(2+5D)}{4-25D^2}\cos 2x - \frac{2D-5}{\left[4-20D+25(-2^2)\right]}\cos 2x$$

$$= x\frac{(2+5D)}{4+100}\cos 2x + \frac{2D-5}{4(5D+24)}\cos 2x$$

$$= \frac{x}{104}(2\cos 2x - 10\sin 2x) + \frac{2D-5}{4(5D+24)}\cdot\frac{(5D-24)}{(5D-24)}\cos 2x$$

$$= \frac{x}{104}(2\cos 2x - 10\sin 2x) + \frac{(10D^2 - 73D + 120)}{4(25D^2 - 576)}\cos 2x$$

$$= \frac{x}{52}(\cos 2x - 5\sin 2x) + \frac{(10D^2 - 73D + 120)}{4(-100-576)}\cos 2x$$

$$= \frac{1}{52}x(\cos 2x - 5\sin 2x) - \frac{1}{2704}(-40\cos 2x + 146\sin 2x + 120\cos 2x)$$

Hence, the general solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{52}x(\cos 2x - 5\sin 2x) - \frac{1}{1352}(40\cos 2x + 73\sin 2x)$$

Example 2

Solve $(D^2-1)y = xe^x$ where $D = \frac{d}{dx}$. **[Summer 2017]**

Solution

The auxiliary equation is

$$m^2 - 1 = 0$$

$$m^2 = 1$$

$$m = \pm 1 \text{ (real and distinct)}$$

$$\text{CF} = c_1 e^x + c_2 e^{-x}$$

$$\text{PI} = \frac{1}{D^2-1} xe^x$$

$$= e^x \frac{1}{(D+1)^2 - 1} x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 1} x$$

$$= e^x \frac{1}{D^2 + 2D} x$$

$$= \frac{e^x}{2D} \frac{1}{1 + \frac{D}{2}} x$$

$$= \frac{e^x}{2D} \left[1 + \frac{D}{2}\right]^{-1} x$$

$$= \frac{e^x}{2D} \left[1 - \frac{D}{2}\right] x$$

$$= \frac{e^x}{2D} \left[x - \frac{1}{2}\right]$$

$$= \frac{e^x}{2} \int \left(x - \frac{1}{2}\right) dx$$

$$= \frac{e^x}{2} \left(\frac{x^2}{2} - \frac{x}{2}\right)$$

$$= \frac{1}{4} e^x (x^2 - x)$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{-x} + \frac{1}{4} e^x (x^2 - x)$$

Example 3

Solve $(D^2 + 2D + 1)\,y = xe^{-x}\cos x.$

Solution

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$
$$(m+1)^2 = 0$$
$$m = -1, -1 \quad \text{(real and repeated)}$$
$$CF = (c_1 + c_2 x)e^{-x}$$
$$PI = \frac{1}{D^2 + 2D + 1} xe^{-x}\cos x$$
$$= \frac{1}{(D+1)^2} xe^{-x}\cos x$$
$$= e^{-x}\frac{1}{(D-1+1)^2} x\cos x$$
$$= e^{-x}\frac{1}{D^2} x\cos x$$
$$= e^{-x}\left[x\frac{1}{D^2}\cos x - \frac{2D}{(D^2)^2}\cos x\right]$$
$$= e^{-x}\cdot\left(x\frac{1}{-1^2}\cos x - \frac{2D}{(-1^2)^2}\cos x\right)$$
$$= e^{-x}(-x\cos x - 2D\cos x)$$
$$= e^{-x}(-x\cos x + 2\sin x)$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-x} + e^{-x}(-x\cos x + 2\sin x)$$

Example 4

Solve $(D^2 + 3D + 2)y = xe^{x}\sin x.$

Solution

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$
$$(m+1)(m+2) = 0$$
$$m = -1, \; -2 \text{ (real and distinct)}$$

$$\text{CF} = c_1e^{-x} + c_2e^{-2x}$$

$$\text{PI} = \frac{1}{(D+1)(D+2)} xe^x \sin x$$

$$= e^x \frac{1}{(D+1+1)(D+1+2)} x\sin x$$

$$= e^x \frac{1}{(D+2)(D+3)} x\sin x$$

$$= e^x \frac{1}{D^2+5D+6} x\sin x$$

$$= e^x \left[x\frac{1}{D^2+5D+6}\sin x - \frac{2D+5}{(D^2+5D+6)^2}\sin x \right]$$

$$= e^x \left[x\frac{1}{-1^2+5D+6}\sin x - \frac{2D+5}{(-1^2+5D+6)^2}\sin x \right]$$

$$= e^x \left[x\frac{1}{5(D+1)}\cdot\frac{(D-1)}{(D-1)}\sin x - \frac{2D+5}{25(D^2+2D+1)}\sin x \right]$$

$$= e^x \left[\frac{x}{5}\cdot\frac{(D-1)}{(D^2-1)}\sin x - \frac{2D+5}{25(-1^2+2D+1)}\sin x \right]$$

$$= e^x \left[\frac{x}{5}\cdot\frac{(D-1)}{(-1^2-1)}\sin x - \frac{2D+5}{25(2D)}\sin x \right]$$

$$= e^x \left[-\frac{x}{10}(\cos x - \sin x) - \frac{1}{25}\left(1+\frac{5}{2D}\right)\sin x \right]$$

$$= e^x \left[-\frac{x}{10}(\cos x - \sin x) - \frac{1}{25}\left(\sin x + \frac{5}{2}\int \sin x\, dx\right) \right]$$

$$= e^x \left[-\frac{x}{10}(\cos x - \sin x) - \frac{1}{25}\left(\sin x - \frac{5}{2}\cos x\right) \right]$$

Hence, the general solution is

$$y = c_1e^{-x} + c_2e^{-2x} - \frac{1}{5}e^x \left[\frac{x}{2}(\cos x - \sin x) + \frac{1}{5}\left(\sin x - \frac{5}{2}\cos x\right) \right]$$

Example 5

Solve $(4D^2+8D+3)y = xe^{-\frac{x}{2}}\cos x.$

Solution

The auxiliary equation is

$$4m^2+8m+3=0$$

$$(2m+1)(2m+3)=0$$

$$m=-\frac{1}{2},\ -\frac{3}{2} \quad \text{(real and distinct)}$$

$$\text{CF}=c_1e^{-\frac{x}{2}}+c_2e^{-\frac{3x}{2}}$$

$$\text{PI}=\frac{1}{4D^2+8D+3}xe^{-\frac{x}{2}}\cos x$$

$$=e^{-\frac{x}{2}}\frac{1}{4\left(D-\frac{1}{2}\right)^2+8\left(D-\frac{1}{2}\right)+3}x\cos x$$

$$=e^{-\frac{x}{2}}\frac{1}{4\left(D^2+\frac{1}{4}-D\right)+8D-4+3}x\cos x$$

$$=e^{-\frac{x}{2}}\frac{1}{(4D^2+4D)}x\cos x$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{D}\left(\frac{1}{D+1}x\cos x\right)$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{D}\left[x\cdot\frac{1}{D+1}\cos x-\frac{1}{(D+1)^2}\cos x\right]$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{D}\left(x\cdot\frac{D-1}{D^2-1}\cos x-\frac{1}{D^2+2D+1}\cos x\right)$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{D}\left[x\frac{(D-1)}{(-1^2-1)}\cos x-\frac{1}{-1^2+2D+1}\cos x\right]$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{D}\left[-\frac{x}{2}(D\cos x-\cos x)-\frac{1}{2}\int\cos x\,dx\right]$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{D}\left[-\frac{x}{2}(-\sin x-\cos x)-\frac{1}{2}\sin x\right]$$

$$=\frac{e^{-\frac{x}{2}}}{8}\left[\int x(\sin x+\cos x)dx+\int\sin x\ dx\right]$$

$$= \frac{e^{-\frac{x}{2}}}{8}\left[x(-\cos x + \sin x) - (-\sin x - \cos x) - \cos x\right]$$

$$= \frac{e^{-\frac{x}{2}}}{8}\left[x(\sin x - \cos x) + \sin x\right]$$

Hence, the general solution is

$$y = c_1 e^{-\frac{x}{2}} + c_2 e^{-\frac{3x}{2}} + \frac{e^{-\frac{x}{2}}}{8}\left[x(\sin x - \cos x) + \sin x\right]$$

Example 6

Solve $(D^2 - 1)y = \sin x + e^x + x^2 e^x$.

Solution

The auxiliary equation is

$$m^2 - 1 = 0$$

$$m = 1, -1 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^x + c_2 e^{-x}$$

$$\text{PI} = \frac{1}{D^2 - 1}(x \sin x + e^x + x^2 e^x)$$

$$= \frac{1}{D^2 - 1} x \sin x + \frac{1}{D^2 - 1} e^x + \frac{1}{D^2 - 1} x^2 e^x$$

$$= \left[x \frac{1}{D^2 - 1} \sin x - \frac{2D}{(D^2 - 1)^2} \sin x\right] + x \frac{1}{2D} e^x + e^x \frac{1}{(D+1)^2 - 1} x^2$$

$$= \left[x \frac{1}{-1^2 - 1} \sin x - \frac{2D}{(-1^2 - 1)^2} \sin x\right] + \frac{x}{2(1)} e^x + e^x \frac{1}{D^2 + 2D} x^2$$

$$= \left[-\frac{x \sin x}{2} - \frac{2D \sin x}{4}\right] + \frac{xe^x}{2} + e^x \frac{1}{2D\left(1 + \frac{D}{2}\right)} x^2$$

$$= \left[-\frac{x \sin x}{2} - \frac{\cos x}{2}\right] + \frac{xe^x}{2} + e^x \frac{1}{2D}\left(1 + \frac{D}{2}\right)^{-1} x^2$$

$$= -\left[\frac{x \sin x}{2} + \frac{\cos x}{2}\right] \frac{xe^x}{2} + e^x \frac{1}{2D}\left(1 - \frac{D}{2} + \frac{D^2}{4} - \frac{D^3}{8} + \cdots\right) x^2$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{xe^x}{2} + e^x \frac{1}{2D}\left(x^2 - \frac{2x}{2} + \frac{2}{4} - 0\right)$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{xe^x}{2} + e^x \frac{1}{2}\int\left(x^2 - x + \frac{1}{2}\right)dx$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{xe^x}{2} + \frac{e^x}{2}\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2}x\right)$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{1}{2}e^x\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{4}\right)$$

Hence, the general solution is

$$y = c_1e^x + c_2e^{-x} - \left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{1}{2}e^x\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{4}\right)$$

EXERCISE 5.2

Solve the following differential equations:

1. $(D^2 + D + 2)y = e^{\frac{x}{2}}$

$$\left[\textbf{Ans.:}\ y = e^{-\frac{x}{2}}\left[c_1\cos\left(\frac{\sqrt{7}x}{2}\right) + c_2\sin\left(\frac{\sqrt{7}x}{2}\right)\right] + -\frac{4}{11}e^x + \frac{1}{4}xe^{\frac{x}{2}}\right]$$

2. $(D^2 - 4)y = (1 + e^x)^2$

$$\left[\textbf{Ans.:}\ y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4} - \frac{2}{3}e^x + \frac{1}{4}xe^{2x}\right]$$

3. $(D^2 + D + 1)y = e^{3x} + 6e^x - 3e^{-2x} + 5$

$$\left[\textbf{Ans.:}\ y = e^{-\frac{x}{2}}\left(c_1\cos\frac{\sqrt{3}x}{2} + c_2\sin\frac{\sqrt{3}x}{2}\right) + \frac{e^{3x}}{13} + 2e^x - e^{-2x} + 5\right]$$

4. $(D^2 + 4D + 5)y = -2\cosh x + 2^x$

$$\left[\textbf{Ans.:}\ y = e^{-2x}(c_1\cos x + c_2\sin x) - \frac{e^x}{10} - \frac{e^{-x}}{2} + \frac{2^x}{(\log 2)^2 + 4(\log 2) + 5}\right]$$

5. $(D^3+D^2+D+1)y=\sin 2x$

$$\left[\text{Ans.: } y=c_1e^{-x}+c_2\cos x+c_3\sin x+\frac{1}{15}(2\cos 2x-\sin 2x)\right]$$

6. $(3D^2-7D+2)y=\sin x+\cos x$

$$\left[\text{Ans.: } y=c_1e^{2x}+c_2e^{\frac{x}{3}}+\frac{1}{25}(3\cos x-4\sin x)\right]$$

7. $(D^3-2D^2+4D)y=e^{2x}+\sin 2x$

$$\left[\text{Ans.: } y=c_1+e^{x}\left(c_2\cos\sqrt{3}x+c_3\sin\sqrt{3}x\right)+\frac{1}{8}(e^{2x}+\sin 2x)\right]$$

8. $(D^3+2D^2+D)y=\sin^2 x$

$$\left[\text{Ans.: } y=c_1+(c_2+c_3x)e^{-x}+\frac{x}{2}+\frac{1}{100}(3\sin 2x+4\cos 2x)\right]$$

9. $(D^2+D-6)y=e^{2x}$

$$\left[\text{Ans.: } y=c_1e^{2x}+c_2e^{-3x}+\frac{xe^{2x}}{5}\right]$$

10. $(9D^2+6D+1)y=e^{-\frac{x}{3}}$

$$\left[\text{Ans.: } y=(c_1+c_2x)e^{-\frac{x}{3}}+\frac{x^2}{18}e^{-\frac{x}{3}}\right]$$

11. $(D^2+4)y=e^{x}+\sin 2x$

$$\left[\text{Ans.: } y=c_1\cos 2x+c_2\sin 2x+\frac{e^{x}}{5}-\frac{x}{4}\cos 2x\right]$$

12. $(D^2-4)y=x^2$

$$\left[\text{Ans.: } y=c_1e^{2x}+c_2e^{-2x}-\frac{1}{4}\left(x^2+\frac{1}{2}\right)\right]$$

13. $(D^2+D)y=x^2+2x+4$

$$\left[\text{Ans.: } y=c_1+c_2e^{-x}+\frac{x^3}{3}+4x\right]$$

14. $(D^2+1)y = e^{2x} + \cosh 2x + x^3$

$$\left[\textbf{Ans.: } y = c_1 \cos x + c_2 \sin x + \frac{e^{2x}}{5} + \frac{1}{5}\cosh 2x + x^3 - 6x\right]$$

15. $(D-1)^2(D+1)^2 y = \sin^2 \frac{x}{2} + e^x + x$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{-x} + \frac{1}{2} - \frac{1}{8}\cos x + \frac{x^2}{8}e^x + x\right]$$

16. $(D^2 - 3D + 2)y = 2e^x \cos \frac{x}{2}$

$$\left[\textbf{Ans.: } y = c_1 e^x + c_2 e^{2x} - \frac{8}{5}e^x\left(2\sin\frac{x}{2} + \cos\frac{x}{2}\right)\right]$$

17. $(D^2 - 2D + 10)y = 16e^x \cos 3x + 24e^x \sin 3x$

$$\left[\textbf{Ans.: } y = e^x(c_1 \cos 3x + c_2 \sin 3x) + \frac{xe^x}{3}(8\sin 3x - 12\cos 3x)\right]$$

18. $(D^3 - 4D^2 + 9D - 10)y = 24e^x \sin 2x$

$$\left[\textbf{Ans.: } y = c_1 e^{2x} + e^x(c_2 \cos 2x + c_3 \sin 2x) - \frac{6xe^x}{5}(2\sin 2x - \cos 2x)\right]$$

19. $(4D^3 - 12D^2 + 13D - 10)y = 16e^{\frac{x}{2}} \cos x$

$$\left[\textbf{Ans.: } y = c_1 e^{2x} + e^{\frac{x}{2}}(c_2 \cos x + c_3 \sin x) - \frac{4xe^{\frac{x}{2}}}{13}(2\cos x + 3\sin x)\right]$$

20. $(D^2 + 4D + 8)y = 12e^{-2x} \sin x \sin 3x$

$$\left[\textbf{Ans.: } y = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{2}e^{-2x}(3x\sin 2x + \cos 4x)\right]$$

21. $(4D^2 + 9D + 2)y = xe^{-2x}$

$$\left[\textbf{Ans.: } y = c_1 e^{-2x} + c_2 e^{-\frac{x}{4}} - \frac{1}{98}(7x^2 + 8x)e^{-2x}\right]$$

22. $(D^2+4)y = x\sin x$

$$\left[\text{Ans.: } y = c_1\cos 2x + c_2\sin 2x + \frac{1}{9}(3x\sin x - 2\cos x)\right]$$

23. $(D^2+9)y = xe^{2x}\cos x$

$$\left[\text{Ans.: } y = c_1\cos 3x + c_2\sin 3x + \frac{e^{2x}}{400}[(30x-11)\cos x + (10x-2)\sin x]\right]$$

24. $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x$

$$\left[\text{Ans.: } y = (c_1+c_2x)e^{2x} - e^{2x}[4x\cos 2x + (2x^2-3)\sin 2x]\right]$$

25. $(D^2-1)y = x\sin x + (1+x^2)e^x$

$$\left[\text{Ans.: } y = c_1e^x + c_2e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{1}{12}xe^x(2x^2-3x+9)\right]$$

5.5 EULER-CAUCHY EQUATIONS

An equation of the form

$$a_0x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_{n-1}x\frac{dy}{dx} + a_ny = Q(x) \quad \ldots(5.10)$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants, is called Euler-Cauchy equation.

To solve Eq. (5.10),

let $x = e^z$, $1 = e^z\frac{dz}{dx}$, $\frac{dz}{dx} = \frac{1}{e^z} = \frac{1}{x}$

Now,
$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{1}{x} = \frac{1}{x}\frac{dy}{dz}$$

$$x\frac{dy}{dx} = \frac{dy}{dz}, \quad xDy = \mathcal{D}y, \quad \text{where } \mathcal{D} \equiv \frac{d}{dz} \text{ and } D = \frac{d}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\cdot\frac{1}{x}$$

$$x^2\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) \quad \text{or} \quad x^2D^2y = \mathcal{D}(\mathcal{D}-1)y$$

Similarly,
$$x^3D^3y = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$$
$$\dots\dots\dots\dots$$
$$\dots\dots\dots\dots$$
$$x^nD^ny = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)\dots[\mathcal{D}-(n-1)]y$$

Substituting these derivatives in Eq. (5.10),
$$\left[a_0\mathcal{D}(\mathcal{D}-1)\dots(\mathcal{D}-n+1)+a_1\mathcal{D}(\mathcal{D}-1)\dots(\mathcal{D}-n+2)\right.$$
$$\left.+\dots+a_{n-1}\mathcal{D}+a_n\right]y = Q(e^z)$$

which is a linear differential equation with constant coefficients and can be solved by the usual methods described in previous sections.

Example 1

Solve $x^2y'' - 20\,y = 0$.

Solution

$$(x^2D^2 - 20)\,y = 0$$

Putting $x = e^z$,
$$[\mathcal{D}(\mathcal{D}-1)-20]y = 0 \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$
$$(\mathcal{D}^2 - \mathcal{D} - 20)y = 0$$

The auxiliary equation is
$$m^2 - m - 20 = 0$$
$$(m-5)(m+4) = 0$$
$$m = 5, -4 \quad \text{(real and distinct)}$$

Hence, the general solution is
$$y = c_1e^{5z} + c_2e^{-4z}$$
$$= c_1x^5 + c_2x^{-4}$$
$$= c_1x^5 + \frac{c_2}{x^4}$$

Example 2

Solve $(x^2D^2 + xD)y = 0$.

Solution

$$(x^2D^2 + xD)y = 0$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1)+\mathcal{D}]y=0 \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^2-\mathcal{D}+\mathcal{D})y=0$$

$$\mathcal{D}^2 y=0$$

The auxiliary equation is

$$m^2=0$$

$$m=0, 0 \quad \text{(real and repeated)}$$

Hence, the general solution is

$$y=(c_1+c_1 z)e^{0z}$$

$$=c_1+c_2 z$$

$$=c_1+c_2 \log x$$

Example 3

Solve $(4x^2D^2+16xD+9)y=0.$

Solution

$$(4x^2\,D^2+16xD+9)y=0$$

Putting $x = e^z$,

$$[4\mathcal{D}(\mathcal{D}-1)+16\mathcal{D}+9]y=0 \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(4\mathcal{D}^2+12\mathcal{D}+9)y=0$$

The auxiliary equation is

$$4m^2+12m+9=0$$

$$(2m+3)^2=0$$

$$m=-\frac{3}{2}, -\frac{3}{2} \quad \text{(real and repeated)}$$

Hence, the general solution is

$$y=(c_1+c_2 z)e^{-\frac{3}{2}z}$$

$$=(c_1+c_2\log x)x^{-\frac{3}{2}}$$

Example 4

Solve $(x^2D^2 - xD + 2)y = 6.$

Solution

$$(x^2D^2 - xD + 2)y = 6$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) - \mathcal{D} + 2]y = 6 \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^2 - 2\mathcal{D} + 2)y = 6$$

The auxiliary equation is

$$m^2 - 2m + 2 = 0$$

$$m = 1 \pm i \text{ (complex)}$$

$$\text{CF} = e^z(c_1 \cos z + c_2 \sin z)$$

$$= x[c_1 \cos(\log x) + c_2 \sin(\log x)]$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 - 2\mathcal{D} + 2} 6e^{0z}$$

$$= \frac{1}{2} \cdot 6$$

$$= 3$$

Hence, the general solution is

$$y = x\left[c_1 \cos(\log x) + c_2 \sin(\log x)\right] + 3$$

Example 5

Solve $x^2 y'' - xy' + y = x.$

Solution

$$(x^2D^2 - xD + 1)y = x$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) - \mathcal{D} + 1]y = e^z \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^2 - 2\mathcal{D} + 1)y = e^z$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1 \text{ (real and repeated)}$$

$$\text{CF} = (c_1 + c_2 z)e^z$$

$$= (c_1 + c_2 \log x)x$$

$$\begin{aligned}
\text{PI} &= \frac{1}{\mathcal{D}^2 - 2\mathcal{D} + 1} e^z \\
&= \frac{1}{(\mathcal{D}-1)^2} e^z \\
&= z \frac{1}{2(\mathcal{D}-1)} e^z \\
&= z^2 \frac{1}{2} e^z \\
&= \frac{(\log x)^2 x}{2} \\
&= \frac{x}{2} (\log x)^2
\end{aligned}$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x)x + \frac{x}{2}(\log x)^2$$

Example 6

Solve $(x^2\text{D}^2 - 7x\text{D} + 12)y = x^2$.

Solution

$$(x^2\text{D}^2 - 7x\text{D} + 12)y = x^2$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) - 7\mathcal{D} + 12\right]y = (e^z)^2 \qquad \text{where } \mathcal{D} \equiv \frac{\text{d}}{\text{dz}}$$

$$(\mathcal{D}^2 - 8\mathcal{D} + 12)y = e^{2z}$$

The auxiliary equation is

$$\begin{aligned}
m^2 - 8m + 12 &= 0 \\
(m-6)(m-2) &= 0 \\
m &= 2, 6 \quad \text{(real and distinct)} \\
\text{CF} &= c_1 e^{2z} + c_2 e^{6z} \\
&= c_1 x^2 + c_2 x^6 \\
\text{PI} &= \frac{1}{\mathcal{D}^2 - 8\mathcal{D} + 12} e^{2z} \\
&= z \frac{1}{2\mathcal{D} - 8} e^{2z} \\
&= z \frac{1}{4-8} 2^{2z}
\end{aligned}$$

$$= -\frac{z}{4}e^{2z}$$

$$= -\frac{\log x}{4}x^2$$

Hence, the general solution is

$$y = c_1x^2 + c_2x^6 - \frac{x^2}{4}\log x$$

Example 7

Solve $\left(xD^2 + D - \frac{1}{x}\right)y = -ax^2.$

Solution

$$\left(xD^2 + D - \frac{1}{x}\right)y = -ax^2$$

Multiplying the given equation by x,

$$(x^2D^2 + xD - 1)y = -ax^3$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) + \mathcal{D} - 1]y = -ae^{3z} \quad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^2 - 1)y = -ae^{3z}$$

The auxiliary equation is

$$m^2 - 1 = 0$$

$$m = \pm 1 \text{ (real and distinct)}$$

$$\text{CF} = c_1e^z + c_2e^{-z}$$

$$= c_1x + c_2x^{-1}$$

$$= c_1x + \frac{c_2}{x}$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 - 1}(-ae^{3z})$$

$$= -a\frac{1}{8}e^{3z}$$

$$= -\frac{a}{8}x^3$$

Hence, the general solution is

$$y = c_1x + \frac{c_2}{x} - \frac{a}{8}x^3$$

Example 8

Solve $x^2\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}+2y=x^2+\dfrac{1}{x^2}$.

Solution

$$(x^2D^2+4xD+2)y=x^2+\frac{1}{x^2}$$

Putting $x=e^z$,

$$[\mathcal{D}(\mathcal{D}-1)+4\mathcal{D}+2]y=e^{2z}+\frac{1}{e^{2z}} \qquad \text{where } \mathcal{D}\equiv\frac{d}{dz}$$

$$(\mathcal{D}^2+3\mathcal{D}+2)y=e^{2z}+e^{-2z}$$

The auxiliary equation is

$$m^2+3m+2=0$$

$$(m+2)(m+1)=0$$

$$m=-2,-1 \quad \text{(real and distinct)}$$

$$\text{CF}=c_1e^{-2z}+c_2e^{-z}$$

$$=\frac{c_1}{x^2}+\frac{c_2}{x}$$

$$\text{PI}=\frac{1}{\mathcal{D}^2+3\mathcal{D}+2}(e^{2z}+e^{-2z})$$

$$=\frac{1}{\mathcal{D}^2+3\mathcal{D}+2}e^{2z}+\frac{1}{\mathcal{D}^2+3\mathcal{D}+2}e^{-2z}$$

$$=\frac{1}{4+3(2)+2}e^{2z}+\frac{1}{(\mathcal{D}+2)(\mathcal{D}+1)}e^{-2z}$$

$$=\frac{e^{2z}}{12}+\frac{1}{(\mathcal{D}+2)}\left[\frac{1}{-2+1}\right]e^{-2z}$$

$$=\frac{e^{2z}}{12}-\frac{1}{(\mathcal{D}+2)}e^{-2z}$$

$$=\frac{e^{2z}}{12}-z\cdot\frac{1}{1}e^{-2z}$$

$$=\frac{x^2}{12}-(\log x)\frac{1}{x^2}$$

Hence, the general solution is

$$y=\frac{c_1}{x^2}+\frac{c_2}{x}+\frac{x^2}{12}-\frac{1}{x^2}\log x$$

Example 9

Solve $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \sin(\log x)$. **[Summer 2017]**

Solution

$$(x^2D^2 - xD + 1)y = \sin(\log x)$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) - \mathcal{D} + 1]y = \sin z \qquad \text{where } \mathcal{D} = \frac{d}{dz}$$

$$(\mathcal{D}^2 - \mathcal{D} - \mathcal{D} + 1)y = \sin z$$

$$(\mathcal{D}^2 - 2\mathcal{D} + 1)y = \sin z$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1 \text{ (real and repeated)}$$

$$\text{CF} = (c_1 + c_2 z)\, e^z$$

$$= (c_1 + c_2 \log x)x$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 - 2\mathcal{D} + 1}\sin z$$

$$= \frac{1}{-1 - 2\mathcal{D} + 1}\sin z$$

$$= -\frac{1}{2\mathcal{D}}\sin z$$

$$= -\frac{1}{2}\int \sin z\, dz$$

$$= -\frac{1}{2}(-\cos z)$$

$$= \frac{1}{2}\cos z$$

$$= \frac{1}{2}\cos(\log x)$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x)x + \frac{1}{2}\cos(\log x)$$

Example 10

Solve $(4x^2D^2+1)y = 19\cos(\log x)+22\sin(\log x)$.

Solution

$$(4x^2D^2+1)y = 19\cos(\log x)+22\sin(\log x)$$

Putting $x = e^z$,

$$[4\mathcal{D}(\mathcal{D}-1)+1]\,y = 19\cos z + 22\sin z \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(4\mathcal{D}^2-4\mathcal{D}+1)y = 19\cos z+22\sin z$$

The auxiliary equation is

$$4m^2-4m+1=0$$

$$(2m-1)^2=0$$

$$m=\frac{1}{2},\frac{1}{2} \text{ (real and repeated)}$$

$$\text{CF} = (c_1+c_2z)e^{\frac{1}{2}z}$$

$$= (c_1+c_2\log x)x^{\frac{1}{2}}$$

$$\text{PI} = \frac{1}{4\mathcal{D}^2-4\mathcal{D}+1}(19\cos z+22\sin z)$$

$$= \frac{1}{4(-1^2)-4\mathcal{D}+1}(19\cos z+22\sin z)$$

$$= \frac{1}{-(4\mathcal{D}+3)}\cdot\frac{(4\mathcal{D}-3)}{(4\mathcal{D}-3)}(19\cos z+22\sin z)$$

$$= \frac{4\mathcal{D}-3}{-(16\mathcal{D}^2-9)}(19\cos z+22\sin z)$$

$$= \frac{4\mathcal{D}-3}{-\left[16(-1^2)-9\right]}(19\cos z+22\sin z)$$

$$= \frac{1}{25}[4(-19\sin z+22\cos z)-3(19\cos z+22\sin z)]$$

$$= \frac{1}{25}(31\cos z-142\sin z)$$

$$= \frac{1}{25}[31\cos(\log x)-142\sin(\log x)]$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x)x^{\frac{1}{2}} + \frac{1}{25}[31\cos(\log x) - 142\sin(\log x)]$$

Example 11

Solve $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = 12\dfrac{\log x}{x^2}$.

Solution

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = 12\frac{\log x}{x^2}$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12\log x$$

$$(x^2D^2 + xD)y = 12\log x$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) + \mathcal{D}]y = 12z \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^2 - \mathcal{D} + \mathcal{D})y = 12z$$

$$\mathcal{D}^2 y = 12z$$

The auxiliary equation is

$$m^2 = 0$$

$$m = 0, 0 \qquad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2 z)e^{0z}$$

$$= c_1 + c_2 z$$

$$= c_1 + c_2 \log x$$

$$\text{PI} = \frac{1}{\mathcal{D}^2}12z$$

$$= 12\frac{1}{\mathcal{D}^2}z$$

$$= 12\frac{1}{\mathcal{D}}\int z\,dz$$

$$= 12\frac{1}{\mathcal{D}}\left[\frac{z^2}{2}\right]$$

$$= 12\int\frac{z^2}{2}dz$$

$$= 12\frac{z^3}{6}$$

$$= 2z^3$$

$$= 2(\log x)^3$$

Hence, the general solution is

$$y = c_1 + c_2 \log x + 2(\log x)^3$$

Example 12

Solve $(4x^2D^2 + 1)y = \log x,\ x > 0.$

Solution

$$(4x^2D^2 + 1)y = \log x,$$

Putting $x = e^z$,

$$[4\mathcal{D}(\mathcal{D}-1)+1]y = z \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(4\mathcal{D}^2 - 4\mathcal{D}+1)y = z$$

The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)^2 = 0$$

$$m = \frac{1}{2}, \frac{1}{2} \text{ (real and repeated)}$$

$$\text{CF} = (c_1 + c_2 z)e^{\frac{1}{2}z}$$

$$= (c_1 + c_2 \log x)x^{\frac{1}{2}}$$

$$\text{PI} = \frac{1}{4\mathcal{D}^2 - 4\mathcal{D}+1} z$$

$$= \frac{1}{(2\mathcal{D}-1)^2} z$$

$$= \frac{1}{(1-2\mathcal{D})^2} z$$

$$= (1-2\mathcal{D})^{-2} z$$

$$= (1+4\mathcal{D}+12\mathcal{D}^2+\ldots)z$$

$$= z + 4\mathcal{D}z + 6\mathcal{D}^2 z + \ldots$$

$$= z + 4 + 0$$

$$= \log x + 4$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x)\sqrt{x} + \log x + 4$$

Example 13

Solve $x^2\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}+2y=x^2\sin(\log x).$ **[Winter 2016]**

Solution

$$(x^2D^2+4xD+2)\,y=x^2\sin(\log x)$$

Putting $x=e^z$,

$$[\mathcal{D}(\mathcal{D}-1)+4\mathcal{D}+2]\,y=(e^z)^2\sin z=e^{2z}\sin z \qquad \text{where } \frac{d}{dx}=\mathcal{D}$$

$$(\mathcal{D}^2+3\mathcal{D}+2)\,y=e^{2z}$$

The auxiliary equation is

$$m^2+3m+2=0$$
$$(m+2)(m+1)=0$$
$$m=-1,-2 \quad \text{(real and distinct)}$$
$$\text{CF}=c_1e^{-z}+c_2e^{-2z}$$
$$=c_1x^{-1}+c_2x^{-2}$$
$$=\frac{c_1}{x}+\frac{c_2}{x^2}$$

$$\text{PI}=\frac{1}{\mathcal{D}^2+3\mathcal{D}+2}e^{2z}\sin z$$
$$=e^{2z}\frac{1}{(\mathcal{D}+2)^2+3(\mathcal{D}+2)+2}\sin z$$
$$=e^{2z}\frac{1}{\mathcal{D}^2+4\mathcal{D}+4+3\mathcal{D}+6+2}\sin z$$
$$=e^{2z}\frac{1}{\mathcal{D}^2+7\mathcal{D}+12}\sin z$$
$$=e^{2z}\frac{1}{(-1)+7\mathcal{D}+12}\sin z$$
$$=e^{2z}\frac{1}{7\mathcal{D}+11}\sin z$$
$$=e^{2z}\frac{7\mathcal{D}-11}{49\mathcal{D}^2-121}\sin z$$
$$=e^{2z}\frac{7\mathcal{D}-11}{-49-121}\sin z$$
$$=-\frac{1}{170}e^{2z}[7\mathcal{D}-11]\sin z$$

$$= -\frac{1}{170}e^{2z}\,[7\cos z - 11\sin z]$$

$$= -\frac{1}{170}x^2[7\cos(\log x) - 11\sin(\log x)]$$

Hence, the general solution is

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} - \frac{1}{170}x^2[7\cos(\log x) - 11\sin(\log x)]$$

Example 14

Solve $(x^2D^2 + 5xD + 3)y = \dfrac{\log x}{x^2}$.

Solution

$$(x^2D^2 + 5xD + 3)y = \frac{\log x}{x^2}$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) + 5\mathcal{D} + 3]y = \frac{z}{e^{2z}} \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^2 + 4\mathcal{D} + 3)y = e^{-2z}z$$

The auxiliary equation is

$$m^2 + 4m + 3 = 0$$

$$(m+1)(m+3) = 0$$

$$m = -1, -3 \text{ (real and distinct)}$$

$$\text{CF} = c_1e^{-z} + c_2e^{-3z}$$

$$= c_1(x)^{-1} + c_2(x)^{-3}$$

$$= \frac{c_1}{x} + \frac{c_2}{x^3}$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 + 4\mathcal{D} + 3}e^{-2z}z$$

$$= e^{-2z}\frac{1}{(\mathcal{D}-2)^2 + 4(\mathcal{D}-2) + 3}z$$

$$= e^{-2z}\frac{1}{\mathcal{D}^2 - 1}z$$

$$= -e^{-2z}(1 - \mathcal{D}^2)^{-1}z$$

$$= -e^{-2z}(1 + \mathcal{D}^2 + \mathcal{D}^4 + \ldots)z$$

$$= -e^{-2z}(z + \mathcal{D}^2 z + \mathcal{D}^4 z + \ldots)$$
$$= -e^{-2z}(z+0)$$
$$= -(x)^{-2}(\log x)$$
$$= -\frac{\log x}{x^2}$$

Hence, the general solution is

$$y = \frac{c_1}{x} + \frac{c_2}{x^3} - \frac{\log x}{x^2}$$

Example 15

Solve $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x\log x.$

Solution

$$(x^2D^2 + 4xD + 2)y = x \log x$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) + 4\mathcal{D} + 2]y = e^z \cdot z \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^2 + 3\mathcal{D} + 2)y = e^z$$

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$
$$(m+2)(m+1) = 0$$
$$m = -1, -2 \qquad \text{(real and distinct)}$$
$$\text{CF} = c_1 e^{-z} + c_2 e^{-2z}$$
$$= c_1 x^{-1} + c_2 x^{-2}$$
$$= \frac{c_1}{x} + \frac{c_2}{x^2}$$
$$\text{PI} = \frac{1}{\mathcal{D}^2 + 3\mathcal{D} + 2} z e^z$$
$$= e^z \frac{1}{(\mathcal{D}+1)^2 + 3(\mathcal{D}+1) + 2} z$$
$$= e^z \frac{1}{\mathcal{D}^2 + 2\mathcal{D} + 1 + 3\mathcal{D} + 3 + 2} z$$
$$= e^z \frac{1}{\mathcal{D}^2 + 5\mathcal{D} + 6} z$$

$$= \frac{e^z}{6} \frac{1}{\left(1+\frac{5\mathcal{D}+\mathcal{D}^2}{6}\right)} z$$

$$= \frac{e^z}{6}\left[1+\frac{5\mathcal{D}+\mathcal{D}^2}{6}\right]^{-1} z$$

$$= \frac{e^z}{6}\left[1-\frac{5\mathcal{D}+\mathcal{D}^2}{6}+\cdots\right] z$$

$$= \frac{e^z}{6}\left[z-\frac{5}{6}\mathcal{D}z-\frac{1}{6}\mathcal{D}^2 z+\cdots\right]$$

$$= \frac{e^z}{6}\left[z-\frac{5}{6}\right]$$

$$= \frac{1}{6}x\left[\log x-\frac{5}{6}\right]$$

Hence, the general solution is

$$y = \frac{c_1}{x}+\frac{c_2}{x^2}+\frac{1}{6}x\left[\log x-\frac{5}{6}\right]$$

Example 16

Solve $x^2\frac{d^2y}{dx^2}-6x\frac{dy}{dx}+6y = x^{-3}\log x.$ **[Winter 2015]**

Solution

$$(x^2 D^2 - 6x D + 6)\, y = x^{-3} \log x$$

Putting $x = e^z$,

$$(\mathcal{D}^2 - 7\mathcal{D} + 6)y = z e^{-3z} \qquad \text{where} \quad \mathcal{D} = \frac{d}{dz}$$

The auxiliary equation is

$$m^2 - 7m + 6 = 0$$
$$m^2 - 6m - m + 6 = 0$$
$$m = 1, 6 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^z + c_2 e^{6z}$$
$$= c_1 x + c_2 x^6$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 - 7\mathcal{D} + 6} z\, e^{-3z}$$

$$= \frac{e^{-3z}}{(\mathcal{D}-3)^2 - 7(\mathcal{D}-3) + 6} z$$

$$= e^{-3z} \frac{1}{\mathcal{D}^2 - 6\mathcal{D} + 9 - 7\mathcal{D} + 21 + 6} z$$

$$= e^{-3z} \frac{1}{\mathcal{D}^2 - 13\mathcal{D} + 36} z$$

$$= \frac{e^{-3z}}{36}\left[1 + \left(-\frac{13\mathcal{D} + \mathcal{D}^2}{36}\right)\right]^{-1} z$$

$$= \frac{e^{-3z}}{36}\left[z + \frac{13}{36}\right]$$

$$= \frac{1}{36} x^{-3}\left(\log x + \frac{13}{36}\right)$$

Hence, the general solution is

$$y = c_1 x + c_2 x^6 + \frac{1}{36} x^{-3}\left(\log x + \frac{13}{36}\right)$$

Example 17

Solve $(x^2\text{D}^2 - x\text{D} + 1)y = \left(\dfrac{\log x}{x}\right)^2$.

Solution

$$(x^2\text{D}^2 - x\text{D} + 1)y = \left(\frac{\log x}{x}\right)^2$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) - \mathcal{D} + 1]y = \left(\frac{z}{e^z}\right)^2 \qquad \text{where } \mathcal{D} \equiv \frac{\text{d}}{\text{d}z}$$

$$(\mathcal{D}^2 - 2\mathcal{D} + 1)y = z^2 e^{-2z}$$

$$(\mathcal{D} - 1)^2 y = z^2 e^{-2z}$$

The auxiliary equation is

$$(m-1)^2 = 0$$

$$m = 1, 1 \quad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2 z)e^z$$

$$= (c_1 + c_2 \log x)x$$

$$\text{PI} = \frac{1}{(\mathcal{D}-1)^2}(z^2 e^{-2z})$$

$$= e^{-2z}\frac{1}{(\mathcal{D}-2-1)^2}z^2$$

$$= e^{-2z}\frac{1}{(\mathcal{D}-3)^2}z^2$$

$$= \frac{e^{-2z}}{9}\frac{1}{\left(1-\frac{\mathcal{D}}{3}\right)^2}z^2$$

$$= \frac{e^{-2z}}{9}\left(1-\frac{\mathcal{D}}{3}\right)^{-2}z^2$$

$$= \frac{e^{-2z}}{9}\left[1+\frac{2\mathcal{D}}{3}+3\frac{\mathcal{D}^2}{9}+3\frac{\mathcal{D}^3}{27}+\cdots\right]z^2$$

$$= \frac{e^{-2z}}{9}\left[z^2+\frac{2}{3}\text{D}z^2+\frac{1}{3}\text{D}^2z^2+\frac{1}{9}\text{D}^3z^2+\cdots\right]$$

$$= \frac{e^{-2z}}{9}\left[z^2+\frac{2}{3}(2z)+\frac{1}{3}(2)+0\right]$$

$$= \frac{e^{-2z}}{9}\left[z^2+\frac{4}{3}z+\frac{2}{3}\right]$$

$$= \frac{1}{9x^2}\left[(\log x)^2+\frac{4}{3}\log x+\frac{2}{3}\right]$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x)x + \frac{1}{9x^2}\left[(\log x)^2+\frac{4}{3}\log x+\frac{2}{3}\right]$$

Example 18

Solve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=\log x\cdot\sin(\log x).$ **[Winter 2017]**

Solution

$$(x^2 D^2 - x D + 1)\, y = \log x \sin(\log x)$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) + \mathcal{D} + 1]y = z \sin z$$
$$(\mathcal{D}^2 + 1)y = z \sin z$$

The auxiliary equation is

$$m^2 + 1 = 0$$
$$m = \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos z + c_2 \sin z$$
$$= c_1 \cos(\log x) + c_2 \sin(\log x)$$

$$\text{PI} = \frac{1}{\mathcal{D}^2+1} z \sin z$$
$$= z\frac{1}{\mathcal{D}^2+1}\sin z - \frac{2\mathcal{D}}{(\mathcal{D}^2+1)^2}\sin z$$
$$= z\left(z\frac{1}{2\mathcal{D}}\sin z\right) - \frac{2}{(\mathcal{D}^2+1)^2}\mathcal{D}\sin z$$
$$= \frac{z^2}{2}\int \sin z\, dz - \frac{2}{(\mathcal{D}^2+1)^2}\cos z$$
$$= \frac{z^2}{2}(-\cos z) + \frac{2}{2(\mathcal{D}^2+1)2\mathcal{D}}\cos z$$
$$= -\frac{z^2}{2}\cos z + \frac{1}{2(\mathcal{D}^3+\mathcal{D})}\cos z$$
$$= -\frac{z^2}{2}\cos z + \frac{z}{2}\frac{1}{3\mathcal{D}^2+1}\cos z$$
$$= -\frac{z^2}{2}\cos z + \frac{z}{2}\frac{1}{3(-1)^2+1}\cos z$$
$$= -\frac{z^2}{2}\cos z - \frac{z}{4}\cos z$$
$$= -\frac{(\log x)^2}{2}\cos(\log x) - \frac{\log x}{4}\cos(\log x)$$

Hence, the general solution is

$$y = c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{(\log x)^2}{2}\cos(\log x) - \frac{\log x}{4}\cos(\log x)$$

Example 19

Solve $(x^3D^3 + x^2D^2 - 2)y = x + \frac{1}{x^3}$.

Solution

$$(x^3D^3 + x^2D^2 - 2)y = x + \frac{1}{x^3}$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2) + \mathcal{D}(\mathcal{D}-1) - 2]y = e^z + e^{-3z} \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^3 - 2\mathcal{D}^2 + \mathcal{D} - 2)y = e^z + e^{-3z}$$

The auxiliary equation is

$$m^3 - 2m^2 + m - 2 = 0$$

$$(m-2)(m^2+1) = 0$$

$$m = 2 \text{ (real)}, \ m = \pm i \text{ (complex)}$$

$$\text{CF} = c_1e^{2z} + c_2\cos z + c_3 \sin z$$

$$= c_1x^2 + c_2\cos(\log x) + c_3\sin(\log x)$$

$$\text{PI} = \frac{1}{\mathcal{D}^3 - 2\mathcal{D}^2 + \mathcal{D} - 2}(e^z + e^{-3z})$$

$$= \frac{1}{1-2+1-2}e^z + \frac{1}{(-3)^3 - 2(-3)^2 - 3 - 2}e^{-3z}$$

$$= -\frac{1}{2}e^z - \frac{1}{50}e^{-3z}$$

$$= -\frac{1}{2}x - \frac{1}{50}(x)^{-3}$$

$$= -\frac{1}{2}x - \frac{1}{50x^3}$$

Hence, the general solution is

$$y = c_1x^2 + c_2\cos(\log x) + c_3\sin(\log x) - \frac{1}{2}x - \frac{1}{50x^3}$$

EXERCISE 5.3

Solve the following differential equations:

1. $(x^2D^2 + xD - 1)y = 0$ $\left[\textbf{Ans.: } y = c_1x + \frac{c_2}{x}\right]$

2. $(9x^2D^2 + 3xD + 10)y = 0$

$$\left[\text{Ans.: } y = x^{\frac{1}{3}}\left[c_1\cos(\log x) + c_2\sin(\log x)\right]\right]$$

3. $(x^3D^3 - 2xD + 4)y = 0$

$$\left[\text{Ans.: } y = \frac{c_1}{x} + (c_2 + c_3\log x)x^2\right]$$

4. $(x^3D^3 + 3x^2D^2 + 14xD + 34)y = 0$

$$\left[\text{Ans.: } \frac{c_1}{x^2} + x[c_2\cos(4\log x) + c_3\sin(4\log x)]\right]$$

5. $(x^2D^2 - 3xD + 4)y = x^3$

$$\left[\text{Ans.: } y = (c_1 + c_2\log x)x^2 + x^3\right]$$

6. $(x^3D^3 + 6x^2D^2 - 12)y = \dfrac{12}{x^2}$

$$\left[\text{Ans.: } y = c_1x^2 + \frac{c_2}{x^2} + \frac{c_3}{x^3} - \frac{3}{x^2}\log x\right]$$

7. $(4x^3D^3 + 12x^2D^2 + xD + 1)y = 50\sin(\log x)$

$$\left[\text{Ans.: } y = (c_1 + c_2\log x)x^{\frac{1}{2}} + \frac{c_3}{x} + \sin(\log x) + 7\cos(\log x)\right]$$

8. $(x^2D^2 - 3xD + 3)y = 2 + 3\log x$

$$\left[\text{Ans.: } y = c_1x + c_2x^3 + \log x + 2\right]$$

9. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \dfrac{\sin(\log x) + 1}{x}$

$$\left[\begin{aligned}\text{Ans.: } y = x^2\left[c_1\cosh\left(\sqrt{3}\log x\right) + c_2\sinh\left(\sqrt{3}\log x\right)\right] + \frac{1}{6x}\\ + \frac{1}{61x}[5\sin(\log x) + 6\cos(\log x)]\end{aligned}\right]$$

10. $(x^2D^2 - 3xD + 5)y = x^2\sin(\log x)$

$$\left[\text{Ans.: } y = x^2[c_1\cos(\log x) + c_2\sin(\log x)] - \frac{x^2}{2}\log x\cos(\log x)\right]$$

11. $(x^2D^3 + 3xD^2 + D)y = x^2 \log x$

$$\left[\text{Ans.: } c_1 + c_2 \log x + c_3 (\log x)^2 + \frac{x^3}{27}(\log x - 1)\right]$$

12. $(x^3D^3 + 2x^2D^2 + 2)y = 10\left(x + \frac{1}{x}\right)$

$$\left[\text{Ans.: } y = \frac{c_1}{x} + x[c_2 \cos(\log x) + c_3 \sin(\log x)] + 5x + \frac{2}{x}\log x\right]$$

13. $(x^2D^2 - 2xD + 2)y = (\log x)^2 - \log x^2$

$$\left[\text{Ans.: } y = c_1 x + c_2 x^2 + \frac{1}{2}[(\log x)^2 + \log x] + \frac{1}{4}\right]$$

14. $(x^2D^2 + 3xD + 1)y = \dfrac{1}{(1-x)^2}$

$$\left[\text{Ans.: } y = \frac{1}{x}(c_1 + c_2 \log x) + \frac{1}{x}\log\frac{x}{x-1}\right]$$

5.6 EXISTENCE AND UNIQUENESS OF SOLUTIONS

Let $p_1, p_2, \ldots, p_n$, and Q all be continuous functions on an open interval I such that $x_0 \in I$. Then the n^{th} order linear ordinary differential equation

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x) y = Q(x) \qquad \ldots (5.11)$$

with the initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$ has a unique solution $y = \phi(x)$ throughout the interval I.

5.7 LINEAR DEPENDENCE AND INDEPENDENCE OF SOLUTIONS

Two solutions $y_1(x)$ and $y_2(x)$ of linear differential equations of second order with constant coefficients are said to be linearly independent if

$$k_1 y_1(x) + k_2 y_2(x) = 0 \qquad \ldots (5.12)$$

implies k_1 and k_2 are both zeros, i.e., when y_1 or y_2 can not be expressed in terms of each other.

Two solutions $y_1(x)$ and $y_2(x)$ are linearly dependent if Eq. (5.12) holds for some constants k_1 and k_2 not both zero.

If Wronskian of $y_1(x)$ and $y_2(x)$, i.e.,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$y_1(x)$ and $y_2(x)$ are linearly independent.

If $W = 0$, $y_1(x)$ and $y_2(x)$ are linearly dependent.

Example 1

Determine if the following solutions y_1 and y_2 are linearly dependent or independent:

(a) $y_1 = \cos ax$, $y_2 = \sin ax$ *with* $a \neq 0$

(b) $y_1 = \log x$, $y_2 = \log x^n$ *with n non-negative integer.*

(c) $y_1 = 9\cos 2x$, $y_2 = 2\cos^2 x - 2\sin^2 x$

(d) $y_1 = 2x^2$, $y_2 = x^4$ *with* $x \neq 0$

(e) $y_1 = 6^x$, $y_2 = 6^{x+2}$

Solution

(a) $y_1 = \cos ax$, $y_2 = \sin ax$ with $a \neq 0$

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} \\ &= a(\cos^2 ax - \sin^2 ax) \\ &= a \\ &\neq 0 \end{aligned}$$

Hence, y_1 and y_2 are linearly independent.

(b) $y_1 = \log x$, $y_2 = \log x^n = n\log x$

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} \log x & n\log x \\ \frac{1}{x} & \frac{n}{x} \end{vmatrix} \\ &= \frac{n}{x}\log x - \frac{n}{x}\log x \\ &= 0 \end{aligned}$$

Hence, y_1 and y_2 are linearly dependent.

(c)
$$y_1 = 9\cos 2x,\ y_2 = 2\cos^2 x - 2\sin^2 x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} 9\cos 2x & 2\cos^2 x - 2\sin^2 x \\ -18\sin 2x & -4\cos x\sin x - 4\sin x\cos x \end{vmatrix}$$

$$= \begin{vmatrix} 9\cos 2x & 2\cos 2x \\ -18\sin 2x & -4\sin 2x \end{vmatrix}$$

$$= -36\cos 2x\sin 2x - (-36\cos 2x\sin 2x)$$

$$= 0$$

Hence, y_1 and y_2 are linearly dependent.

(d)
$$y_1 = 2x^2,\ y_2 = x^4 \text{ with } x \neq 0$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} 2x^2 & x^4 \\ 4x & 4x^3 \end{vmatrix}$$

$$= 8x^5 - 4x^5$$

$$= 4x^5$$

$$\neq 0$$

Hence, y_1 and y_2 are linearly independent.

(e)
$$y_1 = 6^x,\ y_2 = 6^{x+2}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} 6x & 6^{x+2} \\ 6^x \log x & 6^{x+2}\log 6 \end{vmatrix}$$

$$= 6^x 6^{x+2}\log 6 - 6^{x+2} 6^x \log x$$

$$= 0$$

Hence, y_1 and y_2 are linearly independent.

EXERCISE 5.4

Show that the following pair of solutions are linearly independent:

1. $\cos 2\pi x$, $\sin 2\pi x$
2. $e^{ax}\sin bx$, $e^{ax}\cos bx$
3. $1, x$
4. e^x, x^2

5. e^x, xe^x

6. $x^4, x^4 \log x$

7. $x^2, x^2 \log x$

8. $x^{\frac{1}{2}}, x^{-\frac{1}{2}}$

5.8 METHOD OF VARIATION OF PARAMETERS

This method is used to find the particular integral if the complementary function is known. In this method, the particular integral is obtained by varying the arbitrary constants of the complementary function and, hence, is known as variation-of-parameters method.

Consider a linear nonhomogeneous differential equation of second order with constant coefficients

$$\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=Q(x) \quad ...(5.13)$$

Let the complementary function be

$$CF = c_1y_1 + c_2y_2 \quad ...(5.14)$$

where y_1, y_2 are the solution of

$$\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=0 \quad ...(5.15)$$

Let the particular integral be

$$y = u(x)\,y_1 + v(x)\,y_2 \quad ...(5.16)$$

where u and v are unknown functions of x.

Differentiating Eq. (5.16) w.r.t. x,

$$y' = uy_1' + vy_2' + u'y_1 + v'y_2$$

Let u, v satisfy the equation

$$u'y_1 + v'y_2 = 0 \quad ...(5.17)$$

Then $$y' = uy_1' + vy_2'$$

Differentiating y' again w.r.t. x,

$$y'' = uy_1'' + vy_2'' + u'y_1' + v'y_2'$$

Substituting y'', y' and y in Eq. (5.13),

$$uy_1'' + vy_2'' + u'y_1' + v'y_2' + a_1(uy_1' + vy_2') + a_2(uy_1 + vy_2) = Q(x)$$

$$u(y_1'' + a_1y_1' + a_2y_1) + v(y_2'' + a_1y_2' + a_2y_2) + u'y_1' + v'y_2' = Q(x)$$

Since y_1 and y_2 satisfy Eq. (5.15),

$$u'y_1' + v'y_2' = Q \quad ...(5.18)$$

Solving Eqs (5.17) and (5.18) by using Cramer's rule,

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ Q & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 Q}{y_1 y_2' - y_1' y_2}$$

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & Q \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 Q}{y_1 y_2' - y_1' y_2}$$

$$u = \int -\frac{y_2 Q}{y_1 y_2' - y_1' y_2}\,dx = \int -\frac{y_2 Q}{W}\,dx \qquad ...(5.19)$$

$$v = \int \frac{y_1 Q}{y_1 y_2' - y_1' y_2}\,dx = \int \frac{y_1 Q}{W}\,dx \qquad ...(5.20)$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is known as the Wronskian of y_1, y_2.

Hence, the required general solution of Eq. (5.13) is

$$y = CF + PI$$
$$= c_1 y_1 + c_2 y_2 + u y_1 + v y_2$$

where u, v are obtained using equations (5.19) and (5.20).

Note: The above method can also be extended to third-order differential equation. Let the complementary function of a third-order differential equation be

$$CF = c_1 y_1 + c_2 y_2 + c_3 y_3$$

Let PI $= u(x)y_1 + v(x)y_2 + w(x)y_3$

where
$$u(x) = \int \frac{(y_2 y_3' - y_3 y_2')Q}{W}\,dx$$

$$v(x) = \int \frac{(y_3 y_1' - y_1 y_3')Q}{W}\,dx$$

$$w(x) = \int \frac{(y_1 y_2' - y_2 y_1')Q}{W}\,dx$$

Wronskian,
$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Working Rules

1. Find the complementary function as $CF = c_1 y_1 + c_2 y_2$.
2. Find the Wronskian of y_1, y_2 as $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.
3. Assume the particular integral as $PI = u(x)y_1 + v(x)y_2$.
4. Find u and v by evaluating the integrals $u = \int \frac{-y_2 Q}{W}\,dx$, $v = \int \frac{y_1 Q}{W}\,dx$.
5. Substitute u and v in PI and write the general solution as $y = CF + PI$.

Example 1

Find the Wronskian of y_1, y_2 of $y'' - 2y' + y = e^x \log x$.

Solution

$$(D^2 - 2D + 1)\,y = e^x \log x$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1 \quad \text{(real and repeated)}$$

$$CF = (c_1 + c_2 x)e^x = c_1 e^x + c_2 x e^x$$

$$y_1 = e^x, \quad y_2 = xe^x$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}$$

$$= e^x(xe^x + e^x) - xe^{2x}$$

$$= xe^{2x} + e^{2x} - xe^{2x}$$

$$= e^{2x}$$

Example 2

Solve $\dfrac{d^2 y}{dx^2} + y = \sin x$. **[Winter 2017]**

Solution

$$(D^2+1)y = \sin x$$

The auxiliary equation is

$$m^2+1=0$$

$$m = \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos x + c_2 \sin x$$

$$y_1 = \cos x, \qquad y_2 = \sin x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

Let

$$\text{PI} = u\cos x + v \sin x \qquad \dots(1)$$

where

$$u = \int -\frac{y_2 Q}{W} dx$$

$$= \int -\frac{\sin x \sin x}{1} dx$$

$$= -\int \frac{(1-\cos 2x)}{2} dx$$

$$= -\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)$$

and

$$v = \int \frac{y_1 Q}{W} dx$$

$$= \int \frac{\cos x \sin x}{1} dx$$

$$= \int \frac{\sin 2x}{2} dx$$

$$= \frac{1}{2}\left(-\frac{\cos 2x}{2}\right)$$

$$= -\frac{1}{4}\cos 2x$$

Substituting u and v in Eq. (1),

$$\text{PI} = -\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)\cos x - \frac{1}{4}\cos 2x \sin x$$

$$= -\frac{1}{2}x + \frac{1}{4}(\sin 2x \cos x - \cos 2x \sin x)$$

$$= -\frac{1}{2}x + \frac{1}{4}\sin(2x - x)$$

$$= -\frac{1}{2}x + \frac{1}{4}\sin x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{2}x + \frac{1}{4}\sin x$$

Example 3

Solve $(D^2 + 1)y = \operatorname{cosec} x.$

Solution

The auxiliary equation is

$$m^2 = -1$$

$$m = \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos x + c_2 \sin x$$

$$y_1 = \cos x, \qquad y_2 = \sin x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

Let $\quad \text{PI} = u \cos x + v \sin x \qquad \text{...(1)}$

where

$$u = \int -\frac{y_2 Q}{W} dx$$

$$= -\int \frac{\sin x \operatorname{cosec} x}{1} dx$$

$$= -\int dx$$

$$= -x$$

and

$$v = \int \frac{y_1 Q}{W} dx$$

$$= \int \frac{\cos x \operatorname{cosec} x}{1} dx$$

$$= \int \cot x \, dx$$

$$= \log \sin x$$

Substituting u and v in Eq. (1),

$$\text{PI} = -x \cos x + (\log \sin x) \sin x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \cdot \log \sin x$$

Example 4

Solve $y'' + 9y = \sec 3x$. **[Summer 2015]**

Solution

$$(D^2 + 9)y = \sec 3x$$

The auxiliary equation is

$$m^2 + 9 = 0$$

$$m = \pm 3i \quad \text{(complex)}$$

$$CF = c_1 \cos 3x + c_2 \sin 3x$$

$$y_1 = \cos 3x, \quad y_2 = \sin 3x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 3x + 3\sin^2 3x = 3$

Let $\quad PI = u\cos 3x + v\sin 3x \quad$ ...(1)

where

$$u = \int -\frac{y_2 Q}{W} dx$$

$$= -\int \frac{\sin 3x \sec 3x}{3} dx$$

$$= -\frac{1}{3}\int \frac{\sin 3x}{\cos 3x} dx$$

$$= -\frac{1}{3}\int \tan 3x \, dx$$

$$= -\frac{1}{3}\left(-\frac{1}{3}\log\cos 3x\right)$$

$$= \frac{1}{9}\log\cos 3x$$

and

$$v = \int \frac{y_1 Q}{W} dx$$

$$= \int \frac{\cos 3x \sec 3x}{3} dx$$

$$= \frac{1}{3}\int dx$$

$$= \frac{x}{3}$$

Substituting u and v in Eq. (1),

$$PI = \frac{1}{9}\cos 3x \log\cos 3x + \frac{x}{3}\sin 3x$$

Hence, the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{9}\cos 3x \log\cos 3x + \frac{x}{3}\sin 3x$$

Example 5

Solve $y'' + a^2y = \tan ax$. **[Summer 2016]**

Solution

$$(D^2 + a^2)y = \tan ax$$

The auxiliary equation is

$$m^2 + a^2 = 0$$

$$m = \pm ai \quad \text{(complex)}$$

$$CF = c_1 \cos ax + c_2 \sin ax$$

$$y_1 = \cos ax, \qquad y_2 = \sin ax$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a(\cos^2 ax + \sin^2 ax) = a$

Let $\quad PI = u\cos ax + v\sin x \quad$ ...(1)

where
$$u = \int -\frac{y_2 Q}{W}dx$$

$$= \int -\frac{\sin ax \tan ax}{a}dx$$

$$= -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax}dx$$

$$= -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax}dx$$

$$= -\frac{1}{a}\int (\sec ax - \cos ax)dx$$

$$= -\frac{1}{a}\cdot\frac{1}{a}\log(\sec ax + \tan ax) + \frac{1}{a^2}\sin ax$$

$$= \frac{1}{a^2}\sin ax - \frac{1}{a^2}\log(\sec ax + \tan ax)$$

and
$$v = \int \frac{y_1 Q}{W}dx$$

$$= \int \frac{\cos ax \tan ax}{a}dx$$

$$= \frac{1}{a}\int \sin ax\, dx$$

$$= \frac{1}{a}\left(-\frac{1}{a}\cos ax\right)$$

$$= -\frac{1}{a}\cos ax$$

Substituting u and v in Eq. (1),

$$PI = \frac{1}{a^2}\sin ax\cos ax - \frac{1}{a^2}\cos ax\log(\sec ax + \tan ax) - \frac{1}{a^2}\sin ax\cos ax$$

$$= -\frac{1}{a^2}\cos ax \log(\sec ax + \tan ax)$$

Hence, the general solution is

$$y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\cos ax \log(\sec ax + \tan ax)$$

Example 6

Solve $(D^2 + 4)y = \tan 2x$. **[Summer 2014]**

Solution

The auxiliary equation is

$$m^2 + 4 = 0$$

$$m = \pm 2i \quad \text{(complex)}$$

$$CF = c_1 \cos 2x + c_2 \sin 2x$$

$$y_1 = \cos 2x, \quad y_2 = \sin 2x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$

Let $\quad PI = u\cos 2x + v\sin 2x \quad$ ...(1)

where $\quad u = \int -\frac{y_2 Q}{W} dx$

$$= \int -\frac{\sin 2x \tan 2x}{2} dx$$

$$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx$$

$$= -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x} dx$$

$$= -\frac{1}{2}\int (\sec 2x - \cos 2x) dx$$

$$= -\frac{1}{2}\cdot\frac{1}{2}\log(\sec 2x + \tan 2x) + \frac{1}{2}\frac{\sin 2x}{2}$$

$$= \frac{1}{4}[\sin 2x - \log(\sec 2x + \tan 2x)$$

and $\quad v = \int \frac{y_1 Q}{W} dx$

$$= \int \frac{\cos 2x \tan 2x}{2} dx$$

$$= \frac{1}{2}\int \sin 2x dx$$

$$= \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right)$$

$$= -\frac{1}{4}\cos 2x$$

Substituting u and v in Eq. (1),

$$\text{PI} = \frac{1}{4}[\sin 2x - \log(\sec 2x + \tan 2x)]\cos 2x - \frac{1}{4}\cos 2x \sin 2x$$

$$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Hence, the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Example 7

Solve $\dfrac{d^2y}{dx^2} + 9y = \tan 3x.$ **[Winter 2015]**

Solution

$$(D^2 + 9)y = \tan 3x$$

The auxiliary equation is

$$m^2 + 9 = 0$$

$$m = \pm 3i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos 3x + c_2 \sin 3x$$

$$y_1 = \cos 3x, \quad y_2 = \sin 3x$$

Wronskian $W = \begin{vmatrix} y_i & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 3x + 3\sin^2 3x = 3$

Let $\text{PI} = u \cos 3x + v \sin 3x$ ...(1)

where $u = \int -\frac{y_2 Q}{W} dx$

$$= \int -\frac{\sin 3x \tan 3x}{3} dx$$

$$= -\frac{1}{3}\int \frac{\sin^2 3x}{\cos 3x} dx$$

$$= -\frac{1}{3}\int \left(\frac{1 - \cos^2 3x}{\cos 3x}\right) dx$$

$$= -\frac{1}{3}\int (\sec 3x - \cos 3x) dx$$

$$= -\frac{1}{3}\left[\log(\sec 3x + \tan 3x)\cdot\frac{1}{3} - \frac{1}{3}\sin 3x\right]$$

$$= -\frac{1}{9}[\log(\sec 3x + \tan 3x)] + \frac{1}{9}\sin 3x$$

and

$$v = \int \frac{y_1 Q}{W}\,dx$$

$$= \int \frac{\cos 3x \tan 3x}{3}\,dx$$

$$= \frac{1}{3}\int \sin 3x\,dx$$

$$= \frac{1}{3}\left(-\frac{1}{3}\cos 3x\right)$$

$$= -\frac{1}{9}\cos 3x$$

Substituting u and v in Eq. (1)

$$\text{PI} = -\frac{1}{9}\cos 3x[(\log(\sec 3x + \tan 3x)] + \frac{1}{9}\sin 3x\cos 3x - \frac{1}{9}\cos 3x \sin 3x$$

$$= -\frac{1}{9}\cos 3x[\log(\sec 3x + \tan 3x)]$$

Hence, the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{9}\cos 3x\ [\log(\sec 3x + \tan 3x)]$$

Example 8

Solve $(D^2 + 4)y = \cot 2x$.

Solution

The auxiliary equation is

$$m^2 + 4 = 0$$

$$m = \pm 2i \qquad \text{(complex)}$$

$$\text{CF} = c_1 \cos 2x + c_2 \sin 2x \qquad ...(1)$$

$$y_1 = \cos 2x, \qquad y_2 = \sin 2x$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Let $\text{PI} = u\cos 2x + v\sin 2x$

where $u = \int -\frac{y_2 Q}{W}\,dx$

$$= \int -\frac{\sin 2x \cot 2x}{2} \, dx$$

$$= -\frac{1}{2}\int \sin 2x \left(\frac{\cos 2x}{\sin 2x}\right) dx$$

$$= -\frac{1}{2}\int \cos 2x \, dx$$

$$= -\frac{1}{2}\frac{\sin 2x}{2}$$

$$= -\frac{1}{4}\sin 2x$$

and

$$v = \int \frac{y_1 Q}{W} \, dx$$

$$= \int \frac{\cos 2x \cot 2x}{2} \, dx$$

$$= \frac{1}{2}\int \frac{\cos^2 2x}{\sin 2x} \, dx$$

$$= \frac{1}{2}\int \frac{1-\sin^2 2x}{\sin 2x} \, dx$$

$$= \frac{1}{2}\int (\operatorname{cosec} 2x - \sin 2x) \, dx$$

$$= \frac{1}{2}\left[\frac{\log(\operatorname{cosec} 2x - \cot 2x)}{2} + \frac{\cos 2x}{2}\right]$$

$$= \frac{1}{4}[\log(\operatorname{cosec} 2x - \cot 2x) + \cos 2x]$$

Substituting u and v in Eq. (1),

$$\text{PI} = -\frac{1}{4}\sin 2x \cos 2x + \frac{1}{4}[\log(\operatorname{cosec} 2x - \cot 2x) + \cos 2x] \sin 2x$$

Hence, the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\sin 2x \cos 2x + \frac{1}{4}[\log(\operatorname{cosec} 2x - \cot 2x) + \cos 2x] \sin 2x$$

Example 9

Solve $y'' - 3y' + 2y = e^x$. **[Winter 2016]**

Solution

$$(D^2 - 3D + 2)y = e^x$$

The auxiliary equation is

$$m^3 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0$$

$$m = 1, 2 \quad \text{(real and distinct)}$$

$$CF = c_1 e^x + c_2 e^{2x}$$
$$y_1 = e^x, \qquad y_2 = e^{2x}$$
$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^x \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}$$

Let
$$PI = ue^x + ve^{2x} \qquad ...(1)$$

where
$$u = \int -\frac{y_2 Q}{W} dx$$
$$= \int -\frac{e^{2x} e^x}{e^{3x}} dx$$
$$= -\int \frac{e^{3x}}{e^{3x}} dx$$
$$= -\int dx$$
$$= -x$$

and
$$v = \int \frac{y_1 Q}{W} dx$$
$$= \int \frac{e^x e^x}{e^{3x}} dx$$
$$= \int \frac{e^{2x}}{e^{3x}} dx$$
$$= \int e^{-x} dx$$
$$= -e^{-x}$$

Substituting u and v in Eq. (1),
$$PI = -xe^x - e^{2x} e^{-x}$$
$$= -xe^x - e^x$$
$$= -(x+1)e^x$$

Hence, the general solution is
$$y = c_1 e^x + c_2 e^{2x} - (x+1)e^x$$

Example 10

Solve $(D^2 - 3D + 2)y = \dfrac{e^x}{1+e^x}$. **[Winter 2012]**

Solution

The auxiliary equation is
$$m^2 - 3m + 2 = 0$$
$$(m-1)(m-2) = 0$$

$$m = 1, 2 \qquad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^x + c_2 e^{2x}$$

$$y_1 = e^x, \qquad y_2 = e^{2x}$$

Wronskian $\quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}$

Let $\quad \text{PI} = ue^x + ve^{2x}$ ...(1)

where $\quad u = \int -\frac{y_2 Q}{W} \mathrm{d}x$

$$= -\int e^{2x} \cdot \frac{e^x}{1+e^x} \cdot \frac{1}{e^{3x}} \mathrm{d}x$$

$$= -\int \frac{1}{1+e^x} \mathrm{d}x$$

$$= -\int \frac{e^{-x}}{1+e^{-x}} \mathrm{d}x \qquad \left[\text{Multiplying and dividing by } e^{-x}\right]$$

$$= \log(1+e^{-x}) \qquad \left[\because \int \frac{f'(x)}{f(x)} \mathrm{d}x = \log f(x)\right]$$

and $\quad v = \int \frac{y_1 Q}{W} \mathrm{d}x$

$$= \int e^x \cdot \frac{e^x}{1+e^x} \cdot \frac{1}{e^{3x}} \mathrm{d}x$$

$$= \int \frac{1}{e^x(1+e^x)} \mathrm{d}x$$

$$= \int \frac{e^{-x}}{1+e^x} \mathrm{d}x$$

$$= \int \frac{e^{-x} \cdot e^{-x}}{e^{-x}+1} \mathrm{d}x$$

$$= \int \frac{e^{-x}(e^{-x}+1-1)}{e^{-x}+1} \mathrm{d}x$$

$$= \int \left(e^{-x} - \frac{e^{-x}}{1+e^{-x}}\right) \mathrm{d}x$$

$$= \int e^{-x} \mathrm{d}x - \int \frac{e-x}{1+e^{-x}} \mathrm{d}x$$

$$= -e^{-x} + \log(1+e^{-x}) \qquad \left[\because \int \frac{f'(x)}{f(x)} \mathrm{d}x = \log f(x)\right]$$

Substituting u and v in Eq. (1),

$$\text{PI} = \log(1+e^{-x})e^{x} + \left[-e^{-x} + \log(1+e^{-x})\right]e^{2x}$$
$$= \log(1+e^{-x})e^{x} - e^{x} + e^{2x}\log(1+e^{-x})$$

Hence, the general solution is

$$y = c_1e^{x} + c_2e^{2x} + e^{x}\log(1+e^{-x}) - e^{x} + e^{2x}\log(1+e^{-x})$$

Example 11

Solve $(D^2+1)y = x\sin x$.

Solution

The auxiliary equation is

$$m^2 + 1 = 0$$
$$m = \pm i \qquad \text{(complex)}$$
$$\text{CF} = c_1\cos x + c_2\sin x$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Let $\text{PI} = u\cos x + v\sin x$ ...(1)

where
$$u = \int -\frac{y_2Q}{W}\,dx$$
$$= -\int \frac{\sin x \cdot x\sin x}{1}\,dx$$
$$= -\int x\sin^2 x\,dx$$
$$= -\int x\frac{(1-\cos 2x)}{2}\,dx$$
$$= -\frac{1}{2}\int x\,dx + \frac{1}{2}\int x\cos 2x\,dx$$
$$= -\frac{x^2}{4} + \frac{1}{2}\left[x\frac{\sin 2x}{2} - 1\frac{(-\cos 2x)}{4}\right]$$
$$= -\frac{x^2}{4} + \frac{1}{8}(2x\sin 2x + \cos 2x)$$

and
$$v = \int \frac{y_1Q}{W}\,dx$$
$$= \int \frac{\cos x \cdot x\sin x}{1}\,dx$$
$$= \int x\frac{\sin 2x}{2}\,dx$$

$$= \frac{1}{2}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]$$

$$= \frac{1}{8}(-2x\cos 2x + \sin 2x)$$

Substituting u and v in Eq. (1),

$$\text{PI} = \left[-\frac{x^2}{4} + \frac{1}{8}(2x\sin 2x + \cos 2x)\right]\cos x + \left[\frac{1}{8}(-2x\cos 2x + \sin 2x)\right]\sin x$$

$$= -\frac{x^2}{4}\cos x + \frac{1}{8}[2x(\sin 2x\cos x - \cos 2x\sin x) + (\cos 2x\cos x + \sin 2x\sin x)]$$

$$= -\frac{x^2}{4}\cos x + \frac{1}{8}[2x\sin(2x - x) + \cos(2x - x)]$$

$$= -\frac{x^2}{4}\cos x + \frac{1}{8}(2x\sin x + \cos x)$$

Hence, the general solution is

$$y = c_1\cos x + c_2\sin x - \frac{x^2}{4}\cos x + \frac{1}{8}(2x\sin x + \cos x)$$

Example 12

Solve $(D^2 + 1) = \operatorname{cosec} x \cot x$.

Solution

The auxiliary equation is

$$m^2 + 1 = 0$$

$$m = \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1\cos x + c_2\sin x$$

$$y_1 = \cos x, \qquad y_2 = \sin x$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Let $\text{PI} = u\cos x + v\sin x$ ...(1)

where

$$u = \int -\frac{y_2 Q}{W}\,dx$$

$$= \int -\frac{\sin x \operatorname{cosec} x \cot x}{1}\,dx$$

$$= -\int \cot x\,dx$$

$$= -\log(\sin x)$$

and
$$v = \int \frac{y_1 Q}{W} \mathrm{d}x$$
$$= \int \frac{\cos x \operatorname{cosec} x \cot x}{1} \mathrm{d}x$$
$$= \int \cot^2 x \,\mathrm{d}x$$
$$= \int \left(\operatorname{cosec}^2 x - 1\right) \mathrm{d}x$$
$$= -\cot x - x$$

Substituting u and v in Eq. (1),
$$\text{PI} = -\log(\sin x)\cos x + (-\cot x - x)\sin x$$
$$= -\cos x \log(\sin x) - (\cot x + x) \sin x$$

Hence, the general solution is
$$y = c_1 \cos x + c_2 \sin x - \cos x \log(\sin x) - (\cot x + x) \sin x$$

Example 13

Solve $(D^2 - 1)y = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$.

Solution

The auxiliary equation is
$$m^2 - 1 = 0$$
$$m = \pm 1 \text{ (real and distinct)}$$
$$\text{CF} = c_1 e^x + c_2 e^{-x}$$
$$y_1 = e^x, \qquad y_2 = e^{-x}$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^0 - e^0 = -2$

Let
$$\text{PI} = ue^x + ve^{-x} \qquad \ldots(1)$$

where
$$u = \int -\frac{y_2 Q}{W} \mathrm{d}x$$
$$= -\int \frac{e^{-x}\left[e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right]}{-2} \mathrm{d}x$$

Let $e^{-x} = t, \quad -e^{-x}\mathrm{d}x = \mathrm{d}t$,
$$u = -\frac{1}{2}\int (t \sin t + \cos t)\mathrm{d}t$$
$$= -\frac{1}{2}\left[t(-\cos t) - (-\sin t) + \sin t\right]$$

$$= \frac{1}{2} t \cos t - \sin t$$

$$= \frac{1}{2} e^{-x} \cos(e^{-x}) - \sin(e^{-x})$$

and
$$v = \int \frac{y_1 Q}{W}\, dx$$

$$= \int \frac{e^x \left[e^{-x} \sin(e^{-x}) + \cos(e^{-x}) \right]}{-2} dx$$

$$= \int \frac{e^x \left[\cos(e^{-x}) + e^{-x} \sin(e^{-x}) \right]}{-2} dx$$

$$= -\frac{1}{2} e^x \cos(e^{-x}) \quad \left[\because \int e^x \{ f(x) + f'(x) \} dx = e^x f(x) \right. \quad \left. \text{Here } f(x) = \cos e^{-x} \right]$$

Substituting u and v in Eq. (1),

$$\text{PI} = \frac{1}{2}\cos(e^{-x}) - e^x \sin(e^{-x}) - \frac{1}{2}\cos(e^{-x})$$

$$= -e^x \sin(e^{-x})$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{-x} - e^x \sin(e^{-x})$$

Example 14

Solve $(D^2 + 3D + 2)y = e^{e^x}$.

Solution

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$(m + 1)(m + 2) = 0$$

$$m = -1, -2 \text{ (real and distinct)}$$

$$\text{CF} = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_1 = e^{-x}, \qquad y_2 = e^{-2x}$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-2x}e^{-x} + e^{-2x}e^{-x} = -e^{-3x}$

Let
$$\text{PI} = ue^{-x} + ve^{-2x} \qquad \ldots(1)$$

where
$$u = \int -\frac{y_2 Q}{W}\, dx$$

$$= -\int \frac{e^{-2x} e^{e^x}}{-e^{-3x}} dx$$

$$= \int e^{e^x} e^x dx$$

$$= e^{e^x} \qquad \left[\because \int e^{f(x)} f'(x) dx = e^{f(x)} \quad \text{Here, } f(x) = e^x\right]$$

and

$$v = \int \frac{y_1 Q}{W} dx$$

$$= \int \frac{e^{-x} e^{e^x}}{-e^{-3x}} dx$$

$$= -\int e^{2x} e^{e^x} dx$$

$$= -\int e^x e^{e^x} \cdot e^x dx$$

Let $e^x = t$, $\quad e^x dx = dt$

$$v = -\int t e^t dt$$

$$= -(te^t - e^t)$$

$$= -e^x e^{e^x} + e^{e^x}$$

Substituting u and v in Eq. (1),

$$\text{PI} = e^{e^x} e^{-x} + \left(-e^x e^{e^x} + e^{e^x}\right) e^{-2x}$$

$$= e^{-2x} e^{e^x}$$

Hence, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$$

Example 15

Solve $(D^2 + 5D + 6)y = e^{-2x} \sec^2 x(1 + 2\tan x)$.

Solution

The auxiliary equation is

$$m^2 + 5m + 6 = 0$$

$$(m+2)(m+3) = 0$$

$$m = -2, -3 \text{ (real and distinct)}$$

$$\text{CF} = c_1 e^{-2x} + c_2 e^{-3x}$$

$$y_1 = e^{-2x}, \qquad y_2 = e^{-3x}$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix} = -3e^{-5x} + 2e^{-5x} = -e^{-5x}$

Let $\quad \text{PI} = ue^{-2x} + ve^{-3x} \qquad \ldots(1)$

where

$$u = \int -\frac{y_2 Q}{W}\,dx$$

$$= -\int \frac{e^{-3x}e^{-2x}\sec^2 x(1+2\tan x)}{-e^{-5x}}\,dx$$

$$= \int (1+2\tan x)\frac{2\sec^2 x}{2}\,dx$$

[Multiplying and dividing by 2]

$$= \frac{1}{2}\cdot\frac{(1+2\tan x)^2}{2} \qquad \left[\because \int f(x)\cdot f'(x)\,dx = \frac{\{f(x)\}^2}{2} \quad \text{Here, } f(x) = (1+2\tan x)\right]$$

and

$$v = \int \frac{y_1 Q}{W}\,dx$$

$$= \int \frac{e^{-2x}e^{-2x}\sec^2 x(1+2\tan x)}{-e^{-5x}}\,dx$$

$$= -\int e^x \sec^2 x(1+2\tan x)\,dx$$

$$= -\int e^x(\sec^2 x + 2\sec^2 x\tan x)\,dx$$

$$= -e^x\sec^2 x \qquad \left[\because \int e^x\{f(x)+f'(x)\}dx = e^x f(x) \quad \text{Here } f(x) = \sec^2 x\right]$$

Substituting u and v in Eq. (1),

$$\text{PI} = \frac{1}{4}(1+2\tan x)^2 e^{-2x} + \left(-e^x\sec^2 x\right)e^{-3x}$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{4}(1+2\tan x)^2 e^{-2x} - e^{-2x}\sec^2 x$$

Example 16

Solve $(D^2-2D+2)y = e^x \tan x.$

Solution

The auxiliary equation is

$$m^2 - 2m + 2 = 0$$

$$m = 1 \pm i \text{ (complex)}$$

$$\therefore \quad CF = e^x(c_1 \cos x + c_2 \sin x)$$

$$y_1 = e^x \cos x, \qquad y_2 = e^x \sin x$$

Wronskian
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix}$$

$$= e^x \cos x\left(e^x \sin x + e^x \cos x\right) - e^x \sin x\left(e^x \cos x - e^x \sin x\right)$$

$$= e^{2x} \cos x \sin x + e^{2x} \cos^2 x - e^{2x} \cos x \sin x + e^{2x} \sin^2 x$$

$$= e^{2x}\left(\cos^2 x + \sin^2 x\right)$$

$$= e^{2x}$$

Let
$$PI = ue^x \cos x + ve^x \sin x \qquad \ldots(1)$$

where
$$u = \int -\frac{y_2 Q}{W}\,dx$$

$$= \int -\frac{e^x \sin x \cdot e^x \tan x}{e^{2x}}\,dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\,dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= -\int \sec x\,dx + \int \cos x\,dx$$

$$= -\log\left(\sec x + \tan x\right) + \sin x$$

and
$$v = \int \frac{y_1 Q}{W}\,dx$$

$$= \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}} dx$$

$$= \int \sin x \, dx$$

$$= -\cos x$$

Substituting u and v in Eq. (1),

$$\text{PI} = \left[-\log(\sec x + \tan x) + \sin x \right] \cdot e^x \cos x + (-\cos x) \cdot e^x \sin x$$

$$= -e^x \cos x \cdot \log(\sec x + \tan x)$$

Hence, the general solution is

$$y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \cdot \log(\sec x + \tan x)$$

Example 17

Solve $(D^2 + 1)y = \dfrac{1}{1 + \sin x}$.

Solution

The auxiliary equation is

$$m^2 + 1 = 0$$

$$m = \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos x + c_2 \sin x$$

$$y_1 = \cos x, \qquad y_2 = \sin x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

Let $\quad \text{PI} = u \cos x + v \sin x \quad$ ...(1)

where

$$u = \int -\frac{y_2 Q}{W} dx$$

$$= \int -\frac{\sin x}{1} \cdot \frac{1}{1 + \sin x} dx$$

$$= -\int \frac{\sin x}{1 + \sin x} \cdot \frac{(1 - \sin x)}{(1 - \sin x)} dx$$

$$= -\int \frac{\sin x - \sin^2 x}{1 - \sin^2 x} dx$$

$$= -\int \frac{\sin x - \sin^2 x}{\cos^2 x} dx$$

$$= -\int\left(\tan x \sec x - \tan^2 x\right)dx$$

$$= -\int\left(\tan x \sec x - \sec^2 x + 1\right)dx$$

$$= -\left(\sec x - \tan x + x\right)$$

and

$$v = \int \frac{y_1 Q}{W}\, dx$$

$$= \int \frac{\cos x}{1} \cdot \frac{1}{1+\sin x}\, dx$$

$$= \int \frac{\cos x}{1+\sin x}\, dx$$

$$= \log\left(1+\sin x\right) \qquad \left[\because \int \frac{f'(x)}{f(x)}dx = \log\{f(x)\} \text{ Here } f(x) = 1+\sin x\right]$$

Substituting u and v in Eq. (1),

$$\text{PI} = -\left(\sec x - \tan x + x\right)\cos x + \left[\log(1+\sin x)\right]\sin x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x - \left(\sec x - \tan x + x\right)\cos x + \left[\log(1+\sin x)\right]\sin x$$

Example 18

Solve $y'' - 4y' + 4y = \dfrac{e^{2x}}{x}$. **[Summer 2017]**

Solution

$$(D^2 - 4D + 4)y = \frac{e^{2x}}{x}$$

The auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2 \text{ (real and repeated)}$$

$$\text{CF} = (c_1 + c_2 x)e^{2x}$$

$$y_1 = e^{2x}, \quad y_2 = xe^{2x}$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = 2xe^{4x} + e^{4x} - 2xe^{4x} = e^{4x}$$

Let $$\text{PI} = ue^{2x} + vxe^{2x} \qquad ...(1)$$

where
$$u = \int -\frac{y_2 Q}{W}\,dx$$
$$= \int -\frac{xe^{2x}}{e^{4x}} \cdot \frac{e^{2x}}{x}\,dx$$
$$= \int -\,dx$$
$$= -x$$

and
$$v = \int \frac{y_1 Q}{W}\,dx$$
$$= \int \frac{e^{2x}}{e^{4x}} \cdot \frac{e^{2x}}{x}\,dx$$
$$= \int \frac{1}{x}\,dx$$
$$= \log x$$

Substituting u and v in Eq. (1),
$$\text{PI} = -xe^{2x} + x(\log x)e^{2x}$$
$$= xe^{2x}(\log x - 1)$$

Hence, the general solution is
$$y = (c_1 + c_2 x)\,e^{2x} + xe^{2x}(\log x - 1)$$

Example 19

Solve $(\mathrm{D}^3 + \mathrm{D})y = \operatorname{cosec} x.$

Solution

The auxiliary equation is
$$m^3 + m = 0$$
$$m(m^2 + 1) = 0$$
$$m = 0 \text{ (real)}, \quad m = \pm i \text{ (complex)}$$
$$\text{CF} = c_1 + c_2 \cos x + c_3 \sin x$$
$$y_1 = 1, \quad y_2 = \cos x, \quad y_3 = \sin x$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$= 1\left(\sin^2 x + \cos^2 x\right) - \cos x \cdot (0-0) + \sin x \cdot (0-0) = 1$$

Let

$$\text{PI} = u \cdot 1 + v\cos x + w\sin x \qquad \ldots(1)$$

where

$$u = \int \frac{(y_2 y_3' - y_3 y_2')Q}{W}\,dx$$

$$= \int \frac{\left[\cos x \cos x - \sin x(-\sin x)\right]\operatorname{cosec} x}{1}\,dx$$

$$= \int \left(\cos^2 x + \sin^2 x\right)\operatorname{cosec} x\,dx$$

$$= \int \operatorname{cosec} x\,dx$$

$$= \log\left(\operatorname{cosec} x - \cot x\right)$$

$$v = \int \frac{(y_3 y_1' - y_1 y_3')Q}{W}\,dx$$

$$= \int \frac{\left[\sin x \cdot 0 - 1 \cdot \cos x\right]\operatorname{cosec} x}{1}\,dx$$

$$= \int (-\cos x)\operatorname{cosec} x\,dx$$

$$= -\int \cot x\,dx$$

$$= -\log \sin x$$

$$w = \int \frac{(y_1 y_2' - y_2 y_1')Q}{W}\,dx$$

$$= \int \frac{\left[1 \cdot (-\sin x) - \cos x \cdot 0\right]\operatorname{cosec} x}{1}\,dx$$

$$= \int -dx$$

$$= -x$$

Substituting u, v and w in Eq. (1),

$$\text{PI} = \log(\operatorname{cosec} x - \cot x) \cdot 1 + (-\log \sin x)\cos x + (-x)\sin x$$

$$= \log(\operatorname{cosec} x - \cot x) - \cos x \log \sin x - x \sin x$$

Hence, the general solution is

$$y = c_1 + c_2 \cos x + c_3 \sin x + \log(\operatorname{cosec} x - \cot x) - \cos x \log \sin x - x \sin x$$

Example 20

Solve $(D^3 - 6D^2 + 12D - 8)y = \dfrac{e^{2x}}{x}$.

Solution

The auxiliary equation is

$$m^3 - 6m^2 + 12m - 8 = 0$$

$$(m-2)^3 = 0$$

$$m = 2,\ 2,\ 2 \text{ (real and repeated)}$$

$$\text{CF} = (c_1 + c_2x + c_3x^2)e^{2x} = c_1e^{2x} + c_2xe^{2x} + c_3x^2e^{2x}$$

$$y_1 = e^{2x},\ y_2 = xe^{2x},\ y_3 = x^2e^{2x}$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$= \begin{vmatrix} e^{2x} & xe^{2x} & x^2e^{2x} \\ 2e^{2x} & (2x+1)e^{2x} & (2x^2+2x)e^{2x} \\ 4e^{2x} & 4(x+1)e^{2x} & (4x^2+8x+2)e^{2x} \end{vmatrix}$$

$$= e^{2x}\left[(2x+1)e^{2x}\cdot(4x^2+8x+2)e^{2x} - (2x^2+2x)e^{2x}\cdot 4(x+1)e^{2x}\right]$$
$$-xe^{2x}\left[2e^{2x}\cdot(4x^2+8x+2)e^{2x} - 4e^{2x}\cdot(2x^2+2x)e^{2x}\right]$$
$$+x^2e^{2x}\left[2e^{2x}\cdot 4(x+1)e^{2x} - 4e^{2x}\cdot(2x+1)e^{2x}\right]$$

$$= 2e^{6x}$$

Let $\quad \text{PI} = ue^{2x} + vxe^{2x} + wx^2e^{2x} \qquad \text{...(1)}$

where $\quad u = \int \frac{(y_2y_3' - y_3y_2')Q}{W}\,\mathrm{d}x$

$$= \int \frac{\left[xe^{2x}(2x^2+2x)e^{2x} - x^2e^{2x}(2x+1)e^{2x}\right]}{2e^{6x}}\cdot\frac{e^{2x}}{x}\,\mathrm{d}x$$

$$= \int \frac{x}{2}\,\mathrm{d}x$$

$$= \frac{x^2}{4}$$

$$v = \int \frac{(y_3y_1' - y_1y_3')Q}{W}\,\mathrm{d}x$$

$$= \int \frac{\left[x^2e^{2x}\cdot 2e^{2x} - e^{2x}\cdot(2x^2+2x)e^{2x}\right]}{2e^{6x}}\cdot\frac{e^{2x}}{x}\,\mathrm{d}x$$

$$= \int -\mathrm{d}x$$

$$= -x$$

$$w = \int \frac{(y_1 y_2' - y_2 y_1')Q}{W} \mathrm{d}x$$

$$= \int \frac{\left[e^{2x} \cdot (2x+1)e^{2x} - xe^{2x} \cdot 2e^{2x}\right]}{2e^{6x}} \cdot \frac{e^{2x}}{x} \mathrm{d}x$$

$$= \int \frac{1}{2x} \mathrm{d}x$$

$$= \frac{1}{2} \log x$$

Substituting u, v and w in Eq. (1),

$$\mathrm{PI} = \left(\frac{x^2}{4}\right)e^{2x} - (x)xe^{2x} + \left(\frac{1}{2}\log x\right)x^2 e^{2x}$$

$$= -\frac{3x^2}{4}e^{2x} + \frac{x^2}{2}e^{2x}\log x$$

Hence, the general solution is

$$y = (c_1 + c_2 x + c_3 x^2)e^{2x} - \frac{3x^2}{4}e^{2x} + \frac{x^2}{2}e^{2x}\log x$$

EXERCISE 5.5

Solve the following differential equations:

1. $(D^2 + 3D + 2)y = \sin e^x$

$$\left[\text{Ans.: } y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x} \sin e^x\right]$$

2. $(D^2 + 1)y = \operatorname{cosec} x$

$$\left[\text{Ans.: } y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \log(\sin x)\right]$$

3. $(D^2 + 4)y = \tan 2x$

$$\left[\text{Ans.: } y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)\right]$$

4. $(D^2 + 1)y = x - \cot x$

$$\left[\text{Ans.: } y = c_1 \cos x + c_2 \sin x - x\cos^2 x + x\sin^2 x - \sin x \log(\operatorname{cosec} x - \cot x)\right]$$

5. $(D^2 + D)y = \dfrac{1}{1+e^x}$

$$\left[\text{Ans.: } y = c_1 + c_2 e^{-x} - e^{-x}[e^x \log(e^{-x}+1) + \log(e^x+1)]\right]$$

6. $(D^2 - 2D + 2)y = e^x \tan x$

$$\left[\text{Ans.: } y = e^x(c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)\right]$$

7. $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$

$$\left[\text{Ans.: } y = (c_1 + c_2 x)e^{2x} - e^{2x}(2x^2 \sin 2x + 4x \cos 2x - 3\sin 2x)\right]$$

8. $(D^2 + 2D + 1)y = e^{-x} \log x$

$$\left[\text{Ans.: } y = (c_1 + c_2 x)e^{-x} + \frac{x^2}{2} e^{-x}\left(\log x - \frac{3}{2}\right)\right]$$

5.9 METHOD OF UNDETERMINED COEFFICIENTS

This method can be used to find the particular integral only if linearly independent derivatives of $Q(x)$ are finite in number. This restriction implies that $Q(x)$ can only have the terms such as k, x^n, e^{ax}, $\sin ax$, $\cos ax$, and combinations of such terms where k, a are constants and n is a positive integer. However, when $Q(x) = \frac{1}{x}$ or $\tan x$ or $\sec x$, etc., this method fails, since each function has an infinite number of linearly independent derivatives.

In this method, a particular integral is assumed as a linear combination of the terms in $Q(x)$ and all its linearly independent derivatives. Some of the choices of the particular integral are given below.

Sr. No.	Q(x)	Particular Integral
1.	ke^{ax}	Ae^{ax}
2.	$k \sin(ax+b)$ or $k \cos(ax+b)$	$A \sin(ax+b) + B \cos(ax+b)$
3.	$ke^{ax} \sin(bx+c)$ or $ke^{ax} \cos(bx+c)$	$A e^{ax} \sin(bx+c) + Be^{ax} \cos(bx+c)$
4.	kx^n $n = 0, 1, 2, \ldots$	$A_n x^n + A_{n-1} x^{n-1} + \ldots + A_2 x^2 + A_1 x + A_0$
5.	$kx^n e^{ax}$ $n = 0, 1, 2, \ldots.$	$e^{ax}(A_n x^n + A_{n-1} x^{n-1} + \ldots + A_2 x^2 + A_1 x + A_0)$

Sr. No.	Q(x)	Particular Integral
6.	$kx^n \sin(ax+b)$ or $kx^n \cos(ax+b)$	$x^n[A_n \sin(ax+b) + B_n \cos(ax+b)] + x^{n-1}[A_{n-1}\sin(ax+b) + B_{n-1}\cos(ax+b)] + \ldots + x[A_1 \sin(ax+b) + B_1\cos(ax+b)] + [A_0 \sin(ax+b) + B_0 \cos(ax+b)]$
7.	$kx^n e^{ax}\sin(bx+c)$ or $kx^n e^{ax}\cos(bx+c)$	$e^{ax}[x^n\{A_n \sin(ax+b) + B_n \cos(ax+b)\} + x^{n-1}\{A_{n-1}\sin(ax+b) + B_{n-1}\cos(ax+b)\} + \ldots + x\{A_1 \sin(ax+b) + B_1 \cos(ax+b) + \{A_0 \sin(ax+b) + B_0\cos(ax+b)\}]$

In the table, $A_0, A_1, A_2, \ldots, A_n$ are coefficients to be determined. To obtain the values of these coefficients, we use the fact that the particular integral satisfies the given differential equation.

However, before assuming the particular integral, it is necessary to compare the terms of Q(x) with the complementary function. While comparing the terms following different cases arise.

Case I If no terms of Q(x) occur in the complementary function then particular integral is assumed from the table depending on the nature of Q(x).

Case II If a term u of Q(x) is also a term of the complementary function corresponding to an r-fold root then the assumed particular integral corresponding to u should be multiplied by x^r.

Case III If $x^s u$ is a term of Q(x) and only u is a term of the complementary function corresponding to an r-fold root then the assumed particular integral corresponding to $x^s u$ should be multiplied by x^r.

Note: In cases (ii) and (iii), initially similar types of terms appear in the complementary function and in the assumed particular integral. After multiplication by x^r, the terms of the particular integral change. Hence, this method avoids the repetition of similar terms in the complementary function and particular integral.

Example 1

Solve $y'' + 4y = 8x^2$. **[Summer 2016]**

Solution

$$(D^2 + 4)y = 8x^2$$

The auxiliary equation is

$$m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \pm 2i \quad \text{(complex)}$$

$$\text{CF} = c_1 \sin 2x + c_2 \cos 2x$$

$$Q = 8x^2$$

Let the particular integral be

$$y = A_1x^2 + A_2x + A_3$$

$$Dy = 2A_1x + A_2$$

$$D^2y = 2A_1$$

Substituting these derivatives in the given equation,

$$2A_1 + 4(A_1x^2 + A_2x + A_3) = 8x^2$$

$$4A_1x^2 + 4A_2x + (2A_1 + 4A_3) = 8x^2$$

Comparing coefficients on both the sides,

$$4A_1 = 8, \qquad A_1 = 2$$

$$4A_2 = 0, \qquad A_2 = 0$$

$$2A_1 + 4A_3 = 0, \qquad A_3 = -1$$

$$PI = 2x^2 - 1$$

Hence, the general solution is

$$y = c_1 \sin 2x + c_2 \cos 2x + 2x^2 - 1$$

Example 2

Solve $y'' + 9y = 2x^2$. **[Summer 2017]**

Solution

$$(D^2 + 9)y = 2x^2$$

The auxiliary equation is

$$m^2 + 9 = 0$$

$$m = \pm 3i \text{ (complex)}$$

$$CF = c_1 \cos 3x + c_2 \sin 3x$$

$$Q = 2x^2$$

Let the particular integral be

$$y = A_1x^2 + A_2x + A_3$$

$$Dy = 2A_1x + A_2$$

$$D^2y = 2A_1$$

Substituting these derivatives in the given equation,

$$2A_1 + 9(A_1x^2 + A_2x + A_3) = 2x^2$$

$$9A_1x^2 + 9A_2x + (2A_1 + 9A_3) = 2x^2$$

Comparing the coefficient on both the sides,

$$9A_1 = 2, \qquad A_1 = \frac{2}{9}$$

$$9A_2 = 0, \qquad A_2 = 0$$

$$2A_1 + 9A_3 = 0$$

$$2\cdot\frac{2}{9} + 9A_3 = 0, \qquad A_3 = -\frac{4}{81}$$

$$\text{PI} = \frac{2}{9}x^2 - \frac{4}{81}$$

Hence, the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x + \frac{2}{9}x^2 - \frac{4}{81}$$

Example 3

Solve $(D^2 - 2D + 5)y = 25x^2 + 12.$

Solution

The auxiliary equation is

$$m^2 - 2m + 5 = 0$$

$$m = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i \quad \text{(complex)}$$

$$\text{CF} = e^x (c_1 \cos 2x + c_2 \sin 2x)$$

$$Q = 25x^2 + 12$$

Let the particular integral be

$$y = A_1x^2 + A_2x + A_3$$

$$Dy = 2A_1x + A_2$$

$$D^2y = 2A_1$$

Substituting these derivatives in the given equation,

$$2A_1 - 2(2A_1x + A_2) + 5(A_1x^2 + A_2x + A_3) = 25x^2 + 12$$

$$5A_1x^2 + (-4A_1 + 5A_2)x + (2A_1 - 2A_2 + 5A_3) = 25x^2 + 12$$

Comparing coefficients on both the sides,

$$5A_1 = 25, \qquad A_1 = 5$$

$$-4A_1 + 5A_2 = 0, \qquad A_2 = \frac{4}{5}A_1 = 4$$

$$2A_1 - 2A_2 + 5A_3 = 12, \qquad A_3 = \frac{1}{5}(12 - 10 + 8) = 2$$

$$\text{PI} = 5x^2 + 4x + 2$$

Hence, the general solution is

$$y = e^x (c_1 \cos 2x + c_2 \sin 2x) + 5x^2 + 4x + 2$$

Example 4

[Winter 2017]

Solve $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+4y=2x^2+3e^{-x}$.

Solution

$$(D^2+2D+4)y=2x^2+3e^{-x}$$

The auxiliary equation is

$$m^2+2m+4=0$$

$$m=-1\pm i\sqrt{3} \qquad \text{(complex)}$$

$$CF=e^{-x}\left(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x\right)$$

$$Q=2x^2+3e^{-x}$$

Let the particular integral be

$$y=A_1x^2+A_2x+A_3+A_4e^{-x}$$

$$\frac{dy}{dx}=2A_1x+A_2-A_4e^{-x}$$

$$\frac{d^2y}{dx^2}=2A_1+A_4e^{-x}$$

Substituting these derivatives in the given equation,

$$2A_1+A_4e^{-x}+2(2A_1x+A_2-A_4e^{-x})+4(A_1x^2+A_2x+A_3+A_4e^{-x})=2x^2+3e^{-x}$$

$$(3A_4)e^{-x}+(4A_1)x^2+(4A_1+4A_2)x+(2A_1+2A_2+4A_3)=2x^2+3e^{-x}$$

Comparing coefficients on both the sides,

$$3A_4=3, \qquad A_4=1$$

$$4A_1=2, \qquad A_1=\frac{1}{2}$$

$$4A_1+4A_2=0, \qquad A_2=-A_1=-\frac{1}{2}$$

$$2A_1+2A_2+4A_3=0, \qquad A_3=\frac{1}{2}(A_1+A_2)=0$$

$$PI=\frac{1}{2}x^2-\frac{1}{2}x+e^{-x}$$

Hence, the general solution is

$$y=e^{-x}(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x)+\frac{1}{2}x^2-\frac{1}{2}x+e^{-x}$$

Example 5

Solve $y'' - 2y' + 5y = 5x^3 - 6x^2 + 6x.$ **[Summer 2018]**

Solution

$$(D^2 - 2D + 5)y = 5x^3 - 6x^2 + 6x$$

The auxiliary equation is

$$m^2 - 2m + 5 = 0$$

$$m = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i \quad \text{(complex)}$$

$$\text{CF} = e^x \left(c_1 \cos 2x + c_2 \sin 2x\right)$$

$$Q = 5x^3 - 6x^2 + 6x$$

Let the particular integral be

$$y = A_1x^3 + A_2x^2 + A_3x + A_4$$

$$y' = 3A_1x^2 + 2A_2x + A_3$$

$$y'' = 6A_1x + 2A_2$$

Substituting these derivatives in the given equation,

$$(6A_1x + 2A_2) - 2(3A_1x^2 + 2A_2x + A_3) + 5(A_1x^3 + A_2x^2 + A_3x + A_4) = 5x^3 - 6x^2 + 6x$$

$$(5A_1)x^3 + (-6A_1 + 5A_2)x^2 + (6A_1 - 4A_2 + 5A_3)x + (2A_2 - 2A_3 + 5A_4)$$
$$= 5x^3 - 6x^2 + 6x$$

Comparing the coefficients on both the sides,

$$5A_1 = 5, \qquad A_1 = 1$$

$$-6A_1 + 5A_2 = -6, \qquad A_2 = \frac{1}{5}(-6 + 6A_1) = 0$$

$$6A_1 - 4A_2 + 5A_3 = 6, \qquad A_3 = \frac{1}{5}(6 - 6A_1 + 4A_2) = 0$$

$$2A_2 - 2A_3 + 5A_4 = 0, \qquad A_4 = \frac{1}{5}(-2A_2 - 2A_3) = 0$$

$$\text{PI} = x^3$$

Hence, the general solution is

$$y = e^x \left(c_1 \cos 2x + c_2 \sin 2x\right) + x^3$$

Example 6

Solve $(D^2 - 2D + 3)y = x^3 + \sin x.$

Solution

The auxiliary equation is

$$m^2 - 2m + 3 = 0$$

$$m = 1 \pm i\sqrt{2} \qquad \text{(complex)}$$

$$\text{CF} = e^x\left(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right)$$

$$Q = x^3 + \sin x$$

Let the particular integral be

$$y = A_1x^3 + A_2x^2 + A_3x + A_4 + A_5 \sin x + A_6 \cos x$$

$$Dy = 3A_1x^2 + 2A_2x + A_3 + A_5 \cos x - A_6 \sin x$$

$$D^2y = 6A_1x + 2A_2 - A_5 \sin x - A_6 \cos x$$

Substituting these derivatives in the given equation,

$$(6A_1x + 2A_2 - A_5 \sin x - A_6 \cos x) - 2(3A_1x^2 + 2A_2x + A_3 + A_5 \cos x - A_6 \sin x) + 3(A_1x^3 + A_2x^2 + A_3x + A_4 + A_5 \sin x + A_6 \cos x) = x^3 + \sin x$$

$$3A_1x^3 + (-6A_1 + 3A_2)x^2 + (6A_1 - 4A_2 + 3A_3)x + (2A_2 - 2A_3 + 3A_4) - 2(A_5 - A_6)\cos x + 2(A_5 + A_6)\sin x = x^3 + \sin x$$

Comparing coefficients on both the sides,

$$3A_1 = 1, \qquad A_1 = \frac{1}{3}$$

$$-6A_1 + 3A_2 = 0, \qquad A_2 = 2A_1 = \frac{2}{3}$$

$$6A_1 - 4A_2 + 3A_3 = 0, \qquad A_3 = \frac{1}{3}(4A_2 - 6A_1) = \frac{2}{9}$$

$$2A_2 - 2A_3 + 3A_4 = 0, \qquad A_4 = \frac{2}{3}(A_3 - A_2) = -\frac{8}{27}$$

$$2(A_5 - A_6) = 0, \qquad A_5 = A_6$$

$$2(A_5 + A_6) = 1, \qquad 2(A_5 + A_5) = 1, \qquad A_5 = \frac{1}{4}, A_6 = \frac{1}{4}$$

$$\text{PI} = \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4}(\sin x + \cos x)$$

Hence, the general solution is

$$y = e^x(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4}(\sin x + \cos x)$$

Example 7

Solve $(D^2-9)y = x + e^{2x} - \sin 2x$.

Solution

The auxiliary equation is

$$m^2 - 9 = 0$$

$$m = \pm 3 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{3x} + c_2 e^{-3x}$$

$$Q = x + e^{2x} - \sin 2x$$

Let the particular integral be

$$y = A_1 x + A_2 + A_3 e^{2x} + A_4 \sin 2x + A_5 \cos 2x$$

$$Dy = A_1 + 2A_3 e^{2x} + 2A_4 \cos 2x - 2A_5 \sin 2x$$

$$D^2 y = 4A_3 e^{2x} - 4A_4 \sin 2x - 4A_5 \cos 2x$$

Substituting these derivatives in the given equation,

$$4A_3 e^{2x} - 4A_4 \sin 2x - 4A_5 \cos 2x - 9(A_1 x + A_2 + A_3 e^{2x} + A_4 \sin 2x + A_5 \cos 2x)$$
$$= x + e^{2x} - \sin 2x$$

$$(-5A_3)e^{2x} - 9A_1 x - 9A_2 + \sin 2x(-13A_4) + \cos 2x(-13A_5) = x + e^{2x} - \sin 2x$$

Comparing coefficients on both the sides,

$$-5A_3 = 1, \qquad A_3 = -\frac{1}{5}$$

$$-9A_1 = 1, \qquad A_1 = -\frac{1}{9}$$

$$-9A_2 = 0, \qquad A_2 = 0$$

$$-13A_4 = -1, \qquad A_4 = \frac{1}{13}$$

$$-13A_5 = 0, \qquad A_5 = 0$$

$$\text{PI} = -\frac{1}{9}x - \frac{1}{5}e^{2x} + \frac{1}{13}\sin 2x$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-3x} - \frac{x}{9} - \frac{e^{2x}}{5} + \frac{\sin 2x}{13}$$

Example 8

Solve $(D^2 - 2D)y = e^x \sin x$.

Solution

The auxiliary equation is

$$m^2 - 2m = 0$$

$$m = 0,\ -2 \qquad \text{(real and distinct)}$$

$$CF = c_1 + c_2 e^{2x}$$

$$Q = e^x \sin x$$

Let the particular integral be

$$y = A_1 e^x \sin x + A_2 e^x \cos x$$

$$Dy = A_1(e^x \sin x + e^x \cos x) + A_2(e^x \cos x - e^x \sin x)$$

$$= (A_1 - A_2)e^x \sin x + (A_1 + A_2)e^x \cos x$$

$$D^2 y = (A_1 - A_2)(e^x \sin x + e^x \cos x) + (A_1 + A_2)(e^x \cos x - e^x \sin x)$$

$$= -2A_2\, e^x \sin x + 2A_1 e^x \cos x$$

Substituting these derivatives in the given equation,

$$-2A_2 e^x \sin x + 2A_1 e^x \cos x - 2(A_1 - A_2)e^x \sin x - 2(A_1 + A_2)e^x \cos x = e^x \sin x$$

$$-2A_1 e^x \sin x - 2A_2 e^x \cos x = e^x \sin x$$

Comparing coefficients on both the sides,

$$-2A_1 = 1, \qquad A_1 = -\frac{1}{2}$$

$$2A_2 = 0, \qquad A_2 = 0$$

$$PI = -\frac{1}{2} e^x \sin x$$

Hence, the general solution is

$$y = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x$$

Example 9

Solve $(D^3 + 3D^2 + 2D)y = x^2 + 4x + 8$.

Solution

The auxiliary equation is

$$m^3 + 3m^2 + 2m = 0$$

$$m(m+1)(m+2)=0$$

$$m=0,-1,-2 \quad \text{(real and distinct)}$$

$$\text{CF}=c_1+c_2e^{-x}+c_3e^{-2x}$$

$$Q=x^2+4x+8$$

Let the particular integral be

$$y=A_1x^2+A_2x+A_3$$

Since the constant occurs in $Q(x)$ and is also a part of CF corresponding to the 1-fold root $m=0$, multiplying the assumed particular integral by x, we get

$$y=A_1x^3+A_2x^2+A_3x$$

$$Dy=3A_1x^2+2A_2x+A_3$$

$$D^2y=6A_1x+2A_2$$

$$D^3y=6A_1$$

Substituting these derivatives in the given equation,

$$6A_1+3(6A_1x+2A_2)+2(3A_1x^2+2A_2x+A_3)=x^2+4x+8$$

$$6A_1x^2+(18A_1+4A_2)x+(6A_1+6A_2+2A_3)=x^2+4x+8$$

Comparing coefficients on both the sides,

$$6A_1=1, \qquad A_1=\frac{1}{6}$$

$$18A_1+4A_2=4, \qquad A_2=\frac{1}{4}(4-3)=\frac{1}{4}$$

$$6A_1+6A_2+2A_3=8, \qquad A_3=\frac{1}{2}(8-6A_1-6A_2)=\frac{1}{2}\left(8-1-\frac{3}{2}\right)=\frac{11}{4}$$

$$\text{PI}=\frac{1}{6}x^3+\frac{1}{4}x^2+\frac{11}{4}x$$

Hence, the general solution is

$$y=c_1+c_2e^{-x}+c_3e^{-2x}+\frac{x^3}{6}+\frac{x^2}{4}+\frac{11x}{4}$$

EXERCISE 5.6

Solve the following differential equations using the method of undetermined coefficients:

1. $(D^2+6D+8)y=e^{-3x}+e^x$

$$\left[\textbf{Ans.}: y=c_1e^{-2x}+c_2e^{-4x}-e^{-3x}+\frac{e^x}{15}\right]$$

2. $(4D^2-1)y = e^x + e^{3x}$

$$\left[\text{Ans.: } y = c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} + \frac{1}{105}(35e^x + 3e^{3x})\right]$$

3. $(D^2+D-6)y = 39\cos 3x$

$$\left[\text{Ans.: } y = c_1 e^{2x} + c_2 e^{-3x} + \frac{1}{2}(\sin 3x - 5\cos 3x)\right]$$

4. $(D^2+2D+5)y = 6\sin 2x + 7\cos 2x$

$$\left[\text{Ans.: } y = e^{-x}(c_1\cos 2x + c_2\sin 2x) + 2\sin 2x - \cos 2x\right]$$

5. $(D^2+4D-5)y = 34\cos 2x - 2\sin 2x$

$$\left[\text{Ans.: } y = c_1 e^{x} + c_2 e^{-5x} + 2(\sin 2x - \cos 2x)\right]$$

6. $(D^3-D^2+D-1)y = 6\cos 2x$

$$\left[\text{Ans.: } y = c_1 e^{x} + c_2\cos x + c_3\sin x + \frac{2}{5}(\cos 2x - 2\sin 2x)\right]$$

7. $(2D^2-D-3)y = x^3+x+1$

$$\left[\text{Ans.: } y = c_1 e^{-x} + c_2 e^{\frac{3x}{2}} - \frac{1}{27}(9x^3 - 9x^2 + 51x - 20)\right]$$

8. $(D^2+4)y = 8x^2$

$$\left[\text{Ans.: } y = c_1\cos 2x + c_2\sin 2x + 2x^2 - 1\right]$$

9. $(3D^2+2D-1)y = e^{-2x} + x$

$$\left[\text{Ans.: } y = c_1 e^{-x} + c_2 e^{\frac{x}{3}} + \frac{1}{7}(e^{-2x} - 7x - 14)\right]$$

10. $(D^2-2D+3)y = x^2 + \sin x$

$$\left[\text{Ans.: } y = e^{x}\left(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x\right) + \frac{1}{27}(9x^2+6x-8) + \frac{1}{4}(\sin x + \cos x)\right]$$

11. $(D^4-1)y = x^4+1$

$$\left[\text{Ans.: } y = c_1 e^{x} + c_2 e^{-x} + c_3\cos x + c_4\sin x - x^4 - 25\right]$$

12. $(D^2-1)y = e^{3x}\cos 2x - e^{2x}\sin 3x$

$$\left[\text{Ans.: } y = c_1e^{x} + c_2e^{-x} + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x) + \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x)\right]$$

13. $(D^2+3D+2)y = 12e^{-x}\sin^3 x$

$$\left[\text{Ans.: } y = c_1e^{-x} + c_2e^{-2x} + \frac{e^{-x}}{10}[(\cos 3x + 3\sin 3x) - 45(\cos x + \sin x)]\right]$$

14. $(D^2+4D+3)y = 6e^{-x}$

$$\left[\text{Ans.: } y = c_1e^{-x} + c_2e^{-3x} + 3xe^{-x}\right]$$

15. $(D^2-D-6)y = 5e^{-2x} + 10e^{3x}$

$$\left[\text{Ans.: } y = c_1e^{3x} + c_2e^{-2x} + 2xe^{3x} - xe^{-2x}\right]$$

16. $(D^2+16)y = 16\sin 4x$

$$\left[\text{Ans.: } y = c_1\cos 4x + c_2\sin 4x - 2x\cos 4x\right]$$

17. $(D^2+25)y = 50\cos 5x + 30\sin 5x$

$$\left[\text{Ans.: } y = c_1\cos 5x + c_2\sin 5x - x(3\cos 5x - 5\sin 5x)\right]$$

18. $(D^3-2D^2+4D-8)y = 8(x^2+\cos 2x)$

$$\left[\text{Ans.: } y = c_1e^{2x} + c_2\cos 2x + c_3\sin 2x - (x^2+x) - \frac{x}{2}(\cos 2x + \sin 2x)\right]$$

19. $(D^2-4D+5)y = 16e^{2x}\cos x$

$$\left[\text{Ans.: } y = e^{2x}(c_1\cos x + c_2\sin x) + 8xe^{2x}\sin x\right]$$

20. $(D^2-6D+13)y = 6e^{3x}\sin x\cos x$

$$\left[\text{Ans.: } y = e^{3x}(c_1\cos 2x + c_2\sin 2x) - \frac{3x}{4}e^{3x}\cos 2x\right]$$

21. $(D^3+2D^2-D-2)y = e^{x}+x^2$

$$\left[\text{Ans.: } y = c_1e^{x} + c_2e^{-x} + c_3e^{-2x} + \frac{1}{6}xe^{x} - \frac{x^2}{2} + \frac{x}{2} - \frac{5}{4}\right]$$

22. $(D^2-4D+4)y = x^3e^{2x} + xe^{2x}$

$$\left[\text{Ans.: } y = (c_1+c_2x)e^{2x} + \left(\frac{x^5}{20} + \frac{x^3}{6}\right)e^{2x}\right]$$

23. $(D^2 - 3D + 2)y = xe^{2x} + \sin x$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{2x} + \left(\frac{x^2}{2} - x\right)e^{2x} + \frac{1}{10}\sin x + \frac{3}{10}\cos x\right]$$

24. $(D^2 + 1)y = \sin^3 x$

$$\left[\textbf{Ans.: } y = c_1\cos x + c_2\sin x + \frac{1}{32}\sin 3x - \frac{3}{8}x\cos x\right]$$

25. $(D^2 + 2D + 1)y = x^2e^{-x}$

$$\left[\textbf{Ans.: } y = (c_1 + c_2x)e^{-x} + \frac{x^4}{12}e^{-x}\right]$$

26. $(D^3 - D^2 - 4D + 4)y = 2x^2 - 4x$
$-1 + 2x^2e^{2x} + 5xe^{2x} + e^{2x}$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{2x} + c_3e^{-2x} + \frac{x^2}{2} + \frac{x^3}{6}e^{2x}\right]$$

27. $(D^2 - 5D - 6)y = e^{3x}$, $y(0) = 2$, $y'(0) = 1$

$$\left[\textbf{Ans.: } y = \frac{10}{21}e^{6x} + \frac{45}{28}e^{-x} - \frac{1}{12}e^{3x}\right]$$

28. $(D^2 - 5D + 6)y = e^x(2x - 3)$, $y(0) = 1$, $y'(0) = 3$

$$\left[\textbf{Ans.: } y = e^{2x} + xe^x\right]$$

29. $(D^3 - D)y = 4e^{-x} + 3e^{2x}$, $y(0) = 0$, $y'(0) = -1$, $y''(0) = 2$

$$\left[\textbf{Ans.: } y = c_1 + c_2e^x + c_3e^{-x} + 2xe^{-x} + \frac{1}{2}e^{2x}\right]$$

30. $(D^3 - 2D^2 + D)y = 2e^x + 2x$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$

$$\left[\textbf{Ans.: } y = x^2 + 4x + 4 + e^x(x^2 - 4)\right]$$

Points to Remember

Higher Order Linear Ordinary Differential Equations

Homogeneous Linear Ordinary Differential Equations with constant coefficients

Sr. No.	Roots	Complementary Function (CF)
1.	Real and distinct roots $(m_1, m_2, \dots m_n)$	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_2 e^{m_3 x} + \dots + c_n e^{m_n x}$
2.	Real and repeated roots $(m_1 = m_2)$	$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$
3.	Complex roots $(m_1 = \alpha + i\beta, m_2 = \alpha - i\beta)$	$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$
4.	Complex and repeated roots $(m_1 = m_2 = \alpha + i\beta, m_3 = m_4 = \alpha - i\beta)$	$y = e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \dots + c_n e^{m_n x}$

Nonhomogeneous Linear Ordinary Differential Equations with constant coefficients

Sr. No.	Q(x)	Particular Integral (PI)
1.	e^{ax+b}	(i) $\frac{1}{f(a)} e^{ax+b}$ if $f(a) \neq 0$ (ii) $x^r \frac{1}{f^{(r)}(a)} e^{ax+b}$ if $f^{(r-1)}(a) = 0$ and $f^{(r)}(a) \neq 0$
2.	$\sin(ax+b)$ or $\cos(ax+b)$	(i) $\frac{1}{\phi(-a^2)} \sin(ax+b)$ or $\frac{1}{\phi(-a^2)} \cos(ax+b)$ if $\phi(-a^2) \neq 0$ (ii) $x^r \frac{1}{\phi^{(r)}(-a^2)} \cos(ax+b)$, if $\phi^{(r-1)}(-a^2) = 0$ and $\phi^{(r)}(-a^2) \neq 0$
3.	x^m	$[f(D)]^{-1} x^m = [1 + \phi(D)]^{-1} x^m = (a_0 + a_1 D + a_2 D^2 + \dots + a_m D^m) x^m$
4.	$e^{ax} V$	$e^{ax} \cdot \frac{1}{f(D+a)} V$
5.	xV	$x \frac{1}{f(D)} V - \frac{f'(D)}{[f(D)]^2} V$

Euler–Cauchy Equations

An equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} x \frac{dy}{dx} + a_n y = Q(x)$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants, is called *Euler–Cauchy equation.*

Existence and Uniqueness of Solutions

Let $p_1, p_2, \ldots, p_n$, and Q all be continuous functions on an open interval I such that $x_0 \in I$. Then the n^{th} order linear ordinary differential equation

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = Q(x)$$

with the initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$ has a unique solution $y = \phi(x)$ throughout the interval I.

Linear Dependence and Independence of Solutions

Two solutions $y_1(x)$ and $y_2(x)$ of linear differential equations of second order with constant coefficients are said to be linearly independent if

$$k_1 y_1(x) + k_2 y_2(x) = 0 \qquad \ldots (1)$$

implies k_1 and k_2 are both zeros, i.e., when y_1 or y_2 can not be expressed in terms of each other.

Two solutions $y_1(x)$ and $y_2(x)$ are linearly dependent if Eq. (1) holds for some constants k_1 and k_2 not both zero.

If Wronskian of $y_1(x)$ and $y_2(x)$, i.e.,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$y_1(x)$ and $y_2(x)$ are linearly independent.

If $W = 0$, $y_1(x)$ and $y_2(x)$ are linearly dependent.

Method of Variation of Parameters

$$\text{CF} = c_1 y_1 + c_2 y_2$$

$$\text{PI} = y = u(x) y_1 + v(x) y_2$$

where $u = \int \frac{-y_2 Q}{W} dx$, $v = \int \frac{y_1 Q}{W} dx$ and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. Particular integral of $(D^2 + 4)y = \cos 2x$ is **[Summer 2016]**

(a) $\frac{x \sin 2x}{2}$ (b) $x \sin 2x$ (c) $\frac{x \sin 2x}{4}$ (d) $\frac{x \sin x}{4}$

2. The Wronskian of the two functions $\sin 2x$ and $\cos 2x$ is **[Winter 2016]**

(a) 1 (b) 2 (c) -1 (d) -2

3. The solution of $(D^2 + 6D + 9)x = 0$ is **[Winter 2016]**

(a) $(c_1 + c_2 t)e^{-3t}$ (b) $c_1 e^{-3t}$

(c) $c_1 c_2 e^t$ (d) $c_1 e^{-t}$

4. In solving differential equation $\frac{d^2y}{dx^2} + y = \tan x$ by method of variation of parameters, complementary function $= c_1 \cos x + c_2 \sin x$, particular integral $= u \cos x + v \sin x$, then v is equal to

(a) $-\cos x$ (b) $\log(\sec x + \tan x) - \sin x$

(c) $-\log(\sec xx + \tan x)$ (d) $\cos x$

5. On putting $x = e^z$, the transformed differential equation of $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 \sin(\log x)$ using $D = \frac{d}{dz}$ is

(a) $(D^2 - 4D + 5)y = e^{2z} \sin z$ (b) $(D^2 - 4D + 5)y = x^2 \sin(\log x)$

(c) $(D^2 - 4D - 4)y = e^z \sin z$ (d) $(D^2 - 3D + 5)y = e^{z^2} \sin z$

6. Solution of differential equation $x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{1}{x^2}$ is

(a) $(c_1 x + c_2) - \frac{x^2}{4}$ (d) $(c_1 x^2 + c_2) + \frac{x^2}{4}$

(c) $c_1 + c_2 \frac{1}{x} + \frac{1}{2x^2}$ (d) $(c_1 \log x + c_2) + \frac{x^2}{4}$

7. If $y = c_1 y_1 + c_2 y_2 = e^x(c_1 \cos x + c_2 \sin x)$ is a complementary function of a second order differential equation, Wronskian $W(y_1, y_2)$ is **[Winter 2015]**

(a) e^x (b) e^{3x} (c) e^{2x} (d) e^{-2x}

8. The general solution of $(D^2 + D + 1)y = 0$ is **[Winter 2015]**

(a) $e^t\left(c_1 \cos\frac{\sqrt{3}}{2}t + c_2 \frac{\sqrt{3}}{2}t\right)$ (b) $e^{-t}\left(c_1 \cos\frac{\sqrt{3}}{2}t + c_2 \frac{\sqrt{3}}{2}t\right)$

(c) $e^{\frac{1}{2}}\left(c_1 \cos\frac{\sqrt{3}}{2}t + c_2 \frac{\sqrt{3}}{2}t\right)$ (d) $e^{-\frac{1}{2}}\left(c_1 \cos\frac{\sqrt{3}}{2}t + c_2 \frac{\sqrt{3}}{2}t\right)$

9. The order and degree of the differential equation $y'' + 3y^2 = 3\cos x$ are **[Summer 2017]**

(a) 2, 1 (b) 1, 2 (c) 1, 1 (d) 2, 2

10. The solution of the differential equation $y'' + 11y' + 10y = 0$ is

(a) $c_1 e^{-x} + c_2 e^{-10x}$ (b) $c_1 e^{x} + c_2 e^{-10x}$

(c) $c_1 e^{-x} + c_2 e^{10x}$ (d)

11. If $y = (c_1 + c_2 x)e^x$ is the complementary function of a second order differential equation, then Wronskian $W(y_1, y_2)$ is **[Summer 2017]**

(a) e^x (b) e^{-x} (c) e^{2x} (d) e^{-2x}

12. The particular integral of $y''' + y' = e^{2x}$ is **[Summer 2017]**

(a) e^{2x} (b) $\frac{1}{10}e^{2x}$ (c) $\frac{1}{10}e^{x}$ (d) e^x

13. The order of the differential equation $\frac{d^2y}{d^2t} + \left(\frac{dy}{dt}\right)^3 + y^4 = e^{-t}$ is

(a) 1 (b) 2 (c) 3 (d) none of above

14. A solution of the following differential equation is given by

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

(a) $y = e^{2x} + e^{-3x}$ (b) $y = e^{2x} + e^{3x}$

(c) $y = e^{-2x} + e^{3x}$ (d) $y = e^{-2x} + e^{-3x}$

15. For $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$, the particular integral is

(a) $\frac{1}{15}e^{2x}$ (b) $\frac{1}{5}e^{2x}$ (c) $3e^{2x}$ (d) $c^1e^{-x} + c^2e^{-3x}$

16. The differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + \sin y = 0$ is

(a) linear (b) nonlinear (c) homogeneous (d) of degree 2

17. The particular solution for the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 5\cos x$ is

(a) $0.5\cos x + 1.5\sin x$ (b) $1.5\cos x + 0.5\sin x$

(c) $1.5\sin x$ (d) $0.5\cos x$

18. The general solution of the differential equation $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ is

(a) $Ax + Bx^2$ (b) $Ax + B\log x$ (c) $Ax + Bx^2\log x$ (d) $Ax + Bx\log x$

where A and B are constants

19. If $x^2\frac{dy}{dx}+2xy=\frac{2\log x}{x}$ and $y(1)=0$ then what is $y(e)$?

(a) e (b) 1 (c) $\frac{1}{e}$ (d) $\frac{1}{e^2}$

20. Which one of the following does not satisfy the differential equation $\frac{d^3y}{dx^3}-y=0$?

(a) e^x (b) e^{-x}

(c) $e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$ (d) $e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)$

21. If e^{-x} and xe^{-x} are the fundamental solution of $\frac{d^2y}{dx^2}+a\frac{dy}{dx}+y=0$, the value of a is

(a) 1 (b) 3 (c) 2 (d) 4

22. If $D \equiv \frac{d}{dz}$ and $z = \log x$ then the differential equation $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=6x$ becomes

(a) $D(D-1)y=6e^z$ (b) $D(D-1)y=6e^{2z}$

(c) $D(D+1)y=6e^{2z}$ (d) $D(D+1)y=6e^z$

23. The solution of the equation $\frac{d^2y}{dx^2}-y=k$ ($k=a$ nonzero constant) which vanishes when $x=0$ and which tends to a finite limit as x tends to infinity, is

(a) $y=k(1+e^{-x})$ (b) $y=k(e^{-x}-1)$

(c) $y=k(e^x+e^{-x}-2)$ (d) $y=k(e^x-1)$

24. $m=2$ is a double root and $m=-1$ is another root of the auxiliary equation of a homogeneous differential with constant coefficient. The differential equation is

(a) $(D^3+3D^2+4)y=0$ (b) $(D^3+3D^2-4)y=0$

(c) $(D^3-3D^2+4)y=0$ (d) $(D^3-3D^2-4)y=0$

Answers

1. (c)	2. (d)	3. (a)	4. (a)	5. (a)	6. (c)	7. (c)	8. (d)
9. (a)	10. (a)	11. (c)	12. (b)	13. (b)	14. (b)	15. (b)	16. (b)
17. (a)	18. (d)	19. (d)	20. (b)	21. (c)	22. (c)	23. (b)	24. (c)

CHAPTER 6

Series Solutions of Ordinary Differential Equations and Special Functions

Chapter Outline

6.1 INTRODUCTION

In general, the solutions to differential equations with variable coefficients cannot be expressed as a finite linear combination of known elementary functions. In such cases, the solution can be obtained in the form of an infinite convergent series. The power-series method and an extension of the power-series method, called the *Frobenius method*, are used to solve differential equations with variable coefficients.

The series solution of certain classical differential equations give rise to special functions such as Bessel's function, Legendre's function, etc. which have many applications in science and engineering. This chapter covers detailed study of these two methods and also covers solutions of Bessel's equation and Legendre's equation.

6.2 POWER-SERIES METHOD

Consider a homogeneous linear second-order differential equation with variable coefficients

$$P_0(x)\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=0 \quad ...(6.1)$$

where P_0, P_1, and P_2 are polynomials in x.

Dividing Eq. (6.1) by $P_0(x)$,

$$\frac{d^2y}{dx^2}+\frac{P_1(x)}{P_0(x)}\frac{dy}{dx}+\frac{P_2(x)}{P_0(x)}y=0$$

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0 \quad ...(6.2)$$

where $P(x)=\dfrac{P_1(x)}{P_0(x)}$ and $Q(x)=\dfrac{P_2(x)}{P_0(x)}$

Equation (6.2) is known as the standard form (normal form or canonical form) of Eq. (6.1).

The power-series solution of Eq. (6.2) about a point $x = x_0$ depends on the following definitions.

Ordinary Point

A point x_0 is called an ordinary point of Eq. (6.2) if $P(x)$ and $Q(x)$ are both analytic (i.e., differentiable) at x_0.

Note:

(i) If $P_0(x) \neq 0$ at $x = x_0$ then x_0 is an ordinary point.

(ii) The ordinary point is also known as a regular point of the equation.

Singular Point

A point x_0 is called a singular point of Eq. (6.2) if either $P(x)$ or $Q(x)$, or both are not analytic at x_0.

Note: If $P_0(x) = 0$ at $x = x_0$ then x_0 is a singular point.

Regular Singular Point

A singular point x_0 is called a regular singular point of Eq. (6.2) if $(x - x_0)\, P(x)$ and $(x - x_0)^2\, Q(x)$ both are analytic (i.e., differentiable) at x_0.

Irregular Singular Point

A singular point x_0 is called an irregular singular point of Eq. (6.2) if either $(x - x_0)\, P(x)$ or $(x - x_0)^2\, Q(x)$, or both are not analytic at x_0.

Example 1

Classify the singular points of the differential equation $x^2y'' + xy' - 2y = 0$.

Solution

$$x^2y'' + xy' - 2y = 0 \qquad \ldots(1)$$

$$P_0(x) = x^2$$

At singular points,

$$P_0(x) = 0$$

$$x^2 = 0$$

$$x = 0$$

Hence, $x = 0$ is a singular point.

Dividing Eq. (1) by x^2,

$$y'' + \frac{1}{x}y' - \frac{2}{x^2}y = 0$$

Comparing with the standard form,

$$y'' + P(x)y' + Q(x)y = 0$$

$$P(x) = \frac{1}{x}, \qquad Q(x) = -\frac{2}{x^2}$$

$$xP(x) = 1, \qquad x^2Q(x) = -2$$

Since $xP(x)$ and $x^2Q(x)$ are both analytic (i.e., differentiable) at $x = 0$, it is a regular singular point of Eq. (1).

Example 2

Find the ordinary points and singular points of the equation $(1 - x)^2y'' - 6xy' - 4y = 0$.

Solution

$$(1 - x)^2y'' - 6xy' - 4y = 0 \qquad \ldots(1)$$

$$P_0(x) = 1 - x^2$$

At singular points,

$$P_0(x) = 0$$

$$1 - x^2 = 0$$
$$x = \pm 1$$

Hence, $x = \pm 1$ are singular points of Eq. (1).

Dividing Eq. (1) by $(1 - x^2)$,

$$y'' - \frac{6x}{(1-x^2)} y' - \frac{4}{1-x^2} y = 0$$

Comparing with the standard form,

$$y'' + P(x)y' + Q(x) = 0$$

$$P(x) = -\frac{6x}{1-x^2}, \qquad Q(x) = -\frac{4}{(1-x^2)}$$

$$= \frac{6x}{(x+1)(x-1)}, \qquad = \frac{4}{(x+1)(x-1)}$$

(i) For $x = 1$,

$$(x-1)P(x) = \frac{6x}{x+1}, \qquad (x-1)^2 Q(x) = \frac{4(x-1)}{x+1}$$

Since $(x - 1)P(x)$ and $(x - 1)^2Q(x)$ are both analytic (i.e., differentiab
$x = 1$, it is a regular singular point.

(ii) For $x = -1$,

$$(x+1)P(x) = \frac{6x}{x-1}, \qquad (x+1)^2 Q(x) = \frac{4(x+1)}{x-1}$$

Since $(x + 1) P(x)$ and $(x + 1)^2 Q(x)$ are both analytic (i.e., differential
$x = -1$, it is a regular singular point of the equation.

All the values of $x \neq \pm 1$ are the ordinary points of the equation.

Example 3

Classify the singular points of the equation

$$x^3(x-2)y'' + x^3y' + 6y = 0$$

Solution

$$x^3(x-2)y'' + x^3y' + 6y = 0$$

$$P_0(x) = x^3 (x-2)$$

At singular points,

$$P_0(x) = 0$$
$$x^3(x-2) = 0$$
$$x = 0, \qquad x = 2$$

Hence, $x = 0$ and $x = 2$ are singular points of Eq. (1).

Dividing Eq. (1) by $x^3(x-2)$,

$$y'' + \frac{1}{x-2}y' + \frac{6}{x^3(x-2)}y = 0$$

Comparing with the standard form,

$$y'' + P(x)y' + Q(x) = 0$$

$$P(x) = \frac{1}{x-2}, \qquad Q(x) = \frac{6}{x^3(x-2)}$$

(i) For $x = 0$,

$$xP(x) = \frac{x}{x-2}, \qquad x^2Q(x) = \frac{6}{x(x-2)}$$

Since $x^2Q(x)$ is not analytic (i.e., not differentiable) at $x = 0$, it is an irregular singular point of the equation.

(ii) For $x = 2$,

$$(x-2)P(x) = 1, \qquad (x-2)^2 Q(x) = \frac{6(x-2)}{x^3}$$

Since $(x-2)$ $P(x)$ and $(x-2)^2$ $Q(x)$ are both analytic (i.e., differentiable) at $x = 2$, it is a regular singular point of the equation.

Example 4

Discuss about ordinary point, singular point, regular singular point and irregular singular point for the differential equation

$$x^3(x-1)y'' + 3(x-1)y' + 7xy = 0$$

[Summer 2017]

Solution

$$x^3(x-1)y'' + 3(x-1)y' + 7xy = 0 \qquad ...(1)$$

$$P_0(x) = x^3(x-1)$$

At singular points,

$$P_0(x) = 0$$

$$x^3(x-1) = 0$$

$$x = 0, \quad x = 1$$

Hence, $x = 0$ and $x = 1$ are singular points of Eq. (1).

Diving Eq. (1) by $x^3(x-1)$,

$$y'' + \frac{3}{x^3}y' + \frac{7}{x^2(x-1)}y = 0$$

Comparing with the standard form,

$$y'' + P(x)y' + Q(x)y = 0$$

$$P(x) = \frac{3}{x^3}, \quad Q(x) = \frac{7}{x^2(x-1)}$$

(i) For $x = 0$,

$$xP(x) = \frac{3}{x^2}, \quad x^2Q(x) = \frac{7}{x-1}$$

Since $xP(x)$ is not analytic (i.e., not differentiable) at $x = 0$, it is an irregular singular point of the equation.

(ii) For $x = 1$,

$$(x-1)\,P(x) = \frac{3(x-1)}{x^3}, \quad (x-1)^2 Q(x) = \frac{7(x-1)}{x^2}$$

Since $(x - 1)\, P(x)$ and $(x - 1)^2\, Q(x)$ are both analytic (i.e., differentiable) at $x = 1$, it is a regular singular point of the equation.

EXERCISE 6.1

Find ordinary points and singular points of the following differential equations. Also, classify the singular points.

1. $x^2y'' - 5y' + 3x^2y = 0$

[Ans. : $x = 0$, irregular singular point]

2. $e^xy'' + y' - xy = 0$

[Ans. : All values of x are ordinary points]

3. $(x^2 + x - 2)^2y'' + 3(x + 2)y' + (x - 1)y = 0$

[Ans. : $x = -2$, regular singular point, $x = 1$, irregular singular point]

4. $x^3y'' + 3xy' + 6y = 0$

[Ans. : $x = 0$, irregular singular point]

5. $x^2y'' + (\sin x)y' + (\cos x)y = 0$

[Ans. : $x = 0$, regular singular point]

6.3 SERIES SOLUTION ABOUT AN ORDINARY POINT

Let the power-series solution of the equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

about an ordinary point x_0 be given as ...(6.3)

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

The coefficients $a_1, a_2, a_3, \ldots$ are obtained as follows:

(i) Let $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ ...(6.4)
be the series solution of the given equation.

(ii) Differentiate y w.r.t. to x twice and substitute y, y', y'' in Eq. (6.3).

(iii) Shift the summation index to obtain a common power of x in each term.

(iv) Equate the coefficients of various powers of x to zero to obtain $a_1, a_2, a_3, \ldots$ in terms of a_0.

(v) Substitute $a_1, a_2, a_3, \ldots$ in Eq. (6.3) to obtain the required solution of the given equation.

Example 1

Find the series solution of $y' - 2xy = 0$. **[Winter 2014]**

Solution

$$y' - 2xy = 0 \qquad \text{...(1)}$$

$$P(x) = 1, \qquad Q(x) = -2x$$

Since $P(x)$ and $Q(x)$ are both analytic at $x = 0$, it is an ordinary point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n n\, x^{n-1} = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

Substituting in Eq. (1),

$$\sum_{n=1}^{\infty} n\, a_n x^{n-1} - 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n\, a_n x^{n-1} - 2 \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

To obtain common power of x in each term, putting $n-1=m+1$ (i.e., $n=m+2$) in the first term, we get

$$\sum_{m=-1}^{\infty}(m+2)\,a_{m+2}x^{m+1}-2\sum_{n=0}^{\infty}a_n x^{n+1}=0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-1}^{\infty}(n+2)a_{n+2}x^{n+1}-2\sum_{n=0}^{\infty}a_n x^{n+1}=0$$

$$a_1+\sum_{n=0}^{\infty}(n+2)a_{n+2}x^{n+1}-2\sum_{n=0}^{\infty}a_n x^{n+1}=0$$

Equating the constant term and the coefficient of x^{n+1} to zero,

$$a_1=0$$

and $\quad (n+2)a_{n+2}-2a_n=0, \qquad n\ge 0$

$$a_{n+2}=\frac{2}{n+2}a_n, \qquad n\ge 0$$

Putting $n=0, 1, 2, 3, \ldots$

$$a_2=a_0$$

$$a_3=\frac{2}{3}a_1=0 \qquad [\because\ a_1=0]$$

$$a_4=\frac{2}{4}a_2=\frac{1}{2}a_0$$

$$a_5=\frac{2}{5}a_3=0$$

$$a_6=\frac{2}{6}a_4=\frac{2}{6}\cdot\frac{1}{2}a_0=\frac{1}{6}a_0=\frac{1}{3!}a_0$$

and so on.

Substituting in Eq. (2),

$$y=a_0+0.x+a_0x^2+0\cdot x^3+\frac{1}{2}a_0x^4+0\cdot x^5+\frac{1}{3!}a_0x^6+\cdots$$

$$=a_0\left(1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\ldots\right)$$

Example 2

Find the power-series solution of the equation $\dfrac{d^2y}{dx^2}+y=0$ *about* $x_0=0$.

[Summer 2014]

Solution

$$y'' + y = 0 \qquad \text{...(1)}$$

$P_0(x) = 1 \neq 0$ at $x = 0$

Hence, $x = 0$ is an ordinary point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \text{...(2)}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting in Eq. (1),

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

To obtain a common power of x in each term, putting $n - 2 = m$ (i.e., $n = m + 2$) in the first term, we get

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{n=0}^{\infty} a_n x^n = 0$$

Since m is a dummy variable, replacing m by n in the first term,

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Equating the coefficient of x^n to zero,

$$(n+1)(n+2) a_{n+2} + a_n = 0, \qquad n \geq 0$$

$$a_{n+2} = -\frac{1}{(n+1)(n+2)} a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, \dots$

$$a_2 = -\frac{1}{1 \cdot 2} a_0 = -\frac{1}{2!} a_0$$

$$a_3 = -\frac{1}{2 \cdot 3} a_1 = -\frac{1}{3!} a_1$$

$$a_4 = -\frac{1}{3 \cdot 4} a_2 = -\frac{1}{3 \cdot 4}\left(-\frac{1}{1 \cdot 2} a_0\right) = \frac{1}{4!} a_0$$

$$a_5 = -\frac{1}{4 \cdot 5} a_3 = -\frac{1}{4 \cdot 5}\left(-\frac{1}{3!} a_1\right) = \frac{1}{5!} a_1$$

and so on.

Substituting in Eq. (2),

$$y = a_0 + a_1 x - \frac{1}{2!}a_0 x^2 - \frac{1}{3!}a_1 x^3 + \frac{1}{4!}a_0 x^4 + \frac{1}{5!}a_1 x^5 + \cdots$$

$$= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= a_0 \cos x + a_1 \sin x$$

Example 3

Find the series solution of $y'' = 2y'$ in powers of x. [Winter 2012]

Solution

$$y'' = 2y'$$

$$y'' - 2y' = 0 \qquad \text{...(1)}$$

$$P_0(x) = 1 \neq 0 \text{ at } x = 0$$

Hence, $x = 0$ is an ordinary point.

Let the series solution of Eq. (1) in powers of x be

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting in Eq. (1),

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2\sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

To obtain a common power of x in each term, putting $n - 2 = m - 1$ (i.e., $n = m + 1$) in the first term, we get

$$\sum_{m=1}^{\infty} (m+1) m\, a_{m+1} x^{m-1} - 2\sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=1}^{\infty} n(n+1) a_{n+1} x^{n-1} - 2\sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

Equating the coefficient of x^{n-1} to zero,

$$n(n+1)a_{n+1} - 2n\, a_n = 0, \quad n \geq 1$$

$P(x) = 0.$

$$a_{n+1} = \frac{2n}{n(n+1)} a_n, \qquad n \geq 1$$

$$= \frac{2}{n+1} a_n, \qquad n \geq 1$$

Putting $n = 1, 2, 3, \ldots$

$$a_2 = \frac{2}{2} a_1 = a_1$$

$$a_3 = \frac{2}{3} a_2 = \frac{2}{3} a_1$$

$$a_4 = \frac{2}{4} a_3 = \frac{1}{2} \cdot \frac{2}{3} a_1 = \frac{1}{3} a_1$$

and so on.

Substituting in Eq. (2),

$$y = a_0 + a_1 x + a_1 x^2 + \frac{2}{3} a_1 x^3 + \frac{1}{3} a_1 x^4 + \cdots$$

$$= a_0 + a_1 \left(x + x^2 + \frac{2}{3} x^3 + \frac{1}{3} x^4 + \cdots \right)$$

Example 4

Find the power-series solution of $\frac{d^2y}{dx^2} + xy = 0$.

[Winter 2016; Summer 2016]

Solution

$$y'' + xy = 0 \qquad \ldots(1)$$

$$P_0(x) = 1 \neq 0 \text{ at } x = 0$$

Hence, $x = 0$ is an ordinary point about $x = 0$.

Let the series solution of Eq. (1) be

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \ldots(2)$$

$$y' = \sum_{n=0}^{\infty} a_n \cdot n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting in Eq. (1),

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

To obtain the common power of x in each term, putting $n-2 = m+1$ (i.e., $n = m+3$) in the first term, we get

$$\sum_{m=-1}^{\infty} (m+3)(m+2)a_{m+3}x^{m+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-1}^{\infty} (n+3)(n+2)a_{n+3}x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$2a_2 + \sum_{n=0}^{\infty} (n+2)(n+3)a_{n+3}x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Equating the constant term and the coefficient of x^{n+1} to zero,

$$2a_2 = 0, \qquad a_2 = 0$$

and $\quad (n+2)(n+3)\,a_{n+3} + a_n = 0, \qquad n \ge 0$

$$a_{n+3} = -\frac{1}{(n+2)(n+3)}a_n, \qquad n \ge 0$$

Putting $n = 0, 1, 2, 3, 4, \ldots$

$$a_3 = -\frac{1}{2\cdot 3}a_0$$

$$a_4 = -\frac{1}{3\cdot 4}a_1$$

$$a_5 = -\frac{1}{4\cdot 5}a_2 = 0 \qquad [\because\ a_2 = 0]$$

$$a_6 = -\frac{1}{5\cdot 6}a_3 = -\frac{1}{5\cdot 6}\left(-\frac{1}{2\cdot 3}\right)a_0 = \frac{1}{2\cdot 3\cdot 5\cdot 6}a_0$$

$$a_7 = -\frac{1}{6\cdot 7}a_4 = -\frac{1}{6\cdot 7}\left(-\frac{1}{3\cdot 4}a_1\right) = \frac{1}{3\cdot 4\cdot 6\cdot 7}a_1$$

and so on.

Substituting in Eq. (2),

$$y = a_0 + a_1 x + 0\cdot x^2 - \frac{1}{2\cdot 3}a_0 x^3 - \frac{1}{3\cdot 4}a_1 x^4 + 0\cdot x^5 + \frac{1}{2\cdot 3\cdot 5\cdot 6}a_0 x^6 + \frac{1}{3\cdot 4\cdot 6\cdot 7}a_1 x^7 + \cdots$$

$$= a_0\left(1 - \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} - \cdots\right) + a_1\left(x - \frac{x^4}{3\cdot 4} + \frac{x^7}{3\cdot 4\cdot 6\cdot 7} - \cdots\right)$$

Example 5

Solve in series the equation $\dfrac{d^2y}{dx^2}+x^2y=0$.

[Winter 2013; Summer 2018]

Solution

$$y''+x^2y=0 \quad ...(1)$$

$$P_0(x)=1\neq 0 \text{ at } x=0$$

Hence, $x=0$ is an ordinary point.

Let the series solution of Eq. (1) about $x=0$ be

$$y=\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+\cdots \quad ...(2)$$

$$y'=\sum_{n=0}^{\infty}a_nnx^{n-1}=\sum_{n=1}^{\infty}na_nx^{n-1}$$

$$y''=\sum_{n=0}^{\infty}a_n\,n(n-1)x^{n-2}=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$$

Substituting in Eq. (1),

$$\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+x^2\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=0}^{\infty}a_nx^{n+2}=0$$

To obtain a common power of x in each term, putting $n-2=m+2$ (i.e., $n=m+4$) in the first term, we get

$$\sum_{m=-2}^{\infty}(m+4)(m+3)a_{m+4}x^{m+2}+\sum_{n=0}^{\infty}a_nx^{n+2}=0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-2}^{\infty}(n+3)(n+4)a_{n+4}x^{n+2}+\sum_{n=0}^{\infty}a_nx^{n+2}=0$$

$$2a_2+6a_3x+\sum_{n=0}^{\infty}(n+3)(n+4)a_{n+4}x^{n+2}+\sum_{n=0}^{\infty}a_nx^{n+2}=0$$

Equating the constant term, the coefficient of x, and the coefficient of x^{n+2} to zero,

$$2a_2=0, \qquad a_2=0$$

$$6a_3=0, \qquad a_3=0$$

and $\quad (n+3)(n+4)a_{n+4}+a_n=0, \quad n\geq 0$

$$a_{n+4} = -\frac{1}{(n+3)(n+4)}a_n, \quad n \geq 0$$

Putting $n = 0, 1, 2, \ldots$

$$a_4 = -\frac{1}{3 \cdot 4}a_0$$

$$a_5 = -\frac{1}{4 \cdot 5}a_1$$

$$a_6 = -\frac{1}{5 \cdot 6}a_2 = 0 \quad [\because\ a_2 = 0]$$

$$a_7 = -\frac{1}{6 \cdot 7}a_3 = 0 \quad [\because\ a_3 = 0]$$

$$a_8 = -\frac{1}{7 \cdot 8}a_4 = -\frac{1}{7 \cdot 8}\left(-\frac{1}{3 \cdot 4}a_0\right) = \frac{1}{3 \cdot 4 \cdot 7 \cdot 8}a_0$$

$$a_9 = -\frac{1}{8 \cdot 9}a_5 = -\frac{1}{8 \cdot 9}\left(-\frac{1}{4 \cdot 5}a_1\right) = \frac{1}{4 \cdot 5 \cdot 8 \cdot 9}a_1$$

and so on.

Substituting in Eq. (2),

$$y = a_0 + a_1 x + 0 \cdot x^2 + 0 \cdot x^3 - \frac{1}{3 \cdot 4}a_0 x^4 - \frac{1}{4 \cdot 5}a_1 x^5 + 0 \cdot x^6 + 0 \cdot x^7 + \frac{1}{3 \cdot 4 \cdot 7 \cdot 8}a_0 x^8 + \frac{1}{4 \cdot 5 \cdot 8 \cdot 9}a_1 x^9 + \cdots$$

$$= a_0\left(1 - \frac{x^4}{3 \cdot 4} + \frac{x^8}{3 \cdot 4 \cdot 7 \cdot 8} - \cdots\right) + a_1\left(x - \frac{x^5}{4 \cdot 5} + \frac{x^9}{4 \cdot 5 \cdot 8 \cdot 9} - \cdots\right)$$

Example 6

Find the series solution of $(x-2)\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + 9y = 0$ *about* $x_0 = 0$.

[Winter 2015]

Solution

$$(x-2)y'' - x^2 y' + 9y = 0 \qquad \ldots(1)$$

$$P_0(x) = x - 2 \neq 0 \text{ at } x = 0$$

Hence, $x = 0$ is on ordinary point.

Let the series solution of Eq. (1) about $x_0 = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \ldots(2)$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=2}^{\infty} n\,(n-1) a_n\, x^{n-2}$$

Substituting in Eq. (1),

$$(x-2)\sum_{n=2}^{\infty} n\,(n-1)a_n\, x^{n-2} - x^2 \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} + 9\sum_{n=0}^{\infty} a_n\, x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-1} - 2\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-2} - \sum_{n=1}^{\infty} n\, a_n\, x^{n+1} + 9\sum_{n=0}^{\infty} a_n\, x^n = 0$$

To obtain a common power of x in each term, putting $n-1=m$ (i.e., $n=m+1$) and $n-2=t$ (i.e., $n=t+2$) or $n+1=p$ (i.e., $n=p-1$), we get

$$\sum_{m+1=2}^{\infty} (m+1)(m+1-1)a_{m+1}\, x^{m+1-1} - 2\sum_{t+2=2}^{\infty} (t+2)(t+1)a_{t+2}\, x^{t+2-2}$$

$$- \sum_{p-1=1}^{\infty} (p-1)a_{p-1}x^{p-1+1} + 9\sum_{n=0}^{\infty} a_n\, x^n = 0$$

Since m, t and p are dummy variables, replacing m, t, p by n in the first, second and third term respectively,

$$\sum_{n=1}^{\infty}(n+1)na_{n+1}x^n - 2\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n$$

$$-\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n + 9\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2a_2 x + \sum_{n=2}^{\infty} n(n+1)a_{n+1}x^n - 4a_2 - 12a_3 x - 2\sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n$$

$$-\sum_{n=2}^{\infty}(n-1)a_{n-1}\, x^n + 9a_0 + 9a_1 x + \sum_{n=2}^{\infty} 9a_n x^n = 0$$

$$2a_2 x - 4a_2 - 12a_3 x + 9a_0 + 9a_1 x$$

$$+\sum_{n=2}^{\infty}[n(n+1)a_{n+1} - 2(n+1)(n+2)a_{n+2} - (n-1)a_{n-1} + 9a_n]x^n = 0$$

Equating the constant term, the coefficient of x and the coefficient of x^n to zero,

(constant term)

$$9a_0 - 4a_2 = 0$$

$$4a_2 = 9a_0$$

$$a_2 = \frac{9}{4}a_0$$

$$2a_2 - 12a_3 + 9a_1 = 0 \qquad \text{(coefficient of } x\text{)}$$

$$12a_3 = 2a_2 + 9a_1$$

$$= 2\left(\frac{9}{4}a_0\right) + 9a_1$$

$$= \frac{18}{4}a_0 + 9a_1$$

$$a_3 = \frac{3}{8}a_0 + \frac{3}{4}a_1$$

and $n(n+1)\,a_{n+1} - 2(n+1)(n+2)\,a_{n+2} - (n-1)\,a_{n-1} + 9a_n = 0$ (coefficient of x^n)

$n = 2, 3, 4, \ldots$

Putting $n = 2$,

$$6a_3 - 24a_4 - a_1 + 9a_2 = 0$$

$$24a_4 = 6a_3 + 9a_2 - a_1$$

$$a_4 = \frac{6}{24}a_3 + \frac{9}{24}a_2 - \frac{1}{24}a_1$$

$$= \frac{1}{4}a_3 + \frac{3}{8}a_2 - \frac{1}{24}a_1$$

$$= \frac{1}{4}\left(\frac{3}{8}a_0 + \frac{3}{4}a_1\right) + \frac{3}{8}\left(\frac{9}{4}a_0\right) - \frac{1}{24}a_1$$

$$= \frac{3}{32}a_0 + \frac{3}{16}a_1 + \frac{27}{32}a_0 - \frac{1}{24}a_1$$

$$= \frac{30}{32}a_0 + \frac{7}{48}a_1$$

and so on.

Substituting these values in Eq. (2),

$$y = a_0 + a_1x + \frac{9}{4}a_0x^2 + \left(\frac{3}{8}a_0 + \frac{3}{4}a_1\right)x^3 + \left(\frac{30}{32}a_0 + \frac{7}{48}a_1\right)x^4 + \ldots$$

$$= a_0\left(1 + \frac{9}{4}x^2 + \frac{3}{8}x^3 + \frac{15}{16}x^4 + \ldots\right) + a_1\left(x + \frac{3}{4}x^3 + \frac{7}{48}x^4 + \ldots\right)$$

Example 7

Find the series solution of $(1 + x^2)\,y'' + xy' - 9y = 0$.

[Summer 2015]

Solution

$$(1+x^2)y'' + xy' - 9y = 0 \qquad \text{...(1)}$$

$$P_0(x) = 1 + x^2 \neq 0 \text{ at } x = 0$$

Hence, $x = 0$ is an ordinary point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting in Eq. (1),

$$(1+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - 9\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n a_n x^n - 9\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} \{n(n-1) + n - 9\} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} (n^2 - 9) a_n x^n = 0$$

To obtain the common power of x in each term, putting $n - 2 = m$ in the first term, we get

$$\sum_{m=0}^{\infty} (m+1)(m+2)a_{m+2} x^m + \sum_{n=0}^{\infty} (n^2 - 9) a_n x^n = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n + \sum_{n=0}^{\infty} (n^2 - 9) a_n x^n = 0$$

Equating the coefficient of x^n to zero,

$$(n+1)(n+2)a_{n+2} = -(n^2 - 9)a_n, \qquad n \geq 0$$

$$a_{n+2} = -\frac{(n^2 - 9)}{(n+1)(n+2)} a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, 3, 4, ...$

$$a_2 = -\frac{(-9)}{1\cdot 2}a_0 = \frac{9}{2}a_0$$

$$a_3 = -\frac{(-8)}{2\cdot 3}a_1 = \frac{4}{3}a_1$$

$$a_4 = -\frac{(-5)}{3\cdot 4}a_2 = \frac{5}{12}a_2 = \frac{5}{12}\cdot\frac{9}{2}a_0 = \frac{15}{8}a_0$$

$$a_5 = 0$$

$$a_6 = -\frac{7}{5.6}a_4 = -\frac{7}{30}a_4 = -\frac{7}{30}\cdot\frac{15}{8}a_0 = -\frac{7}{16}a_0$$

and so on.

Substituting these values in Eq. (2),

$$y = a_0 + a_1 x + \frac{9}{2}a_0 x^2 + \frac{4}{3}a_1 x^3 + \frac{15}{8}a_0 x^4 - \frac{7}{16}a_0 x^6 + ...$$

$$= a_0\left(1 + \frac{9}{2}x^2 + \frac{15}{8}x^4 - \frac{7}{16}x^6 + \cdots\right) + a_1\left(x + \frac{4}{3}x^3 + \cdots\right)$$

Example 8

Using the power-series method, solve $(1-x^2)y'' - 2xy' + 2y = 0.$

[Winter 2017, 2013]

Solution

$$(1-x^2)y'' - 2xy' + 2y = 0 \qquad ...(1)$$

$$P_0(x) = 1 - x^2 \neq 0 \quad \text{at } x = 0$$

Hence, $x = 0$ is an ordinary point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad ...(2)$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n\, n(n-1) x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting in Eq. (1),

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n\, a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

To obtain a common power of x in each term, putting $n-2=m$ in the first term, we get

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} na_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} na_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2a_1 x - 2\sum_{n=2}^{\infty} na_n x^n$$

$$+ 2a_0 + 2a_1 x + 2\sum_{n=2}^{\infty} a_n x^n = 0$$

$$2a_0 + 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+1)(n+2)a_n + 2 - n(n-1)a_n - 2na_n + 2a_n\right] x^n = 0$$

$$2(a_0 + a_2) + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+1)(n+2)a_{n+2} - (n-1)(n+2)a_n\right] x^n = 0$$

Equating the constant term, coefficient of x, and coefficient of x^n to zero,

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

$$3\cdot 2a_3 x = 0$$

$$a_3 = 0$$

and

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + 2a_n = 0, \quad n \ge 0$$

$$(n+1)(n+2)a_{n+2} - (n^2 + n - 2)a_n = 0, \quad n \ge 0$$

$$a_{n+2} = \frac{(n-1)(n+2)}{(n+1)(n+2)} a_n, \quad n \ge 0$$

$$= \frac{n-1}{n+1} a_n, \quad n \ge 0$$

Putting $n = 2, 3, \ldots$

$$a_4 = \frac{1}{3}a_2 = -\frac{1}{3}a_0$$

$$a_5 = \frac{2}{4}a_3 = 0$$

$$a_6 = \frac{3}{5}a_4 = -\frac{1}{5}a_0$$

and so on.

Substituting these values in Eq. (2),

$$y = a_0 + a_1x - a_0x^2 + 0\cdot x^3 - \frac{1}{3}a_0x^4 + 0\cdot x^5 - \frac{1}{5}a_0x^6 + \cdots$$

$$y = a_0 + a_1x - a_0x^2 - \frac{1}{3}a_0x^4 - \frac{1}{5}a_0x^6 \ldots$$

$$= a_1x + a_0\left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 \ldots\right)$$

where a_0 and a_1 are arbitrary constants.

Example 9

Find the power-series solution of the equation

$$(x^2 + 1)\,y'' + xy' - xy = 0 \text{ about } x = 0.$$

[Winter 2012; Summer 2017, 2013]

Solution

$$(x^2 + 1)\,y'' + xy' - xy = 0 \qquad \ldots(1)$$

$$P_0(x) = 1 + x^2 = 1 \neq 0 \text{ at } x = 0$$

$$\text{At } x = 0,\ P_0(0) = 1 \neq 0$$

Hence, $x = 0$ is an ordinary point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + \cdots \qquad \ldots(2)$$

$$y' = \sum_{n=0}^{\infty} a_n n\, x^{n-1} = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

Substituting in Eq. (1),

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n\,a_n x^{n-1} - x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n (x^n + x^{n-2}) + \sum_{n=1}^{\infty} n\,a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n\,a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

To obtain a common power of x in each term, putting $n-2=m$ in the first term and $n+1=t$ in the fourth term, we get

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n\,a_n x^n - \sum_{t=1}^{\infty} a_{t-1}x^t = 0$$

Since m and t are dummy variables, replacing m and t by n,

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n\,a_n x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$\left[2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+1)(n+2)a_{n+2}x^n\right] + \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

$$+\left[a_1 x + \sum_{n=2}^{\infty} n\,a_n x^n\right] - \left[a_0 x + \sum_{n=2}^{\infty} a_{n-1}x^n\right] = 0$$

$$2a_2 + (6a_3 + a_1 - a_0)x + \sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2} + n(n-1)a_n + na_n - a_{n-1}\right]x^n = 0$$

$$2a_2 + (6a_3 + a_1 - a_0)x + \sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2} + n^2 a_n - a_{n-1}\right]x^n = 1$$

Equating the constant term, and the coefficients of x and x^n to zero,

$$2a_2 = 0, \qquad a_2 = 0$$

$$6a_3 + a_1 - a_0 = 0, \qquad a_3 = \frac{1}{6}(a_0 - a_1)$$

and $(n+1)(n+2)\,a_{n+2} + n^2 a_n - a_{n-1} = 0,\ n \ge 2$

$$a_{n+2} = \frac{a_{n-1} - n^2 a_n}{(n+1)(n+2)}, \qquad n \ge 2$$

Putting $n = 2, 3, 4, \ldots$

$$a_4 = \frac{a_1 - 4a_2}{12} = \frac{a_1}{12}$$

$$a_5 = \frac{a_2 - 9a_3}{20} = -\frac{9}{20}\left[\frac{1}{6}(a_0 - a_1)\right] = \frac{3}{40}(a_1 - a_0)$$

and so on.

Substituting in Eq. (2),

$$y = a_0 + a_1 x + 0 \cdot x^2 + \frac{1}{6}(a_0 - a_1)x^3 + \frac{a_1}{12}x^4 + \frac{3}{40}(a_1 - a_0)x^5 + \cdots$$

$$= a_0\left(1 + \frac{x^3}{6} - \frac{3}{40}x^5 + \cdots\right) + a_1\left(x - \frac{x^3}{6} + \frac{x^4}{12} + \frac{3}{40}x^5 + \cdots\right)$$

Example 10

Solve the initial-value problem

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = 0 \qquad \text{with } y(1) = 2,\ y'(1) = 4$$

Solution

$$xy'' + y' + 2y = 0 \qquad \text{...(1)}$$

Since the initial conditions are given at $x = 1$, a power-series solution of Eq. (1) in powers of $(x - 1)$ is obtained.

$$P_0(x) = x \neq 0 \text{ at } x = 1$$

Let the series solution of Eq. (1) be

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0 + a_1(x-1) + a_2(x-1)^2 + \cdots \qquad \text{...(2)}$$

Let $x - 1 = t$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot 1 = \frac{dy}{dt}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right)\cdot\frac{dt}{dx} = \frac{d^2y}{dt^2}$$

Substituting in Eq. (1),

$$(t+1)\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = 0 \qquad \text{...(3)}$$

Putting $x - 1 = t$ in Eq. (2), the series solution of Eq. (3) is

$$y = \sum_{n=0}^{\infty} a_n n = a_0 + a_1 t + a_2 t^2 + \cdots \qquad \text{...(4)}$$

$$\frac{dy}{dt} = \sum_{n=0}^{\infty} a_n \cdot n t^{n-1} = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

$$\frac{d^2y}{dt^2} = \sum_{n=0}^{\infty} a_n \cdot n(n-1)t^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$$

Substituting in Eq. (3),

$$(1+t)\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=1}^{\infty} n a_n t^{n-1} + 2\sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n t^{n-1} + \sum_{n=1}^{\infty} n\, a_n t^{n-1} + 2\sum_{n=0}^{\infty} a_n t^n = 0$$

To obtain a common power of t, putting $n-2=m_1$ in the first term and $n-1=m_2$ in the second and third terms, we get

$$\sum_{m_1=0}^{\infty} (m_1+2)(m_1+1)a_{m_1+2}t^{m_1} + \sum_{m_2=1}^{\infty} (m_2+1)m_2 a_{m_2+1}t^{m_2}$$

$$+ \sum_{m_2=0}^{\infty} (m_2+1)a_{m_2+1}t^{m_2} + 2\sum_{n=0}^{\infty} a_n t^n = 0$$

Since m_1 and m_2 are dummy variables, replacing m_1 and m_2 by n,

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}t^n + \sum_{n=1}^{\infty} n(n+1)a_{n+1}t^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + 2\sum_{n=0}^{\infty} a_n t^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+1)(n+2)a_{n+2}t^n + \sum_{n=1}^{\infty} n(n+1)a_{n+1}t^n + a_1$$

$$+ \sum_{n=1}^{\infty}(n+1)a_{n+1}t^n + 2a_0 + 2\sum_{n=1}^{\infty} a_n t^n = 0$$

$$2a_2 + a_1 + 2a_0 + \sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2} + (n+1)^2 a_{n+1} + 2a_n\right]t^n = 0$$

Equating the constant term and the coefficient of t^n to zero,

$$2a_2 + a_1 + 2a_0 = 0$$

$$a_2 = -\frac{1}{2}(a_1 + 2a_0)$$

and $$(n+1)(n+2)a_{n+2} + (n+1)^2 a_{n+1} + 2a_n = 0, \quad n \ge 1$$

$$a_{n+2} = -\frac{(n+1)^2 a_{n+1} + 2a_n}{(n+1)(n+2)}, \quad n \ge 1$$

Putting $n = 1, 2, 3, \ldots$

$$a_3 = -\frac{4a_2 + 2a_1}{2\cdot 3} = \frac{1}{6}\left[-2(a_1 + 2a_0) + 2a_1\right] = \frac{2}{3}a_0$$

$$a_4 = -\frac{9a_3 + 2a_2}{3\cdot 4} = -\frac{1}{12}\left[9\cdot\frac{2}{3}a_0 - (a_1 + 2a_0)\right] = -\frac{1}{12}(4a_0 - a_1)$$

and so on.

Substituting in Eq. (4),

$$y = a_0 + a_1 t - \left(\frac{a_1 + 2a_0}{2}\right)t^2 + \frac{2}{3}a_0 t^3 - \frac{1}{12}(4a_0 - a_1)t^4 + \cdots$$

$$= a_0 + a_1(x-1) - \frac{1}{2}(a_1 + 2a_0)(x-1)^2 + \frac{2}{3}a_0(x-1)^3 - \frac{1}{12}(4a_0 - a_1)(x-1)^4 + \cdots \quad ...(5)$$

$$y' = a_1 - (a_1 + 2a_0)(x-1) + 2a_0(x-1)^2 - \frac{1}{3}(4a_0 - a_1)(x-1)^3 + \cdots \quad ...(6)$$

Initially, at $x = 1$, $y = 2$, and $y' = 4$

Substituting in Eqs (5) and (6),

$$2 = a_0$$

and $4 = a_1$

Putting in Eq. (5),

$$y = 2 + 4(x-1) - 4(x-1)^2 + \frac{4}{3}(x-1)^3 - \frac{1}{3}(x-1)^4 + \cdots$$

EXERCISE 6.2

Find the power-series solutions about the origin of the following equations:

1. $y' - 4y = 0$

$$\left[\textbf{Ans.}: y = a_0 e^{4x}\right]$$

2. $(1+x)y' + xy = 0$

$$\left[\textbf{Ans.}: y = a_0\left(1 - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{8} + \cdots\right)\right]$$

3. $(1-x^2)y' = 2xy$

$$\left[\begin{aligned}\textbf{Ans.}: y &= a_0(1 + x^2 + x^4 + \cdots \\ &= a_0(1-x^2)^{-1}\end{aligned}\right]$$

4. $(x-1)y' = xy$

$$\left[\textbf{Ans.}: y = a_0\left(1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{8} - \cdots\right)\right]$$

5. $y'' - 3x^2 y' = 0$

$$\left[\textbf{Ans.}: y = a_0 + a_1\left(x + \frac{x^4}{4} + \frac{x^7}{14} + \cdots\right)\right]$$

6. $(1-x^2)y'' - 4xy' + 2y = 0$

$$\left[\textbf{Ans.}: y = a_0\left(1 - x^2 - \frac{2}{3}x^4 - \cdots\right) + a_1\left(x + \frac{1}{3}x^3 + \frac{4}{15}x^5 + \cdots\right)\right]$$

7. $(1+x^2)y'' - 9y = 0$

$$\left[\textbf{Ans.}: y = a_0\left(1 + \frac{9}{2}x^2 + \frac{21}{8}x^4 + \cdots\right) + a_1\left(x + \frac{3}{2}x^3 + \frac{9}{40}x^5 + \cdots\right)\right]$$

8. $(x^2+4)y'' - 6xy' + 8y = 0$

$$\left[\textbf{Ans.}: y = a_0\left(1 - x^2 - \frac{1}{24}x^4 - \cdots\right) + a_1\left(x - \frac{1}{12}x^3 - \frac{1}{240}x^5 - \cdots\right)\right]$$

9. $y'' - xy' + (2x^2+1)y = 0$

$$\left[\textbf{Ans.}: y = a_0\left(1 - \frac{x^2}{2} - \frac{5}{24}x^4 + \cdots\right) + a_1\left(x - \frac{x^5}{10} + \cdots\right)\right]$$

Find the power-series solutions of the following equations about the given point.

10. $xy' - y = 0, \quad x_0 = 1$

$$\left[\textbf{Ans.}: y = a_0[1 + (x-1)]\right]$$

11. $y'' + xy' + y = 0, \quad x_0 = 2$

$$\left[\begin{aligned}\textbf{Ans.}: y = a_0&\left[1 - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 + \cdots\right]\\ &+ a_1\left[(x-2) - (x-2)^2 + \frac{1}{3}(x-2)^2 \cdots\right]\end{aligned}\right]$$

12. $(x+1)y' - (x+2)y = 0, \quad x_0 = -2$

$$\left[\textbf{Ans.}: y = a_0\left[1 - \frac{1}{2}(x+2)^2 - \frac{1}{3}(x-2)^3 - \cdots\right]\right]$$

Find the power-series solutions of the following initial-value equations.

13. $y'' - xy = 0, y(1) = 2, y'(1) = 0$

$$\left[\textbf{Ans.}: y = 2 + (x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots\right]$$

14. $(x^2+2)y'' - 2xy' + 3y = 0, y(1) = 1, y'(1) = -1$

$$\left[\textbf{Ans.}: y = 1 - (x-1) - \frac{5}{6}(x-1)^2 + \cdots\right]$$

6.4 FROBENIUS METHOD

In the previous section, the power-series solution for differential equations is obtained when x_0 is an ordinary point. To obtain the solution near a regular singular point x_0, an extension of the power-series method, known as the Frobenius method (or generalised power-series method), is used.

Let x_0 be a regular singular point of the differential equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \qquad \text{...(6.5)}$$

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x) = \dfrac{P_1(x)}{P_0(x)}, \quad Q(x) = \dfrac{P_2(x)}{P_0(x)}$

(i) Let the series solution of Eq. (6.5) about x_0 be

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r} = (x - x_0)^r \left[a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots\right] \qquad \text{...(6.6)}$$

(ii) Differentiate twice and substitute y, y' and y'' in Eq. (6.5).

(iii) Equating to zero the coefficients of the lowest degree term in $(x - x_0)$, a quadratic equation, known as *indicial equation*, is obtained. The roots of the indicial equation are called *indicial roots.*

(iv) Equating to zero the coefficients of other powers of x, a recurrence relation relating the coefficients a_n is obtained.

(v) Using the recurrence relation for each indicial root separately, two linearly independent solutions $y_1(x)$ and $y_2(x)$ of Eq. (6.5) are obtained.
The general solution of Eq. (6.5) is given as

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x)$$

where c_1 and c_2 are arbitrary constants.

(vi) One of the solutions $y_1(x)$ or $y_2(x)$ is in the form of Eq. (6.6). The form of the other solution depends upon the nature of the indicial roots.

Let r_1 and r_2 be the roots of the indicial equation. There are three cases.

Case I Distinct Roots not Differing by an Integer

$$r_1 - r_2 \neq \text{an integer}$$

Then $y_1 = (y)_{r=r_1}$ and $y_2 = (y)_{r=r_2}$

The general solution is

$$y = c_1 y_1 + c_2 y_2$$

Case II Double Root (Repeated Root)

$$r_1 = r_2 = b, \text{ say}$$

$$y_1 = (y)_{r=b} \quad \text{and} \quad y_2 = \left(\frac{\partial y}{\partial r}\right)_{r=b}$$

The general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 (y)_{r=b} + c_2 \left(\frac{\partial y}{\partial r}\right)_{r=b}$$

Case III Roots Differing by an Integer

$$r_1 - r_2 = \text{an integer}, \quad r_1 < r_2$$

In this case, solutions corresponding to r_1 and r_2 may or may not be linearly independent. This leads to two possibilities:

(i) One of the coefficients becomes infinite for the smaller indicial root $r = r_1$. The procedure is modified by putting $a_0 = c_0 (r - r_1)$, $c_0 \neq 0$

$$y_1 (y)_{r=r_1} \quad \text{and} \quad y_2 = \left(\frac{\partial y}{\partial r}\right)_{r=r_1}$$

The solution, corresponding to the second indicial root r_2, is usually a multiple of y_1 or a part of $\left(\frac{\partial y}{\partial r}\right)_{r=r_1}$. Hence, it produces a linearly dependent solution.

The general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 (y)_{r=r_1} + c_2 \left(\frac{\partial y}{\partial r}\right)_{r=r_1}$$

(ii) One of the coefficients becomes indeterminate for the smaller indicial root $r = r_1$. This root produces the complete solution as it contains two arbitrary constants. The second indicial root r_2 produces a linearly dependent solution.

Example 1

Solve in series the differential equation $4x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0.$

[Winter 2017, 2014]

Solution

$$4xy'' + 2y' + y = 0 \quad \text{...(1)}$$

$$P_0(x) = 4x = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{1}{2x}y' + \frac{1}{4x}y = 0$$

$$P(x) = \frac{1}{2x}, \qquad Q(x) = \frac{1}{4x}$$

Since $xP(x)=\frac{1}{2}$ and $x^2Q(x)=\frac{x}{4}$ are analytic (i.e., differentiable) at $x=0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x=0$ be

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}=x^r(a_0+a_1x+a_2x^2+\cdots) \qquad \text{...(2)}$$

$$y'=\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$$

Substituting in Eq. (1),

$$4x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$4\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}+2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(2n+2r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

To obtain a common power of x in each term, putting $n=m+1$ in the first term, we get

$$\sum_{m=-1}^{\infty}2(m+1+r)(2m+2+2r-1)a_{m+1}x^{m+r}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

Since m is a dummy variable, replacing m by n

$$\sum_{n=-1}^{\infty}2(n+r+1)(2n+2r+1)a_{n+1}x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$2r(-2+2r+1)a_0x^{-1+r}+\sum_{n=0}^{\infty}2(n+r+1)(2n+2r+1)a_{n+1}x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$2r(2r-1)a_0x^{r-1}+\sum_{n=0}^{\infty}\left[2(n+r+1)(2n+2r+1)a_{n+1}+a_n\right]x^{n+r}=0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$2a_0r(2r-1)=0$$

$$r=0, \qquad r=\frac{1}{2} \quad [\because a_0\neq 0]$$

Equating the coefficient of x^{n+r} to zero,

$$2(n+r+1)(2n+2r+1)a_{n+1}+a_n=0, \qquad n\geq 0$$

$$a_{n+1}=-\frac{1}{2(n+r+1)(2n+2r+1)}a_n, \qquad n\geq 0 \qquad \text{...(3)}$$

For the first solution, putting $r = 0$ in Eq. (3),

$$a_{n+1} = -\frac{1}{2(n+1)(2n+1)} a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, \ldots$

$$a_1 = -\frac{1}{2} a_0$$

$$a_2 = -\frac{1}{2 \cdot 2 \cdot 3} a_1 = -\frac{1}{12}\left(-\frac{1}{2} a_0\right) = \frac{1}{24} a_0$$

and so on.

Substituting $r = 0$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$y_1 = a_0 - \frac{1}{2} a_0 x + \frac{1}{24} a_0 x^2 - \cdots$$

$$= a_0 \left(1 - \frac{x}{2} + \frac{x^2}{24} - \cdots\right)$$

For the second solution, putting $r = \frac{1}{2}$ in Eq. (3),

$$a_{n+1} = -\frac{1}{2\left(n + \frac{1}{2} + 1\right)(2n+1+1)} a_n, \qquad n \geq 0$$

$$= -\frac{1}{2\left(n + \frac{3}{2}\right)(2n+2)} a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, \ldots$

$$a_1 = -\frac{1}{2 \cdot \frac{3}{2} \cdot 2} a_0 = -\frac{1}{6} a_0$$

$$a_2 = -\frac{1}{2 \cdot \frac{5}{2} \cdot 4} a_1 = -\frac{1}{20}\left(-\frac{1}{6} a_0\right) = \frac{1}{120} a_0$$

and so on.

Substituting $r = \frac{1}{2}$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$y_2 = x^{\frac{1}{2}}\left(a_0 - \frac{1}{6} a_0 x + \frac{1}{120} a_0 x^2 - \cdots\right)$$

$$= a_0 \sqrt{x}\left(1 - \frac{x}{6} + \frac{x^2}{120} - \cdots\right)$$

Hence, the general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 a_0\left(1-\frac{x}{2}+\frac{x^2}{24}+-\cdots\right)+c_2 a_0\sqrt{x}\left(1-\frac{x}{6}+\frac{x^2}{120}-\cdots\right) \\ &= A\left(1-\frac{x}{2}+\frac{x^2}{24}-\cdots\right)+B\sqrt{x}\left(1-\frac{x}{6}+\frac{x^2}{120}-\cdots\right) \\ &= A\cos\sqrt{x}+B\sin\sqrt{x} \end{aligned}$$

where $A = c_1 a_0$, $B = c_2 a_0$

Example 2

Find the series solution of $2x(x-1)y'' - (x+1)y' + y = 0,\ x_0 = 0.$

[Summer 2015]

Solution

$$2x(x-1)y'' - (x+1)y' + y = 0 \qquad \text{...(1)}$$

$$P_0(x) = 2x(x-1) = 0 \quad \text{at} \quad x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' - \frac{x+1}{2x(x-1)}y' + \frac{1}{2x(x-1)}y = 0$$

$$P(x) = -\frac{x+1}{2x(x-1)}, \qquad Q(x) = \frac{1}{2x(x-1)}$$

Since $xP(x) = -\frac{x+1}{2(x-1)}$ and $x^2Q(x) = \frac{x}{2(x-1)}$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the power series solution of Eq. (1) be

$$y = \sum_{n=0}^{\infty} a_n x^{m+n} = x^m(a_0 + a_1x + a_2x^2 + \cdots) \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n (m+n)x^{m+n-1}$$

$$y'' = \sum_{n=0}^{\infty} (m+n)(m+n-1)\, a_n x^{m+n-2}$$

Substituting in Eq. (1),

$$2x(x-1)\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n-2}$$

$$-(x+1)\sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$2\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n} - 2\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n-1}$$

$$-\sum_{n=0}^{\infty}(m+n)a_n x^{m+n} - \sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1} + \sum_{n=0}^{\infty}a_n x^{m+n} = 0$$

$$2\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n} - \sum_{n=0}^{\infty}(m+n)a_n x^{m+n} + \sum_{n=0}^{\infty}a_n x^{m+n}$$

$$-2\sum_{n=-1}^{\infty}(m+n+1)(m+n)a_{n+1}x^{m+n} - \sum_{n=-1}^{\infty}(m+n+1)a_{n+1}x^{m+n} = 0$$

$$\sum_{n=0}^{\infty}\{2(m+n)(m+n-1)-(m+n)+1\}\, a_n x^{m+n} - 2m(m-1)x^{m-1}a_0$$

$$-2\sum_{n=0}^{\infty}(m+n+1)(m+n)a_{n+1}x^{m+n} - ma_0x^{m-1} - \sum_{n=0}^{\infty}(m+n+1)a_{n+1}x^{m+n} = 0$$

Equating the coefficient of x^{m-1} to zero,

$$[-2m\,(m-1) - m]\,a_0 = 0$$

$$-2m^2 + 2m - m = 0 \quad (\because\ a_0 \neq 0)$$

$$2m^2 - m = 0$$

$$m(2m-1) = 0$$

$$m = 0 \text{ and } m = \frac{1}{2}$$

Equating the coefficient of x^{m+n} to zero,

$\{2(m+n)(m+n-1) - (m+n) + 1\}a_n\ \{-2(m+n+1)(m+n) - (m+n+1)\}a_{n+1} = 0$

$\{2(m+n)\,(m+n-1) - (m+n) + 1\}a_n = \{2(m+n+1)(m+n) + (m+n+1)\}a_{n+1}$

$$a_{n+1} = \frac{2(m+n)(m+n-1)-(m+n)+1}{(m+n+1)(2m+2n+1)}a_n, \qquad m \geq 0 \qquad \text{...(3)}$$

Putting $m = 0$,

$$a_{n+1} = \frac{n(2n-3)+1}{(n+1)(2n+1)}a_n$$

Putting $n = 0, 1, 2, \ldots$

$$a_1 = \frac{1}{(1)(1)}a_0 = a_0$$

$$a_2 = 0$$

and so on.

Substituting $m = 0$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$y_1 = a_0 x^0 + a_1 x + a_2 x^2 + \ldots$$
$$= a_0 - a_0 x$$

Putting $m = \frac{1}{2}$ in Eq. (3),

$$a_{n+1} = \frac{2\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2}-1\right)-\left(n+\frac{1}{2}\right)+1}{\left(n+\frac{1}{2}+1\right)(2n+1+1)} a_n$$

$$= \frac{(2n+1)(2n-3)+2}{(2n+3)(n+1)} a_n$$

Putting $n = 0, 1, 2, \ldots$

$$a_1 = -\frac{1}{3} a_0$$

$$a_2 = -\frac{1}{10} a_1 = \frac{1}{30} a_0$$

$$a_3 = \frac{1}{3} a_2 = \frac{1}{90} a_0$$

$$a_4 = \frac{23}{36} a_3 = \frac{23}{36}\left(\frac{1}{90} a_0\right)$$

and so on.

Substituting $m = \frac{1}{2}$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$y_2 = a_0 x^{\frac{1}{2}} + a_1 x^{\frac{3}{2}} + a_2 x^{\frac{5}{2}} + a_3 x^{\frac{7}{2}} + \cdots$$

$$= a_0 x^{\frac{1}{2}} - \frac{1}{3} a_0 x^{\frac{3}{2}} + \frac{1}{30} a_0 x^{\frac{5}{2}} + \cdots$$

Hence, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1[(1-x)] + c_2\left[x^{\frac{1}{2}} - \frac{1}{3} x^{\frac{3}{2}} + \frac{1}{30} x^{\frac{5}{2}} + \cdots\right]$$

Example 3

Using the Forbenius method, obtain the series solution for

$$2x(1-x)y''+(1-x)y'+3y=0 \quad \text{about} \quad x_0=0$$

Solution

$$2x(1-x)y''+(1-x)y'+3y=0 \qquad \text{...(1)}$$

$$P_0(x)=2x(1-x)=0 \text{ at } x=0$$

Hence, $x=0$ is a singular point.

$$y''+\frac{1}{2x}y'+\frac{3}{2x(1-x)}y=0$$

$$P(x)=\frac{1}{2x}, \qquad Q(x)=\frac{3}{2x(1-x)}$$

Since $xP(x)=\frac{1}{2}$ and $x^2Q(x)=\frac{3x}{2(1-x)}$ are analytic (i.e., differentiable) at $x=0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x=0$ be

$$y=\sum_{n=0}^{\infty} a_n x^{n+r} = x^r(a_0+a_1x+a_2x^2+\cdots) \qquad \text{...(2)}$$

$$y'=\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$$

Substituting in Eq. (1),

$$2x(1-x)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+(1-x)\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+3\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}-2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}$$

$$-\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}+3\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-2+1)a_nx^{n+r-1}-\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2+1)-3\right]a_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-1)a_nx^{n+r-1}-\sum_{n=a}^{\infty}\left[(n+r)(2n+2r-1)-3\right]a_nx^{n+r}=0$$

To obtain a common power of x in each term, putting $n = m + 1$ in the first term, we get

$$\sum_{m=-1}^{\infty} (m+1+r)(2m+2+2r-1)a_{m+1}x^{m+r} - \sum_{n=0}^{\infty}\left[(n+r)(2n+2r-1)-3\right]x^{n+r} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-1}^{\infty} (n+r+1)(2n+2r+1)a_{n+1}x^{n+r} - \sum_{n=0}^{\infty}\left[(n+r)(2n+2r-1)-3\right]a_n x^{n+r} = 0$$

$$r(2r-1)a_0x^{r-1} + \sum_{n=0}^{\infty}(n+r+1)(2n+2r+1)a_{n+1}x^{n+r}$$

$$-\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-1)-3\right]a_n x^{n+r} = 0$$

Equating the coefficient of the lowest degree term (i.e., x^{r-1}) to zero, the indicial equation is

$$a_0 r(2r-1) = 0$$

$$r = 0,\ r = \frac{1}{2} \qquad [\because a_0 \neq 0]$$

Equating the coefficient of x^{n+r} to zero,

$$(n+r+1)(2n+2r+1)a_{n+1} - \{(n+r)(2n+2r-1)-3\}a_n = 0, \qquad n \geq 0$$

$$a_{n+1} = \frac{(n+r)(2n+2r-1)-3}{(n+r+1)(2n+2r+1)}a_n, \qquad n \geq 0 \qquad ...(3)$$

For the first solution, putting $r = 0$ in Eq. (3),

$$a_{n+1} = \frac{n(2n-1)-3}{(n+1)(2n+1)}a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, 3, ...$

$$a_1 = -3a_0$$

$$a_2 = \frac{(2-1)-3}{(2)(3)}a_1 = -\frac{2}{6}(-3a_0) = a_0$$

and so on.

Substituting $r = 0$ and values of $a_1, a_2, ...$ in Eq. (2),

$$y_1 = a_0 - 3a_0x + a_0x^2 + \cdots$$

$$= a_0(1 - 3x + x^2 + \cdots)$$

For the second solution, putting $r = \frac{1}{2}$ in Eq. (3),

$$a_{n+1} = \frac{\left(n+\frac{1}{2}\right)2n-3}{\left(n+\frac{3}{2}\right)(2n+2)} a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, ...$

$$a_1 = \frac{-3}{\frac{3}{2}\cdot 2} a_0 = -a_0$$

$$a_2 = \frac{\frac{3}{2}\cdot 2-3}{\frac{5}{2}\cdot 4} = 0$$

Since $a_2 = 0$, $a_3 = a_4 = a_5 = ... = 0$

Substituting $r = \frac{1}{2}$ and values of $a_1, a_2, ...$ in Eq. (2),

$$y_2 = x^{\frac{1}{2}}(a_0 - a_0 x + 0)$$
$$= a_0\sqrt{x}(1-x)$$

Hence, the general solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 a_0 (1 - 3x + x^2 - \cdots) + c_2 a_0 \sqrt{x}(1-x)$$
$$= A(1 - 3x + x^2 - \cdots) + B\sqrt{x}\,(1-x)$$

where $A = c_1 a_0$, $B = c_2 a_0$

Example 4

Find the series solution of the equation $xy'' + y' - y = 0$ *about* $x_0 = 0$.

Solution

$$xy'' + y' - y = 0 \qquad \text{...(1)}$$
$$P_0(x) = x = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{1}{x}y' - \frac{1}{x}y = 0$$

$$P(x) = \frac{1}{x}, \qquad Q(x) = -\frac{1}{x}$$

Since $xP(x) = 1$ and $x^2Q(x) = -x$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

Substituting in Eq. (1),

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1+1)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)^2 a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

To obtain the common power of x in each term, putting $n = m + 1$ in the first term, we get

$$\sum_{m=-1}^{\infty}(m+1+r)^2 a_{m+1} x^{m+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-1}^{\infty}(n+r+1)^2 a_{n+1} x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$a_0 r^2 x^{-1+r} + \sum_{n=0}^{\infty}(n+r+1)^2 a_{n+1} x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$a_0 r^2 = 0$$

$$r = 0, 0 \quad [\because \ a_0 \neq 0]$$

which is a double root.

Equating the coefficient of x^{n+r} to zero.

$$a_{n+1}(n+r+1)^2 - a_n = 0, \qquad n \geq 0$$

$$a_{n+1} = \frac{1}{(n+r+1)^2} a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, \ldots$

$$a_1 = \frac{1}{(r+1)^2} a_0$$

$$a_2 = \frac{1}{(r+2)^2} a_1 = \frac{1}{(r+2)^2 (r+1)^2} a_0$$

and so on.

Substituting in Eq. (2),

$$y = x^r \left[a_0 + \frac{1}{(r+1)^2} a_0 x + \frac{1}{(r+1)^2 (r+2)^2} a_0 x^2 + \cdots \right]$$

$$= a_0 x^r \left[1 + \frac{1}{(r+1)^2} x + \frac{1}{(r+1)^2 (r+2)^2} x^2 + \cdots \right] \qquad \ldots(3)$$

For the first solution, putting $r = 0$ in Eq. (3),

$$y_1 = (y)_{r=0} = a_0 \left(1 + x + \frac{x^2}{4} + \cdots \right)$$

For the second solution, differentiating Eq. (3) w.r.t. r,

$$\frac{\partial y}{\partial r} = a_0 x^r \log x \left[1 + \frac{1}{(r+1)^2} x + \frac{1}{(r+1)^2 (r+2)^2} x^2 + \cdots \right]$$

$$+ a_0 x^r \left[-\frac{2}{(r+1)^3} x - \frac{2}{(r+1)^3} \cdot \frac{x^2}{(r+2)^2} - \frac{2}{(r+2)^3} \cdot \frac{x^2}{(r+1)^2} + \cdots \right]$$

Putting $r = 0$,

$$y_2 = \left(\frac{\partial y}{\partial r} \right)_{r=0}$$

$$= a_0 \log x \left(1 + x + \frac{x^2}{4} + \cdots \right) - 2a_0 \left(x + \frac{x^2}{4} + \frac{x^2}{8} + \cdots \right)$$

$$= a_0 \log x \left(1 + x + \frac{x^2}{4} + \cdots \right) - 2a_0 \left(x + \frac{3}{8} x^2 + \cdots \right)$$

Hence, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 a_0 \left(1 + x + \frac{x^2}{4} + \cdots \right) + c_2 a_0 \log x \left(1 + x + \frac{x^2}{4} + \cdots \right) - 2c_2 a_0 \left(x + \frac{3}{8} x^2 + \cdots \right)$$

$$= (A + B \log x) \left(1 + x + \frac{x^2}{4} + \cdots \right) - 2B \left(x + \frac{3}{8} x^2 + \cdots \right)$$

where $A = c_1 a_0$, $B = c_2 a_0$

Example 5

Solve the differential equation by Frobenius method

$$x(x-1)y'' + (3x-1)y' + y = 0 \quad \text{at } x = 0$$

Solution

$$x(x-1)y'' + (3x-1)y' + y = 0 \qquad \text{...(1)}$$

$$P_0(x) = x(x-1) = 0 \quad \text{at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{(3x-1)}{x(x-1)}y' + \frac{1}{x(x-1)}y = 0$$

$$P(x) = \frac{3x-1}{x(x-1)}, \qquad Q(x) = \frac{1}{x(x-1)}$$

Since $xP(x) = \frac{3x-1}{x-1}$ and $x^2Q(x) = \frac{x}{x-1}$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r(a_0 + a_1x + a_2x^2 + \cdots) \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$$

Substituting in Eq. (1),

$$(x^2-x)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + (3x-1)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + 3\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$

$$- \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$-\sum_{n=0}^{\infty}(n+r)(n+r-1+1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}\left[(n+r)(n+r-1) + 3(n+r) + 1\right]a_n x^{n+r} = 0$$

To obtain a common power of x in each term, putting $n = m + 1$ in the first term, we get

$$-\sum_{m=-1}^{\infty}(m+1+r)^2 a_{m+1}x^{m+r} + \sum_{n=0}^{\infty}\left[(n+r)(n+r+2)+1\right]a_n x^{n+r} = 0$$

Since m is a dummy variable, replacing m by n,

$$-\sum_{n=-1}^{\infty}(n+r+1)^2 a_{n+1}x^{n+r}+\sum_{n=0}^{\infty}\left[(n+r)(n+r+2)+1\right]a_n x^{n+r}=0$$

$$-r^2 a_0 x^{-1+r}-\sum_{n=0}^{\infty}(n+r+1)^2 a_{n+1}x^{n+r}+\sum_{n=0}^{\infty}\left[(n+r)(n+r+2)+1\right]a_n x^{n+r}=0$$

Equating the coefficient of the lowest degree term (i.e., x^{r-1}) to zero,

$$-r^2 a_0=0$$

$$r=0,0 \quad [\because a_0\neq 0]$$

which is a double root.

Equating the coefficient of x^{n+r} to zero,

$$-(n+r+1)^2 a_{n+1}+\left[(n+r)(n+r+2)+1\right]a_n=0, \qquad n\geq 0$$

$$a_{n+1}=\frac{(n+r)(n+r+2)+1}{(n+r+1)^2}a_n, \qquad n\geq 0$$

$$=\frac{(n+r)^2+2(n+r)+1}{(n+r+1)^2}a_n, \qquad n\geq 0$$

$$=\frac{(n+r+1)^2}{(n+r+1)^2}a_n, \qquad n\geq 0$$

$$=a_n$$

Putting $n = 0, 1, 2, \ldots$

$$a_1=a_0$$

$$a_2=a_1=a_0$$

and so on.

Substituting in Eq. (2),

$$y=x^r(a_0+a_0x+a_0x^2+\cdots)$$

$$=a_0x^r(1+x+x^2+\cdots) \qquad \ldots(3)$$

For the first solution, putting $r = 0$ in Eq. (3),

$$y_1=(y)_{r=0}=a_0(1+x+x^2+\ldots)$$

For the second solution, differentiating Eq. (3) w.r.t. r,

$$\frac{\partial y}{\partial r}=a_0x^r\log x(1+x+x^2\cdots)$$

Putting $r = 0$,

$$y_2=\left(\frac{\partial y}{\partial r}\right)_{r=0}$$

$$=a_0\log x(1+x+x^2+\cdots)$$

Hence, the general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 a_0 (1 + x + x^2 \cdots) + c_2 a_0 \log x (1 + x + x^2 + \cdots) \\ &= (A + B \log x)(1 + x + x^2 + \cdots) \\ &= (A + B \log x)(1 - x)^{-1} \end{aligned}$$

where $A = c_1 a_0$, $B = c_2 a_0$

Example 6

Obtain the series solution of the differential equation $xy'' + y' + xy = 0$.

Solution

$$xy'' + y' + xy = 0 \qquad \text{...(1)}$$

$$P_0(x) = x = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{1}{x} y' + y = 0$$

$$P(x) = \frac{1}{x}, \qquad Q(x) = 1$$

Since $xP(x) = 1$ and $x^2 Q(x) = x^2$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting in Eq. (1),

$$x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + x \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1+1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}(n+r)^2 a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

To obtain a common power of x in each term, putting $n = m + 2$ in the first term, we get

$$\sum_{m=-2}^{\infty}(m+2+r)^2 a_{m+2} x^{m+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-2}^{\infty}(n+r+2)^2 a_{n+2} x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$r^2 a_0 x^{r-1} + (r+1)^2 a_1 x^r + \sum_{n=0}^{\infty}(n+r+2)^2 a_{n+2} x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$a_0 r^2 = 0$$

$$r = 0, 0 \quad [\because \quad a_0 \neq 0]$$

which is a double root.

Equating the coefficient of x^r and x^{n+r+1} to zero,

$$a_1(r+1)^2 = 0$$

$$a_1 = 0 \quad [\because \quad r \neq -1]$$

and $a_{n+2}(n+r+2)^2 + a_n = 0, \qquad n \geq 0$

$$a_{n+2} = -\frac{1}{(n+r+2)^2} a_n, \quad n \geq 0$$

Putting $n = 0, 1, 2, \ldots$

$$a_2 = -\frac{1}{(r+2)^2} a_0$$

$$a_3 = -\frac{1}{(r+3)^2} a_1 = 0$$

$$a_4 = -\frac{1}{(r+4)^2} a_2 = -\frac{1}{(r+4)^2}\left[-\frac{1}{(r+2)^2} a_0\right] = \frac{1}{(r+2)^2 (r+4)^2} a_0$$

and so on.

Substituting in Eq. (2),

$$y = x^r\left[a_0 + 0 \cdot x - \frac{1}{(r+2)^2} a_0 x^2 + 0 \cdot x^3 + \frac{1}{(r+2)^2 (r+4)^2} a_0 x^4 + \cdots\right]$$

$$= a_0 x^r\left[1 - \frac{1}{(r+2)^2} x^2 + \frac{1}{(r+2)^2 (r+4)^2} x^4 - \cdots\right] \qquad \ldots(3)$$

For the first solution, putting $r = 0$ in Eq. (3),

$$y_1 = (y)_{r=0}$$

$$= a_0\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right)$$

For the second solution, differentiating Eq. (3) w.r.t. r,

$$\frac{\partial y}{\partial r} = a_0 x^r \log x\left[1-\frac{1}{(r+2)^2}x^2+\frac{1}{(r+2)^2(x+4)^2}x^4-\cdots\right]$$

$$+a_0 x^r\left[1+\frac{2}{(r+2)^3}x^2-\frac{2}{(r+2)^3(r+4)^2}x^4-\frac{2}{(r+2)^2(r+4)^3}x^4+\cdots\right]$$

Putting $r = 0$,

$$y_2 = \left(\frac{\partial y}{\partial r}\right)_{r=0}$$

$$= a_0 \log x\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right)+a_0\left(1+\frac{x^2}{2^2}-\frac{x^4}{2^2\cdot 4^2}-\frac{x^4}{2\cdot 4^3}+\cdots\right)$$

$$= a_0 \log x\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right)+a_0\left(1+\frac{x^2}{2^2}-\frac{3}{2^3\cdot 4^2}x^4+\cdots\right)$$

Hence, the general solution is

$$y = c_1y_1 + c_2y_2$$

$$= c_1a_0\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right)+c_2a_0\log x\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right)$$

$$+c_2a_0\left(1+\frac{x^2}{2^2}-\frac{3}{2^3\cdot 4^2}x^4+\cdots\right)$$

$$= (A+B\log x)\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right)+B\left(1+\frac{x^2}{2^2}-\frac{3}{2^3\cdot 4^2}x^4+\cdots\right)$$

where $A = c_1\, a_0$, $B = c_2\, a_0$

Example 7

Find the roots of the indicial equation to $x^2y''+xy'-(2-x)y=0.$

[Winter 2015]

Solution

$$x^2y''+xy'-(2-x)y=0 \tag{1}$$

$$P_0(x) = x^2 = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{y'}{x} - \frac{(2-x)}{x^2} y = 0$$

$$P(x) = \frac{1}{x}, \qquad Q(x) = -\frac{(2-x)}{x^2} = \frac{x-2}{x^2}$$

Since $x\,P(x) = 1$ and $x^2\,Q(x) = x - 2$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r (a_0 + a_1 x + a_2 x^2 + \ldots) \qquad \ldots(2)$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)\, x^{n+r-2}$$

Substituting in Eq. (1),

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n\, x^{n+r-2} + x \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} - (2-x) \sum_{n=0}^{\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - 2\sum_{n=0}^{\infty} a_n\, x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

To obtain a common power of x in each term, putting $n = m + 1$ in first, second and third term, we get

$$\sum_{m=-1}^{\infty} (m+r+1)(m+r) a_{m+1} x^{m+r+1} + \sum_{m=-1}^{\infty} (m+r+1) a_{m+1} x^{m+r+1} - 2\sum_{m=-1}^{\infty} a_{m+1} x^{m+r+1} + \sum_{n=0}^{\infty} a_n\, x^{n+r+1} = 0$$

$$\sum_{m=-1}^{\infty} \{(m+r+1)(m+r) + (m+r+1) - 2\} a_{m+1} x^{m+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-1}^{\infty} \{(n+r+1)(n+r) + (n+r+1) - 2\} a_{n+1} x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\{r(r-1)+r-2\}a_0\,x^r+\sum_{n=0}^{\infty}\{(n+r+1)(n+r)+(n+r+1)-2\}a_{n+1}x^{n+r+1}$$
$$+\sum_{n=0}^{\infty}a_n\,x^{n+r+1}=0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$[r(r-1)+r-2]a_0=0$$
$$r^2-r+r-2=0 \qquad [\because a_0\neq 0]$$
$$r^2-2=0$$
$$r^2=2$$
$$r=\pm\sqrt{2}$$
$$r=\sqrt{2}, \quad r=-\sqrt{2}$$

Equating the coefficient of x^{r+n+1} to zero,

$$\{(n+r+1)(n+r)+(n+r+1)-2\}a_{n+1}+a_n=0$$
$$a_{n+1}=-\frac{1}{(n+r+1)(n+r)+(n+r+1)-2}a_n$$
$$=-\frac{1}{(n+r+1)^2-2}a_n,\ n\geq 0 \qquad \text{...(3)}$$

For the first solution, putting $r=\sqrt{2}$ in Eq. (3),

$$a_{n+1}=-\frac{1}{\left(n+\sqrt{2}+1\right)^2-2}a_n,\ n\geq 0$$

Putting $n = 0, 1, 2, 3...,$

$$a_1=-\frac{1}{\left(\sqrt{2}+1\right)^2-2}a_0=-\frac{1}{2+2\sqrt{2}+1-2}a_0=-\frac{1}{2\sqrt{2}+1}a_0$$

$$a_2=-\frac{1}{\left(2+\sqrt{2}\right)^2-2}a_1=-\frac{1}{4+2\sqrt{2}+2-2}a_1=-\frac{1}{4+2\sqrt{2}}a_1=\frac{1}{\left(4+2\sqrt{2}\right)\left(2\sqrt{2}+1\right)}a_0$$

and so on.

Substituting $r=\sqrt{2}$ and values of $a_1, a_2...$ in Eq. (2),

$$y_1=x^{\sqrt{2}}(a_0+a_1x+a_2x^2+...)$$

$$= a_0 x^{\sqrt{2}}\left(1-\frac{x}{1+2\sqrt{2}}+\frac{x^2}{(1+2\sqrt{2})(4+2\sqrt{2})}+\dots\right)$$

For the second solution, putting $r = -\sqrt{2}$ in Eq. (3),

$$a_{n+1} = -\frac{1}{(n-\sqrt{2}+1)^2-2}a_n, \quad n \geq 0$$

Putting $n = 0, 1, 2, 3 \dots,$

$$a_1 = -\frac{1}{\left(1-\sqrt{2}\right)^2-2}a_0 = -\frac{1}{1-2\sqrt{2}+2-2}a_0 = -\frac{1}{1-2\sqrt{2}}a_0$$

$$a_2 = -\frac{1}{\left(2-\sqrt{2}\right)^2-2}a_1 = -\frac{1}{4-2\sqrt{2}+2-2}a_1 = -\frac{1}{4-2\sqrt{2}}a_1$$

$$= \frac{1}{\left(4-2\sqrt{2}\right)\left(1-2\sqrt{2}\right)}a_0$$

and so on.

Substituting $r = -\sqrt{2}$ and values of $a_1, a_2, \dots$ in Eq. (2),

$$y_2 = x^{-\sqrt{2}}(a_0 + a_1 x + a_2 x^2 + \dots)$$

$$= a_0\, x^{-\sqrt{2}}\left(1-\frac{x}{1-2\sqrt{2}}+\frac{x^2}{(1-2\sqrt{2})(4-2\sqrt{2})}+\dots\right)$$

Hence, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 a_0 x^{\sqrt{2}}\left(1-\frac{x}{2\sqrt{2}+1}+\frac{x^2}{(1+2\sqrt{2})(4+2\sqrt{2})}+\dots\right)$$

$$+ c_2\, a_0\, x^{-\sqrt{2}}\left(1-\frac{x}{1-2\sqrt{2}}+\frac{x^2}{(1-2\sqrt{2})(4-2\sqrt{2})}+\dots\right)$$

Example 8

Find a series solution of the differential equation

$$x^2y'' + x^3y' + (x^2-2)y = 0 \textit{ about } x = 0$$

Solution

$$x^2y'' + x^3y' + (x^2-2)y = 0 \qquad \dots(1)$$

$$P_0(x) = x^2 = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + xy' + \frac{x^2 - 2}{x^2} y = 0$$

$$P(x) = x, \qquad Q(x) = \frac{x^2 - 2}{x^2}$$

Since $xP(x) = x^2$ and $x^2 Q(x) = x^2 - 2$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

Substituting in Eq. (1),

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x^3 \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + (x^2 - 2) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r+2} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - 2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 2 \right] a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r+1) a_n x^{n+r+2} = 0$$

To obtain a common power of x in each term, putting $n = m - 2$ in the second term, we get

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 2 \right] a_n x^{n+r} + \sum_{m=2}^{\infty} (m+r-1) a_{m-2} x^{m+r} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 2 \right] a_n x^{n+r} + \sum_{n=2}^{\infty} (n+r-1) a_{n-2} x^{n+r} = 0$$

$$\left[r(r-1) - 2 \right] a_0 x^r + \left[(1+r)r - 2 \right] a_1 x^{r+1} + \sum_{n=2}^{\infty} \left[(n+r)(n+r-1) - 2 \right] a_n x^{n+r} + \sum_{n=2}^{\infty} (n+r-1) a_{n-2} x^{n+r} = 0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$a_0[r(r-1)-2]=0$$

$$a_0(r+1)(r-2)=0$$

$$r=-1,\ r=2 \qquad [\because\ a_0 \neq 0]$$

Equating the coefficient of x^{r+1} to zero,

$$[r(r+1)-2]a_1=0$$

$$(r-1)(r+2)a_1=0$$

$$a_1=0 \qquad [\because\ r\neq 1,\ r\neq -2]$$

Equating the coefficient of x^{n+r} to zero,

$$[(n+r)(n+r-1)-2]a_n+(n+r-1)a_{n-2}=0, \qquad n\geq 2$$

$$a_n=-\frac{(n+r-1)}{(n+r-1)(n+r)-2}a_{n-2}, \qquad n\geq 2 \qquad \text{...(3)}$$

For the first solution, putting $r=-1$ in Eq. (3),

$$a_n=-\frac{(n-2)}{(n-2)(n-1)-2}a_{n-2}, \qquad n\geq 2$$

$$=-\frac{(n-2)}{(n^2-3n+2-2)}a_{n-2}, \qquad n\geq 2$$

$$=-\frac{(n-2)}{n(n-3)}a_{n-2}, \qquad n\geq 2$$

Putting $n = 2, 3, 4, \ldots$

$$a_2=0$$

$$a_3=\lim_{n\to 3}\left[-\frac{(n-2)}{n(n-3)}a_1\right] \qquad \left[\frac{0}{0}\text{ form, } \because\ a_1=0\right]$$

$$=\lim_{n\to 3}\left[-\frac{a_1}{2n-3}\right] \qquad \text{[Using L'Hospital's rule]}$$

$$=-\frac{1}{3}a_1=0 \qquad [\because\ a_1=0]$$

$$a_4=-\frac{2}{4\cdot 1}a_2=0$$

$$a_5=-\frac{3}{5\cdot 2}a_3=0$$

and so on.

Putting $r = -1$ and values of $a_1, a_2, a_3, \dots$ in Eq. (2),

$$\begin{aligned} y_1 &= (y)_{r=-1} \\ &= x^{-1}(a_0 + 0\cdot x + 0\cdot x^2 + \cdots) \\ &= \frac{a_0}{x} \end{aligned}$$

For the second solution, putting $r = 2$ in Eq. (3),

$$\begin{aligned} a_n &= -\frac{(n+1)}{(n+1)(n+2)-2}a_{n-2}, \qquad n \ge 2 \\ &= -\frac{(n+1)}{(n^2+3n)}a_{n-2}, \qquad n \ge 2 \\ &= -\frac{n+1}{n(n+3)}a_{n-2}, \qquad n \ge 2 \end{aligned}$$

Putting $n = 2, 3, 4, \dots$

$$a_2 = -\frac{3}{2\cdot 5}a_0 = -\frac{3}{10}a_0$$

$$a_3 = -\frac{4}{3\cdot 6}a_1 = 0 \qquad [\because a_1 = 0]$$

$$a_4 = -\frac{5}{4\cdot 7}a_2 = -\frac{5}{28}\left(-\frac{3}{10}a_0\right) = \frac{3}{56}a_0$$

and so on.

Putting $r = 2$ and values of $a_1, a_2, a_3, \dots$ in Eq. (2),

$$\begin{aligned} y_2 &= (y)_{r=2} \\ &= x^2\left(a_0 + 0\cdot x - \frac{3}{10}a_0x^2 + 0\cdot x^3 + \frac{3}{56}a_0x^4 + \cdots\right) \\ &= a_0x^2\left(1 - \frac{3}{10}x^2 + \frac{3}{56}x^4 - \cdots\right) \end{aligned}$$

Hence, the general solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\frac{a_0}{x} + c_2a_0x^2\left(1 - \frac{3}{10}x^2 + \frac{3}{56}x^4 - \cdots\right) \\ &= \frac{A}{x} + Bx^2\left(1 - \frac{3}{10}x^2 + \frac{3}{56}x^4 - \cdots\right) \end{aligned}$$

where $A = c_1 a_0$, $B = c_2 a_0$

Example 9

Find the general solution of $2x^2y'' + xy' + (x^2 - 1)y = 0$ *by using Frobenius method.*

[Winter 2016]

Solution

$$2x^2y'' + xy' + (x^2 - 1)y = 0 \qquad \text{...(1)}$$

$$P_0(x) = 2x^2 = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{1}{2x}y' + \left(\frac{x^2 - 1}{2x^2}\right)y = 0$$

$$P(x) = \frac{1}{2x}, \quad Q(x) = \frac{1}{2}\left(1 - \frac{1}{x^2}\right)$$

Since $xP(x) = \dfrac{1}{2}$ and $x^2Q(x) = \dfrac{1}{2}(x^2 - 1)$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^2(a_0 + a_1x + a_2x^2 + \cdots) \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n(n+r)\, x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)\, x^{n+r-2}$$

Substituting in Eq. (1),

$$2x^2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)\, x^{n+r-2} + x\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} + (x^2 - 1)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)\, x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} a_n x^{n+r+2} + \sum_{n=0}^{\infty}[2(n+r)(n+r-1) + (n+r) - 1]\, a_n x^{n+r} = 0$$

To obtain a common power of x in each term, putting $n = m - 2$ in the first term, we get

$$\sum_{m=2}^{\infty} a_{m-2}\, x^{m+r} + \sum_{n=0}^{\infty}[2(n+r)(n+r-1) + (n+r) - 1]\, a_n x^{n+r} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=2}^{\infty} a_{n-2}\, x^{n+r} + \sum_{n=0}^{\infty} [2(n+r)(n+r-1)+(n+r)-1]\, a_n\, x^{n+r} = 0$$

$$[2r(r-1)+r-1]a_0 x^r + [2(r+1)r+(r+1)-1]a_1 x^{r+1}$$

$$+\sum_{n=2}^{\infty}\left[a_{n-2}+2(n+r)(n+r-1)+(n+r)-1\right]a_n x^{n+r} = 0$$

Equating the coefficient of the lowest degree term (i.e. x^r) to zero, the indicial equation is

$$[2r(r-1)+r-1]a_0 = 0$$

When $a_0 \neq 0$, we get

$$2r^2 - r - 1 = 0$$

$$2r^2(r-1)+(r-1) = 0$$

$$(r-1)(2r+1) = 0$$

$$r = 1,\ r = -\frac{1}{2}$$

Equating the coefficient of the degree term (i.e., x^{r+1}) to zero,

$$[2r^2+2r+r+1-1]\, a_1 = 0$$

$$(2r^2+3r)\, a_1 = 0$$

$$\because \quad (2r^2+3r) \neq 0, \quad a_1 = 0$$

Equating the coefficient of x^{n+r} to zero,

$$a_n = \frac{a_{n-2}}{2(n+r)(n+r-1)+(n+r)-1}, \quad n \geq 2 \qquad \text{...(3)}$$

For the first solution, putting $r = -\frac{1}{2}$ in Eq. (3),

$$a_n = -\frac{a_{n-2}}{2\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)+\left(n-\frac{1}{2}\right)-1}, \quad n \geq 2$$

Putting $n = 2, 3, 4, \ldots$

$$a_1 = 0$$

$$a_2 = -\frac{1}{2\cdot\frac{3}{2}\cdot\frac{1}{2}+\frac{3}{2}-1}a_0 = -\frac{1}{3-1}a_0 = -\frac{1}{2}a_0$$

$$a_3 = -\frac{1}{9}a_1 = 0$$

$$a_4 = -\frac{1}{40}a_0$$

$$a_5 = 0$$

$$a_6 = -\frac{1}{2160}a_0$$

and so on.

Putting $r = -\frac{1}{2}$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$y_1 = x^r \sum_{n=0}^{\infty} a_n x^n$$

$$= x^{-\frac{1}{2}} (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$

$$= x^{-\frac{1}{2}} \left(a_0 - \frac{1}{2} a_0 x^2 + \frac{1}{40} a_0 x^4 - \frac{1}{2160} a_0 x^6 + \cdots \right)$$

$$= a_0 x^{-\frac{1}{2}} \left(1 - \frac{x^2}{2} + \frac{x^4}{40} - \frac{x^6}{2160} + \cdots \right)$$

For the second solution, putting $r = 1$ in Eq. (3),

$$a_n = -\frac{a_{n-2}}{2(n+1)n + n}, \qquad n \geq 2$$

$$a_n = -\frac{a_{n-2}}{n(2n+3)}, \qquad n \geq 2$$

Putting $n = 2, 3, 4, \ldots$

$$a_1 = 0$$

$$a_2 = -\frac{1}{2 \cdot (4+3)} a_1 = -\frac{1}{14} a_0$$

$$a_3 = -\frac{1}{3 \cdot 9} a_1 = 0$$

$$a_4 = -\frac{1}{4(11)} a_2 = -\frac{1}{44} a_2 = \frac{1}{616} a_0$$

and so on.

Substituting $r = 1$ and values of a_1, a_2, a_3, a_4 in Eq. (2),

$$y_2 = x^r \sum_{n=0}^{\infty} a_n x^n = x(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$= x \left(a_0 - \frac{1}{14} a_0 x^2 + \frac{1}{616} a_0 x^4 + \cdots \right) \qquad \ldots(5)$$

Hence, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 a_0 x^{-\frac{1}{2}} \left(1 - \frac{x^2}{2} + \frac{x^4}{40} - \frac{x^6}{2160} + \cdots \right)$$

$$+ c_2 a_0 x \left(1 - \frac{x^2}{14} + \frac{x^4}{616} + \cdots \right) \quad \text{[From Eq. (4) and Eq. (5)]}$$

$$y = Ax^{-\frac{1}{2}}\left(1-\frac{x^2}{2}+\frac{x^4}{40}-\frac{x^6}{2160}+\cdots\right)+Bx\left(1-\frac{x^2}{14}+\frac{x^4}{616}+\cdots\right)$$

where $A = c_1 a_0 \quad B = c_2 a_0$

Example 10

Find the series solution of $8x^2y'' + 10xy' - (1+x)y = 0.$ **[Summer 2018]**

Solution

$$8x^2y'' + 10xy' - (1+x)y = 0 \qquad \text{...(1)}$$

$$P_0(x) = 8x^2 = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{5}{4x}y' - \left(\frac{1+x}{8x^2}\right)y = 0$$

$$P(x) = \frac{5}{4x}, \quad Q(x) = -\frac{1+x}{8x^2}$$

Since $xP(x) = \frac{5}{4}$ and $x^2Q(x) = -\frac{1+x}{8}$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r(a_0 + a_1x + a_2x^2 + \cdots) \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r)\, x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2}$$

Substituting in Eq. (1),

$$8x^2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)\,x^{n+r-2} + 10x\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} - (1+x)\sum_{n=0}^{\infty} a_n\, x^{n+r} = 0$$

$$8\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)\,x^{n+r} + 10\sum_{n=0}^{\infty} a_n(n+r)x^{n+r} - \sum_{n=0}^{\infty} a_n\, x^{n+r} - \sum_{n=0}^{\infty} a_n\, x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}[8(n+r)(n+r-1)+10(n+r)-1]\,a_n\,x^{n+r}-\sum_{n=0}^{\infty}a_n\,x^{n+r+1}=0$$

To obtain a common power of x in each term, putting $n=m-1$ in the second term, we get

$$\sum_{n=0}^{\infty}[2(n+r)(4n+4r+1)-1]\,a_n\,x^{n+r}-\sum_{m=1}^{\infty}a_{m-1}\,x^{m+r}=0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=0}^{\infty}[2(n+r)(4n+4r+1)-1]\,a_n\,x^{n+r}-\sum_{n=1}^{\infty}a_{n-1}\,x^{n+r}=0$$

$$[2r(4r+1)-1]a_0x^r+\sum_{n=1}^{\infty}[2(n+r)(4n+4r+1)-1]a_nx^{n+r}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r}=0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$a_0[2r(4r+1)-1]=0$$

$$a_0[8r^2+2r-1]=0$$

$$r=\frac{1}{4},\ r=-\frac{1}{2} \qquad [\because a_0\neq 0]$$

Equating the coefficient of x^{n+r} to zero,

$$[2(n+r)(4n+4r+1)-1]a_n-a_{n-1}=0, \quad n\geq 1$$

$$a_n=\frac{1}{2(n+r)(4n+4r+1)-1}a_{n-1} \qquad \text{...(3)}$$

For the first solution, putting $r=\frac{1}{4}$ in Eq. (3),

$$a_n=\frac{1}{2\left(n+\frac{1}{4}\right)(4n+1+1)-1}a_{n-1}, \quad n\geq 1$$

$$=\frac{2}{(4n+1)(4n+2)-2}a_{n-1}, \quad n\geq 1$$

$$=\frac{1}{(4n+1)(2n+1)-1}a_{n-1}, \quad n\geq 1$$

Putting $n=1, 2, 3, \ldots$

$$a_1=\frac{1}{14}a_0$$

$$a_2 = \frac{1}{44}a_1 = \frac{1}{44}\cdot\frac{1}{14}a_0 = \frac{1}{616}a_0$$

and so on.

Substituting $r = \frac{1}{4}$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$y_1 = x^{\frac{1}{4}}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$= a_0 x^{\frac{1}{4}}\left(1 + \frac{1}{14}x + \frac{1}{616}x^2 + \cdots\right)$$

For the second solution, putting $r = -\frac{1}{2}$ in Eq. (3),

$$a_n = \frac{1}{2\left(n - \frac{1}{2}\right)(4n - 2 + 1) - 1}, \quad n \geq 1$$

$$= \frac{1}{(2n-1)(4n-1)-1}a_{n-1}, \quad n \geq 1$$

Putting $n = 1, 2, 3, \ldots$

$$a_1 = \frac{1}{2}a_0$$

$$a_2 = \frac{1}{20}a_0$$

and so on.

Substituting $r = -\frac{1}{2}$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$y_2 = x^{-\frac{1}{2}}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$= a_0 x^{-\frac{1}{2}}\left(1 + \frac{1}{2}x + \frac{1}{20}x^2 + \cdots\right)$$

Hence, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1\, a_0 x^{\frac{1}{4}}\left(1 + \frac{1}{14}x + \frac{1}{616}x^2 + \cdots\right) + c_2\, a_0 x^{-\frac{1}{2}}\left(1 + \frac{1}{2}x + \frac{1}{20}x^2 + \cdots\right)$$

$$y = Ax^{\frac{1}{4}}\left(1 + \frac{1}{14}x + \frac{1}{616}x^2 + \cdots\right) + Bx^{-\frac{1}{2}}\left(1 + \frac{1}{2}x + \frac{1}{20}x^2 + \cdots\right)$$

where $A = c_1 a_0, \quad B = c_2 a_0$

Example 11

Find the series solution of the differential equation

$$(x^2 - x)y'' - xy' + y = 0 \text{ about } x = 0$$

Solution

$$(x^2 - x)y'' - xy' + y = 0 \quad \text{...(1)}$$

$$P_0(x) = x^2 - x = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' - \frac{1}{x-1}y' + \frac{1}{x(x-1)}y = 0$$

$$P(x) = -\frac{1}{x-1}, \quad Q(x) = \frac{1}{x(x-1)}$$

Since $xP(x) = -\frac{x}{x-1}$ and $x^2Q(x) = \frac{x}{x-1}$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r(a_0 + a_1 x + a_2 x^2 + \cdots) \quad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r)x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2}$$

Substituting in Eq. (1),

$$(x^2 - x)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) - (n+r) + 1\right]a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} = 0$$

$$-\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}\left[(n+r)(n+r-2) + 1\right]a_n x^{n+r} = 0$$

To obtain a common power of x in each term, putting $n = m + 1$ in the first term, we get

$$-\sum_{m=-1}^{\infty}(m+1+r)(m+r)a_{m+1}x^{m+r}+\sum_{n=0}^{\infty}\left[(n+r)(n+r-2)+1\right]a_n x^{n+r}=0$$

Since m is a dummy variable, replacing m by n and multiplying by the negative sign,

$$\sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r}-\sum_{n=0}^{\infty}\left[(n+r)(n+r-2)+1\right]a_n x^{n+r}=0$$

$$r(r-1)a_0x^{r-1}+\sum_{n=0}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r}-\sum_{n=0}^{\infty}\left[(n+r)(n+r-2)+1\right]a_n x^{n+r}=0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$r(r-1)\,a_0=0$$

$$r=0,\quad r=1 \qquad [\because\; a_0\neq 0]$$

Equating the coefficient of x^{n+r} to zero,

$$(n+r+1)(n+r)a_{n+1}-\left[(n+r)(n+r-2)+1\right]a_n=0,\quad n\geq 0$$

$$a_{n+1}=\frac{(n+r)(n+r-2)+1}{(n+r)(n+r+1)}a_n,\qquad n\geq 0$$

$$=\frac{(n+r)^2-2(n+r)+1}{(n+r)(n+r+1)}a_n,\quad n\geq 0$$

$$=\frac{(n+r-1)^2}{(n+r)(n+r+1)}a_n,\qquad n\geq 0$$

Putting $n = 0, 1, 2, \ldots$

$$a_1=\frac{(r-1)^2}{r(r+1)}a_0$$

$$a_2=\frac{r^2}{(r+1)(r+2)}a_1$$

and so on.

At $r = 0$, $a_1 = \infty$ and, hence, $a_2 = a_3 = \ldots = \infty$

To obtain the solution of Eq. (1), the procedure is modified by assuming $a_0 = c_0r$, $c_0 \neq 0$. Substituting $a_0 = c_0r$ in $a_1, a_2, \ldots$

$$a_1=\frac{(r-1)^2}{r(r+1)}c_0r=\frac{(r-1)^2}{(r+1)}c_0$$

$$a_2 = \frac{r^2}{(r+1)(r+2)} \frac{(r-1)^2}{(r+1)} c_0 = \frac{r^2(r-1)^2}{(r+1)^2(r+2)} c_0$$

and so on.

Substituting in Eq. (2),

$$y = c_0 x^r \left[r + \frac{(r-1)^2}{(r+1)} x + \frac{r^2(r-1)^2}{(r+1)^2(r+2)} x^2 + \cdots \right] \quad ...(3)$$

For the first solution, putting $r = 0$ in Eq. (3),

$$y_1 (y)_{r=0} = c_0(0 + x + 0) = c_0 x$$

For the second solution, putting $r = 1$ in Eq. (3),

$$y_2 = (y)_{r=1} = c_0 x = y_1$$

Hence, differentiating Eq. (3) w.r.t. r to obtain the second solution,

$$\frac{\partial y}{\partial r} = c_0 x^r \log x \left[r + \frac{(r-1)^2}{(r+1)} x + \begin{matrix}\text{(terms containing higher powers of}\\ r \text{ in the numerator)}\end{matrix} \right]$$

$$+ c_0 x^r \left[1 + \frac{2(r-1)}{r+1} x - \frac{(r-1)^2}{(r+1)^2} x + \begin{matrix}\text{(terms containing higher powers of}\\ r \text{ in the numerator)}\end{matrix} \right]$$

Putting $r = 0$,

$$y_2 = \left(\frac{\partial y}{\partial r} \right)_{r=0} = c_0 \log x(0 + x + 0) + c_0(1 - 2x - x + 0)$$

$$= c_0(x \log x + 1 - 3x)$$

Hence, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 c_0 x + c_2 c_0 (x \log x + 1 - 3x)$$

$$= Ax + B(x \log x + 1 - 3x)$$

$$= (A + B \log x)x + B(1 - 3x)$$

where $A = c_1 c_0$, $B = c_2 c_0$

Example 12

Solve in series the differential equation $xy'' + 2y' + xy = 0$.

Solution

$$xy'' + 2y' + xy = 0 \quad ...(1)$$

$$P_0(x) = x = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{2}{x}y' + y = 0$$

$$P(x) = \frac{2}{x}, \qquad Q(x) = 1$$

Since $xP(x) = 2$ and $x^2Q(x) = x^2$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r(a_0 + a_1 x + a_2 x^2 + \cdots) \qquad \ldots(2)$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

Substituting in Eq. (1),

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + x\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1+2)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r+1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

To obtain a common power of x in each term, putting $n = m + 2$ in the first term, we get

$$\sum_{m=-2}^{\infty}(m+2+r)(m+2+r+1)a_{m+2}x^{m+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-2}^{\infty}(n+r+2)(n+r+3)a_{n+2}x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$r(r+1)a_0 x^{r-1} + (r+1)(r+2)a_1 x^r + \sum_{n=0}^{\infty}(n+r+2)(n+r+3)a_{n+2}x^{n+r+1}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$a_0 r(r+1) = 0$$

$$r = 0, \quad r = -1 \quad [\because \quad a_0 \neq 0]$$

Equating the coefficient of x^r to zero,

$$a_1(r+1)(r+2) = 0$$

$$a_1 \neq 0 \qquad [\because r = -1]$$

Thus, for the smaller root $r = -1$, a_1 is an arbitrary constant.

For $r = 0$, $a_1 = 0$.

Equating the coefficient of x^{n+r+1} to zero,

$$(n+r+2)(n+r+3)a_{n+2} + a_n = 0, \qquad n \geq 0$$

$$a_{n+2} = -\frac{1}{(n+r+2)(n+r+3)}a_n, \qquad n \geq 0 \qquad \text{...(3)}$$

For the first solution, putting $r = -1$,

$$a_{n+2} = -\frac{1}{(n+1)(n+2)}a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, \ldots$

$$a_2 = -\frac{1}{2}a_0 = -\frac{1}{2!}a_0$$

$$a_3 = -\frac{1}{2\cdot 3}a_1 = -\frac{1}{3!}a_1$$

$$a_4 = -\frac{1}{3\cdot 4}a_2 = -\frac{1}{3\cdot 4}\left(-\frac{1}{2!}a_0\right) = \frac{1}{4!}a_0$$

$$a_5 = -\frac{1}{4\cdot 5}a_3 = -\frac{1}{4\cdot 5}\left(-\frac{1}{3!}a_1\right) = \frac{1}{5!}a_1$$

and so on.

Substituting $r = -1$ and values of $a_1, a_2, a_3, \ldots$ in Eq. (2),

$$y_1 = (y)_{r=-1}$$

$$= x^{-1}\left(a_0 + a_1 x - \frac{1}{2!}a_0 x^2 - \frac{1}{3!}a_1 x^3 + \frac{1}{4!}a_0 x^4 + \frac{1}{5!}a_1 x^5 - \cdots\right)$$

$$= \frac{a_0}{x}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right) \qquad \text{...(4)}$$

For the second solution, putting $r = 0$ in Eq. (3),

$$a_{n+2} = -\frac{1}{(n+2)(n+3)}a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, \ldots$

$$a_2 = -\frac{1}{2\cdot 3}a_0 = -\frac{1}{3!}a_0$$

$$a_3 = -\frac{1}{3\cdot 4}a_1 = 0 \quad [\because\ a_1 = 0]$$

$$a_4 = -\frac{1}{4\cdot 5}a_2 = -\frac{1}{4\cdot 5}\left(-\frac{1}{3!}a_0\right) = \frac{1}{5!}a_0$$

$$a_5 = -\frac{1}{5\cdot 6}a_3 = 0$$

and so on.

Substituting $r = 0$ and values of $a_1, a_2, a_3, \ldots$ in Eq. (2),

$$y_2 = (y)_{r=0}$$

$$= x^0\left(a_0 + 0\cdot x - \frac{1}{3!}a_0x^2 + 0\cdot x^3 + \frac{1}{5!}a_0x^4 + 0\cdot x^5 - \cdots\right)$$

$$= a_0\left(1 - \frac{x^2}{3!} - \frac{x^4}{5!} - \cdots\right)$$

But this solution is a constant multiple of the first solution in Eq. (4).

Hence, the solution represented by Eq. (4) is the required general solution with two arbitrary constants as a_0 and a_1.

EXERCISE 6.3

Find the series solutions of the following differential equations by the Frobenius method:

1. $4x^2y'' - 8xy' + 5y = 0$

$$\left[\text{Ans.: } y = Ax^{\frac{1}{2}} + Bx^{\frac{5}{2}}\right]$$

2. $2x^2y'' + (2x^2 - x)y' + y = 0$

$$\left[\text{Ans.: } y = A\sqrt{x}\left(1 - x + \frac{x^2}{2} - \cdots\right) + Bx\left(1 - \frac{2}{3}x + \frac{2^2}{3\cdot 5}x^2 - \cdots\right)\right]$$

3. $(2x + x^3)y'' - y' - 6xy = 0$

$$\left[\text{Ans.: } y = c_1\left(1 + 3x^2 + \frac{3}{5}x^4 + \cdots\right) + c_2x^{\frac{3}{2}}\left(1 + \frac{3}{8}x^2 - \frac{3}{128}x^4 + \cdots\right)\right]$$

4. $2xy'' + (x + 1)y' + 3y = 0$

$$\left[\text{Ans.: } y = c_1(1 - 3x + 2x^2 - \cdots) + c_2\sqrt{x}\left(1 - \frac{7}{6}x + \frac{21}{40}x^2 - \cdots\right)\right]$$

5. $3xy'' - (x-2)y' + 2y = 0$

$$\left[\textbf{Ans.: } y = A\left(1 - x + \frac{1}{10}x^2 - \cdots\right) + Bx^{\frac{1}{3}}\left(1 - \frac{5}{12}x + \frac{5}{252}x^2 - \cdots\right)\right]$$

6. $x^2y'' + x(x-1)y' + (1-x)y = 0$

$$\left[\textbf{Ans.: } y = x(A + B\log x) - Bx^2\left(1 - \frac{x}{4} + \frac{x^2}{18} - \cdots\right)\right]$$

7. $xy'' + (1-2x)y' + (x-1)y = 0$

$$\left[\textbf{Ans.: } y = (A + B\log x)\left(1 + x + \frac{2}{1^2 \cdot 2^2}x^2 + \cdots\right)\right]$$

8. $(x + x^2)y'' + (1+x)y' - y = 0$

$$\left[\textbf{Ans.: } y = (1+x)(A + B\log x) - B\left(2x + \frac{x^2}{2} - \frac{x^3}{6} - \cdots\right)\right]$$

9. $x^2y'' + 4xy' + (x^2 + 2)y = 0$

$$\left[\textbf{Ans.: } y = \frac{1}{x^2}(A\cos x + B\sin x)\right]$$

10. $x^2y'' + 6xy + (6 - 4x^2)y = 0$

$$\left[\textbf{Ans.: } y = Ax^{-3}\left(1 + 2x^2 + \frac{2}{3}x^4 + \cdots\right) + c_1x^{-3}\left(x + \frac{2}{3}x^3 + \frac{2}{15}x^5 + \cdots\right)\right]$$

11. $x^2y'' + xy' + (x^2 - 1)y = 0$

$$\left[\textbf{Ans.: } y = (A + B\log x)\left(x - \frac{x^3}{2 \cdot 4} + \cdots\right) + \frac{B}{x}\left[1 + \frac{x^2}{2^2} - \left(\frac{2}{2^3 \cdot 4} + \frac{1}{2^2 \cdot 4^2}\right)x^4 + \cdots\right]\right]$$

12. $x(1+x)y'' + (x+5)y' - 4y = 0$

$$\left[\textbf{Ans.: } y = A\left(1 + \frac{4}{5}x + \frac{x^2}{5} + \cdots\right) + Bx^{-4}(1 + 4x + 5x^2 + \cdots)\right]$$

13. $x^2y'' + x^3y' + (x^2 - 2)y = 0$

$$\left[\textbf{Ans.: } y = A\left(x^2 - \frac{3}{10}x^4 + \frac{3}{56}x^4 - \cdots\right)\right] + \frac{B}{x}$$

6.5 BESSEL'S EQUATION

The linear second order differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \quad ...(6.7)$$

is called Bessel's equation, where n is a nonnegative real constant.

The solutions of Bessel's equation are called Bessel functions. These functions have diverse applications in physics and engineering as propagation of electromagnetic waves, potential theory, and diffusion involving circular symmetry. They also appear in the study of oscillations of a hanging chain, motion of planets, and vibrations of a circular drumhead.

6.6 BESSEL'S FUNCTIONS OF THE FIRST KIND

In Eq. (6.7), $x = 0$ is a regular singular point. Hence, it is solved using Frobenius method.

Let the series solution of Eq. (6.7) about $x = 0$ be

$$y=\sum_{m=0}^{\infty}a_m x^{m+r}=x^r(a_0+a_1x+a_2x^2+\cdots) \quad ...(6.8)$$

$$\frac{dy}{dx}=\sum_{m=0}^{\infty}a_m(m+r)x^{m+r-1}$$

$$\frac{d^2y}{dx^2}=\sum_{m=0}^{\infty}a_m(m+r)(m+r-1)x^{m+r-2}$$

Substituting in Eq. (6.7),

$$x^2\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2}x+\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1}+(x^2-n^2)\sum_{m=0}^{\infty}a_m x^{m+r}=0$$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r}+\sum_{m=0}^{\infty}(m+r)a_m x^{m+r}+\sum_{m=0}^{\infty}a_m x^{m+r+2}-n^2\sum_{m=0}^{\infty}a_m x^{m+r}=0$$

$$\sum_{m=0}^{\infty}\left[(m+r)^2-n^2\right]a_m x^{m+r}+\sum_{m=0}^{\infty}a_m x^{m+r+2}=0$$

To obtain a common power of x in each term, putting $m=t+2$ in the first term,

$$\sum_{t=-2}^{\infty}\left[(t+2+r)^2-n^2\right]a_{t+2}x^{t+2+r}+\sum_{m=0}^{\infty}a_m x^{m+r+2}=0$$

Since t is a dummy variable, replacing t by m,

$$\sum_{m=-2}^{\infty}\left[(m+r+2)^2-n^2\right]a_{m+2}x^{m+r+2}+\sum_{m=0}^{\infty}a_m x^{m+r+2}=0$$

$$(r^2-n^2)a_0x^r+\left[(r+1)^2-n^2\right]a_1x^{r+1}+\sum_{m=0}^{\infty}\left[\left\{(m+r+2)^2-n^2\right\}a_{m+2}+a_m\right]x^{m+r+2}=0$$

Equating the coefficient of lowest degree term (i.e., x^r) to zero, the indicial equation is

$$(r^2-n^2)a_0=0$$

$$r^2-n^2=0 \qquad [\because\ a_0\neq 0]$$

$$r=\pm n$$

Equating the coefficient of x^{r+1} to zero,

$$\left[(r+1)^2-n^2\right]a_1=0$$

$a_1=0$ for $r=\pm n$

Equating the coefficient of x^{m+r+2} to zero,

$$\left[(m+r+2)^2-n^2\right]a_{m+2}+a_m=0, \quad m\geq 0$$

$$a_{m+2}=-\frac{1}{\left[(m+r+2)^2-n^2\right]}a_m, \qquad m\geq 0$$

$$=-\frac{1}{(m+r+2-n)(m+r+2+n)}a_m,\ m\geq 0 \qquad \text{...(6.9)}$$

Since $a_1=0$, all odd coefficients are zero i.e., $a_3=a_5=\cdots=0$.

For the first solution, putting $r=n$ in Eq. (6.9),

$$a_{m+2}=-\frac{1}{(m+2)(m+2n+2)}a_m, \quad m\geq 0$$

Putting $m=0, 2, \ldots$

$$a_2=-\frac{1}{2(2n+2)}a_0=-\frac{1}{2^2(n+1)}a_0$$

$$a_4=-\frac{1}{4(2n+4)}a_2=-\frac{1}{2^3(n+2)}\left[-\frac{1}{2^2(n+1)}a_0\right]=\frac{1}{2^4 2!(n+1)(n+2)}a_0$$

and so on.

Substituting $r=n$ and values of $a_1, a_2, a_3, \ldots$ in Eq. (6.8),

$$y_1=(y)_{r=n}=x^n\left[a_0-\frac{1}{2^2(n+1)}a_0x^2+\frac{1}{2^4 2!(n+1)(n+2)}a_0x^4-\cdots\right]$$

$$=a_0x^n\left[1-\frac{1}{1!(n+1)}\left(\frac{x}{2}\right)^2+\frac{1}{2!(n+1)(n+2)}\left(\frac{x}{2}\right)^4-\cdots\right]$$

Multiplying and dividing each term by $\lceil n+1$,

$$y_1 = a_0 x^n \left[\frac{\lceil n+1}{\lceil n+1} - \frac{\lceil n+1}{1!(n+1)\lceil n+1}\left(\frac{x}{2}\right)^2 + \frac{\lceil n+1}{2!(n+2)(n+1)\lceil n+1}\left(\frac{x}{2}\right)^4 - \dots \right]$$

$$= a_0 x^n \lceil n+1 \left[\frac{1}{\lceil n+1} - \frac{1}{1!\lceil n+2}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\lceil n+3}\left(\frac{x}{2}\right)^4 - \dots \right]$$

Since a_0 is arbitrary, assuming $a_0 = \dfrac{1}{2^n \lceil n+1}$, the solution is reduced to the simpler form as

$$y_1 = \frac{x^n}{2^n}\left[\frac{1}{\lceil n+1} - \frac{1}{1!\lceil n+2}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\lceil n+3}\left(\frac{x}{2}\right)^4 - \dots \right]$$

$$= \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\lceil n+m+1}\left(\frac{x}{2}\right)^{2m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\lceil n+m+1}\left(\frac{x}{2}\right)^{2m+n}$$

which represents one of the linearly independent solutions of Bessel's equation. This is called Bessel function of the first kind of order n and is denoted by $J_n(x)$.

Hence,
$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\lceil n+m+1}\left(\frac{x}{2}\right)^{2m+n} \qquad \text{...(6.10)}$$

For the second solution, replacing n by $-n$ in Eq. (6.10),

$$y_2 = (y)_{r=-n} = J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\lceil -n+m+1}\left(\frac{x}{2}\right)^{2m-n} \qquad \text{...(6.11)}$$

If n is not an integer, a general solution of Bessel's equation for all $x \neq 0$ is

$$y(x) = A\,J_n(x) + B\,J_{-n}(x) \qquad \text{...(6.12)}$$

If n is an integer then Eq. (6.12) is not a general solution of Eq. (6.7) because of linear dependence of $J_n(x)$ and $J_{-n}(x)$.

In this case

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\lceil m-n+1}\left(\frac{x}{2}\right)^{2m-n}$$

$$= \sum_{m=n}^{\infty} \frac{(-1)^m}{m!(m-n)!}\left(\frac{x}{2}\right)^{2m-n} \qquad [\because n \text{ is an integer}]$$

Putting $m-n=k$,

$$J_{-n}(x)=\sum_{k=0}^{\infty}\frac{(-1)^{n+k}}{(n+k)!k!}\left(\frac{x}{2}\right)^{2(n+k)-n}$$

$$=(-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{2k+n}$$

$$=(-1)^n J_n(x)$$

Hence, if n is an integer

$$J_{-n}(x)=(-1)^n J_n(x) \qquad \text{...(6.13)}$$

Bessel Functions of the First Kind of Order Zero and One

For $n=0$, Eq. (6.10) reduces to

$$J_0(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}$$

$$=1-\frac{x^2}{2^2(1!)^2}+\frac{x^4}{2^4(2!)^2}-\ldots\ldots$$

which is known as Bessel functions of order zero.

At $x=0$, $J_0(0)=1$

Hence, the graph of $y=J_0(x)$ intersects the y axis at the point (0, 1).

For $n=1$, Eq. (6.10) reduces to

$$J_1(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!(m+1)!}\left(\frac{x}{2}\right)^{2m+1}$$

$$=\frac{x}{2}-\frac{x^3}{2^3(1!)(2!)}+\frac{x^5}{2^5(2!)(3!)}-\ldots\ldots$$

which is known as Bessel functions of order one.

At $x=0$, $J_1(0)=0$

Hence, the graph of $y=J_1(x)$ passes through the origin.

Since both the functions have alternating series, their graphs are oscillatory in nature. The height of the waves decreases as x increases (Fig. 6.1).

The zeros of these functions are not regularly spaced.

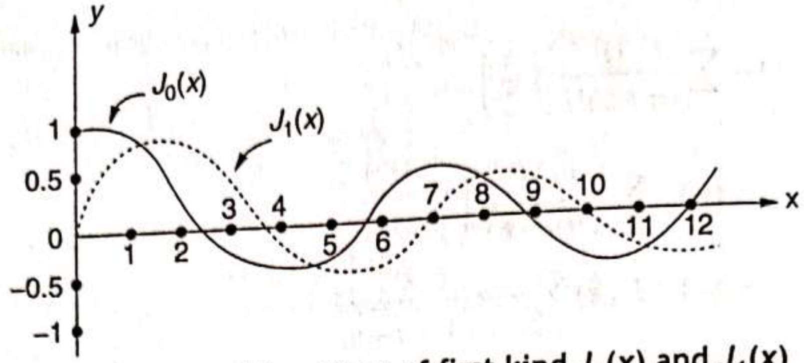

Fig. 6.1 Bessel functions of first kind $J_0(x)$ and $J_1(x)$

6.7 RECURRENCE FORMULAE FOR $J_n(x)$

Bessel's function of the first kind satisfies the following recurrence formulae. These formulae are very useful in establishing the various properties and relations of the function and are very important in applications.

(1) $\dfrac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

Proof

$$x^n J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\left|\overline{n+m+1}\right.}\left(\frac{x}{2}\right)^{2m+n}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\left|\overline{n+m+1}\right.}\frac{x^{2m+2n}}{2^{2m+n}}$$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\left|\overline{n+m+1}\right.}\frac{(2m+2n)x^{2m+2n-1}}{2^{2m+n}}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)\left|\overline{n+m}\right.}\frac{(m+n)x^{2m+n-1}}{2^{2m+n-1}}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\left|\overline{n+m}\right.}\left(\frac{x}{2}\right)^{2m+n-1} = x^n J_{n-1}(x) \qquad \text{...(6.14)}$$

Note: This formula can also be written as

$$\int x^n J_{n-1}(x)\,dx = x^n J_n(x) + c \qquad \text{...(6.15)}$$

(2) $\dfrac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

Proof

$$x^{-n} J_n(x) = x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\left|\overline{n+m+1}\right.}\left(\frac{x}{2}\right)^{2m+n}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \frac{x^{2m}}{2^{2m+n}}$$

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \frac{2m\, x^{2m-1}}{2^{2m+n}}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m-1)!\,\Gamma(n+m+1)} \frac{x^{2m-1}}{2^{2m+n-1}}$$

Putting $m - 1 = k$ and multiplying and dividing by x^{-n},

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^{k+1}}{k!\,\Gamma(n+k+1+1)} \frac{x^{2(k+1)-1}}{x^{-n}2^{2(k+1)+n-1}}$$

$$= -x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+1+k+1)} \left(\frac{x}{2}\right)^{2k+n+1}$$

$$= -x^{-n}J_{n+1}(x) \qquad \text{...(6.16)}$$

Note: This formula can also be written as

$$\int x^{-n}J_{n+1}(x)\, dx = -x^{-n}J_n(x) + c \qquad \text{...(6.17)}$$

(3) $J_n(x) = \dfrac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

Proof From Eq. (6.14),

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

$$nx^{n-1}J_n(x) + x^n J_n'(x) = x^n J_{n-1}(x)$$

$$\frac{n}{x}J_n(x) + J_n'(x) = J_{n-1}(x) \qquad \text{...(6.18)}$$

From Eq. (6.16),

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$$

$$-nx^{-n-1}J_n(x) + x^{-n}J_n'(x) = -x^{-n}J_{n+1}(x)$$

$$-\frac{n}{x}J_n(x) + J_n'(x) = -J_{n+1}(x) \qquad \text{...(6.19)}$$

Subtracting Eq. (6.19) from Eq. (6.18),

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right] \qquad \text{...(6.20)}$$

(4) $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

Proof Adding Eqs (6.18) and (6.19),

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right] \quad \text{...(6.21)}$$

(5) $J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$

Proof From Eq. (6.19),

$$J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x) \quad \text{...(6.22)}$$

(6) $J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$

Proof Equating Eqs (6.21) and (6.22),

$$\frac{1}{2}J_{n-1}(x) - \frac{1}{2}J_{n+1}(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$$

$$\frac{1}{2}J_{n+1}(x) = \frac{n}{x}J_n(x) - \frac{1}{2}J_{n-1}(x)$$

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

EXAMPLE 1

Show that (a) $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ *(b)* $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$ *and deduce the values of* $J_{\frac{3}{2}}(x), J_{-\frac{3}{2}}(x), J_{\frac{5}{2}}(x)$ *and* $J_{-\frac{5}{2}}(x)$. *Also, prove that*

$$\left[J_{\frac{1}{2}}(x)\right]^2 + \left[J_{-\frac{1}{2}}(x)\right]^2 = \frac{2}{\pi x}.$$

Solution

(a) $$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\overline{|m+n+1}}\left(\frac{x}{2}\right)^{2m+n}$$

$$=\left(\frac{x}{2}\right)^n\left[\frac{1}{\overline{|n+1}}-\frac{1}{1!\overline{|n+2}}\left(\frac{x}{2}\right)^2+\frac{1}{2!\overline{|n+3}}\left(\frac{x}{2}\right)^4-\cdots\right]$$

$$=\frac{x^n}{2^n\overline{|n+1}}\left[1-\frac{x^2}{2^2(n+1)}+\frac{x^4}{2!2^4(n+1)(n+2)}-\cdots\right] \quad \ldots(1)$$

Putting $n=\frac{1}{2}$,

$$J_{1/2}(x)=\frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}\overline{\left|\frac{3}{2}\right.}}\left[1-\frac{x^2}{2^2\cdot\frac{3}{2}}+\frac{x^4}{2!\,2^4\cdot\frac{3}{2}\cdot\frac{5}{2}}-\cdots\cdots\right]$$

$$=\frac{\sqrt{x}}{\sqrt{2}.\frac{1}{2}\overline{\left|\frac{1}{2}\right.}}\left[1-\frac{x^2}{2\cdot 3}+\frac{x^4}{2\cdot 3\cdot 4\cdot 5}-\cdots\right]$$

$$=\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}\,x}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right]$$

$$=\sqrt{\frac{2}{\pi x}}\sin x$$

(b) Putting $n=-\frac{1}{2}$ in Eq. (1),

$$J_{-\frac{1}{2}}(x)=\frac{x^{-\frac{1}{2}}}{2^{-\frac{1}{2}}\overline{\left|\frac{1}{2}\right.}}\left[1-\frac{x^2}{2^2\cdot\frac{1}{2}}+\frac{x^4}{2!\cdot 2^4\cdot\frac{1}{2}\cdot\frac{3}{2}}-\cdots\right]$$

$$=\sqrt{\frac{2}{\pi x}}\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right]$$

$$=\sqrt{\frac{2}{\pi x}}\cos x$$

Deductions

(i) From recurrence formula (3),

$$2nJ_n(x)=x\left[J_{n-1}(x)+J_{n+1}(x)\right]$$

$$J_{n+1}(x)=\frac{2n}{x}J_n(x)-J_{n-1}(x) \quad \ldots(2)$$

Putting $n = \frac{1}{2}$,

$$J_{\frac{3}{2}}(x) = \frac{1}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}} \sin x - \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$$

(ii) From recurrence formula (3),

$$J_{n-1}(x) = \frac{2n}{x} J_n(x) - J_{n+1}(x) \qquad \ldots(3)$$

Putting $n = -\frac{1}{2}$,

$$J_{-\frac{3}{2}}(x) = -\frac{1}{x} J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$= -\frac{1}{x}\sqrt{\frac{2}{\pi x}} \cos x - \sqrt{\frac{2}{\pi x}} \sin x$$

$$= -\sqrt{\frac{2}{\pi x}}\left[\frac{\cos x}{x} + \sin x\right]$$

(iii) Putting $n = \frac{3}{2}$ in Eq. (2),

$$J_{\frac{5}{2}}(x) = \frac{3}{x} J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$= \frac{3}{x}\sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right] - \sqrt{\frac{2}{\pi x}} \sin x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3 - x^2}{x^2}\right)\sin x - \frac{3}{x}\cos x\right]$$

(iv) Putting $n = -\frac{3}{2}$ in Eq. (3),

$$J_{-\frac{5}{2}}(x) = -\frac{3}{x} J_{-\frac{3}{2}}(x) - J_{-\frac{1}{2}}(x)$$

$$= \frac{3}{x} \cdot \sqrt{\frac{2}{\pi x}}\left[\frac{\cos x}{x} + \sin x\right] - \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3 - x^2}{x^2}\right)\cos x + \frac{3}{x}\sin x\right]$$

$$\left[J_{\frac{1}{2}}(x)\right]^2+\left[J_{-\frac{1}{2}}(x)\right]^2=\frac{2}{\pi x}\left(\sin^2 x+\cos^2 x\right)=\frac{2}{\pi x}$$

EXAMPLE 2

Prove that $J_2'(x)=\left(1-\frac{4}{x^2}\right)J_1(x)+\frac{2}{x}J_0(x)$ *where* $J_n(x)$ *is the Bessel function of first kind.*

Solution

From recurrence formula (4),

$$J_n'(x)=\frac{1}{2}\left[J_{n-1}(x)-J_{n+1}(x)\right] \quad ...(1)$$

From recurrence formula (5),

$$J_{n+1}(x)=\frac{n}{x}J_n(x)-J_n'(x) \quad ...(2)$$

Eliminating $J_{n+1}(x)$ from Eqs (1) and (2),

$$xJ_n'(x)=-n\,J_n(x)+xJ_{n-1}(x)$$

Putting $n=2$,

$$x\,J_2'(x)=-2J_2(x)+x\,J_1(x) \quad ...(3)$$

From recurrence formula (3),

$$2n\,J_n(x)=xJ_{n-1}(x)+xJ_{n+1}(x)$$

Putting $n=1$,

$$2J_1(x)=x\,J_0(x)+x\,J_2(x)$$

$$J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$$

Substituting in Eq. (3),

$$x\,J_2'(x)=-2\left[\frac{2}{x}J_1(x)-J_0(x)\right]+xJ_1(x)$$

$$J_2'(x)=\left(1-\frac{4}{x^2}\right)J_1(x)+\frac{2}{x}J_0(x)$$

EXAMPLE 3

Prove that $\frac{d}{dx}\left[J_n^2(x)\right]=\frac{x}{2n}\left[J_{n-1}^2(x)-J_{n+1}^2(x)\right]$.

Solution

$$\frac{d}{dx}\left[J_n^2(x)\right] = 2J_n(x)\cdot J_n'(x) \qquad \text{...(1)}$$

From recurrence formulae (3) and (4),

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

Substituting in Eq. (1),

$$\frac{d}{dx}\left[J_n^2(x)\right] = 2\cdot\frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]\cdot\frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

$$= \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$$

EXAMPLE 4

Prove that $\frac{d}{dx}\left[x^2 J_{n-1}(x)\, J_{n+1}(x)\right] = 2x^2 J_n(x) J_n'(x)$.

Solution

$$\frac{d}{dx}\left[x^2 J_{n-1}(x) J_{n+1}(x)\right] = \frac{d}{dx}\left[x^{-(n-1)} J_{n-1}(x)\, x^{n+1} J_{n+1}(x)\right]$$

$$= x^{n+1} J_{n+1}(x)\cdot\frac{d}{dx}\left[x^{-(n-1)} J_{n-1}(x)\right] + x^{-(n-1)} J_{n-1}(x)\,\frac{d}{dx}\left[x^{n+1} J_{n+1}(x)\right]$$

$$= x^{n+1} J_{n+1}(x)\left[-x^{-(n-1)} J_n(x)\right] + x^{-(n-1)} J_{n-1}(x)\cdot x^{n+1} J_n(x)$$

[Using recurrence formulae (5) and (6)]

$$= x^2 J_n(x)\left[-J_{n+1}(x) + J_{n-1}(x)\right]$$

$$= x^2 J_n(x)\cdot 2J_n'(x) \qquad \text{[Using recurrence formula (4)]}$$

$$= 2x^2 J_n(x) J_n'(x)$$

EXAMPLE 5

Evaluate $\int x^3 J_3(x)\, dx$.

Solution

$$\int x^3 J_3(x)\,dx = \int x^5 \cdot x^{-2} J_3(x)\,dx$$

$$= x^5 \cdot \int x^{-2} J_3(x)\,dx - \int \left\{5x^4 \int x^{-2} J_3(x)\,dx\right\} dx$$

$$= x^5\left[-x^{-2}J_2(x)\right] - 5\int x^4\left[-x^{-2}J_2(x)\right] dx \quad \text{[Using Eq. (6.17)]}$$

$$= -x^3 J_2(x) + 5\int x^2 J_2(x)\,dx \quad \text{...(1)}$$

Proceeding in the same manner,

$$\int x^2 J_2(x)\,dx = \int x^3 x^{-1} J_2(x)\,dx$$

$$= -x^2 J_1(x) + 3\int x J_1(x)\,dx$$

$$= -x^2 J_1(x) + 3\left[\int x\, x^o J_1(x)\,dx\right]$$

$$= -x^2 J_1(x) + 3\left[-x J_o(x) + \int J_o(x)\,dx\right] \quad \text{[Using Eq. (6.17)]}$$

$$= -x^2 J_1(x) - 3x J_o(x) + 3\int J_o(x)\,dx$$

Substituting in Eq. (1),

$$\int x^3 J_3(x)\,dx = -x^3 J_2(x) - 5x^2 J_1(x) - 15x J_0(x) + 15\int J_o(x)\,dx$$

Note: $\int J_o(x)\,dx$ cannot be integrated but its values are tabulated.

EXAMPLE 6

Prove that $\int J_3(x)\,dx = -J_2(x) - \dfrac{2J_1(x)}{x}$.

Solution

$$\int J_3(x)\,dx = \int x^{+2}\left[x^{-2}J_3(x)\right] dx$$

$$= x^2 \cdot \int x^{-2} J_3(x)\,dx - \int 2x \cdot \left\{\int x^{-2} J_3(x)\,dx\right\} dx$$

$$= x^2\left[-x^{-2}J_2(x)\right] - \int 2x\left\{-x^{-2}J_2(x)\right\} dx \quad \text{[Using Eq. (6.17)]}$$

$$= -J_2(x) + 2\int x^{-1} J_2(x)\,dx$$

$$= -J_2(x) + 2\left[-x^{-1}J_1(x)\right] \quad \text{[Using Eq. (6.17)]}$$

$$= -J_2(x) - \frac{2J_1(x)}{x}$$

EXAMPLE 7

Prove that $\int_0^1 x(1-x^2)J_0(ax)\,dx = -\frac{2}{a^2}J_0(a) + \frac{4}{a^3}J_1(a).$

Solution

Putting $ax = t,\quad dx = \frac{dt}{a},$

When $x = 0, t = 0$
When $x = 1, t = a$

$$\int_0^1 x(1-x^2)J_0(ax)dx = \int_0^a \frac{t}{a}\left(1-\frac{t^2}{a^2}\right)J_0(t)\frac{dt}{a}$$

$$= \frac{1}{a^4}\int_0^a (a^2-t^2)t\,J_0(t)\,dt$$

$$= \frac{1}{a^4}\left[\left|(a^2-t^2)t\,J_1(t)\right|_0^a - \int_0^a (-2t)tJ_1(t)\,dt\right] \qquad \text{[Using Eq. (6.15)]}$$

$$= \frac{1}{a^4}\left[0 + 2\int_0^a t^2 t^0 J_1(t)\,dt\right] \qquad ...(1)$$

$$= \frac{2}{a^4}\left[\left|t^2\left\{-t^0J_0(t)\right\}\right|_0^a - \int_0^a 2t\left\{-t^0J_0(t)\,dt\right\}\right] \qquad \text{[Using Eq. (6.14)]}$$

$$= \frac{2}{a^4}\left[-a^2J_0(a) + \left|2tJ_1(t)\right|_0^a\right]$$

$$= \frac{2}{a^4}\left[-a^2J_0(a) + 2aJ_1(a)\right]$$

$$= -\frac{2}{a^2}J_0(a) + \frac{4}{a^3}J_1(a)$$

EXAMPLE 8

Prove that $\int J_0(x)\cos x\,dx = xJ_0(x)\cos x + xJ_1(x)\sin x + c$

Solution

$$\int 1\cdot J_0(x)\cos x\,dx = xJ_0(x)\cos x - \int x\left[J_0'(x)\cos x - J_0(x)\sin x\right]dx$$

$$= xJ_0(x)\cos x - \int xJ_0'(x)\cos x\,dx + \int \sin x\cdot xJ_0(x)\,dx$$

$$= xJ_0(x)\cos x + \int xJ_1(x)\cos x\,dx + \sin x\int xJ_0(x)dx$$

$$- \int\left\{\cos x\cdot\int xJ_0(x)\,dx\right\}dx \qquad \left[\because\ J_0'(x) = -J_1(x)\right]$$

$$= xJ_0(x)\cos x + \int xJ_1(x)\cos x\,dx + \sin x\; xJ_1(x)$$

$$- \int \cos x \cdot xJ_1(x)\,dx + c \qquad \text{[Using Eq. (6.15)]}$$

$$= xJ_0(x)\cos x + xJ_1(x)\sin x + c$$

6.8 GENERATING FUNCTION FOR $J_n(x)$

Bessel function of first kind $J_n(x)$ can also be obtained by expanding the exponential function $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}$. This function is known as generating function for $J_n(x)$, where n is an integer.

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = e^{\frac{xt}{2}}\, e^{-\frac{x}{2t}}$$

$$= \left[1 + \left(\frac{xt}{2}\right) + \frac{1}{2!}\left(\frac{xt}{2}\right)^2 + \frac{1}{3!}\left(\frac{xt}{2}\right)^3 + \cdots + \frac{1}{n!}\left(\frac{xt}{2}\right)^n + \frac{1}{(n+1)!}\left(\frac{xt}{2}\right)^{n+1} + \cdots\right]$$

$$\left[1 - \left(\frac{x}{2t}\right) + \frac{1}{2!}\left(\frac{x}{2t}\right)^2 - \frac{1}{3!}\left(\frac{x}{2t}\right)^3 + \cdots + \frac{(-1)^n}{n!}\left(\frac{x}{2t}\right)^n + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2t}\right)^{n+1} + \cdots\right]$$

$$\text{Coefficient of } t^n = \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \cdots$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!}\left(\frac{x}{2}\right)^{n+2m} \qquad \text{...(6.23)}$$

$$= J_n(x)$$

$$\text{Coefficient of } t^{-n} = \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^{n+2}}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} + \cdots$$

$$= \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n - \frac{(-1)^n}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^n}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \cdots$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m(-1)^n}{m!\,(n+m)!}\left(\frac{x}{2}\right)^{n+2m}$$

$$= (-1)^n J_n(x) = J_{-n}(x) \qquad \text{...(6.24)}$$

Combining Eqs (6.23) and (6.24),

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

Note: $$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{-1} J_n(x)\,t^n + \sum_{n=0}^{\infty} J_n(x)\,t^n$$

$$= \sum_{-m=-\infty}^{-1} J_{-m}(x)t^{-m} + \sum_{n=0}^{\infty} J_n(x)\,t^n \qquad \text{[Putting } n=-m]$$

$$= \sum_{m=1}^{\infty}(-1)^m J_m(x)\,t^{-m} + J_0(x) + \sum_{n=1}^{\infty} J_n(x)\,t^n$$

$$= J_0(x) + \sum_{n=1}^{\infty}\left[t^n + (-1)^n t^{-n}\right] J_n(x) \qquad \text{...(6.25)}$$

EXAMPLE 1

Prove that $J_0^2(x)+2\left[J_1^2(x)+J_2^2(x)+J_3^2(x)+\cdots\right]=1.$

Solution

From Eq. (6.25),

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = J_0(x) + \sum_{n=1}^{\infty}\left[t^n + (-1)^n t^{-n}\right] J_n(x) \qquad \text{...(1)}$$

Replacing x by $-x$,

$$e^{-\frac{x}{2}\left(t-\frac{1}{t}\right)} = J_0(-x) + \sum_{n=1}^{\infty}\left[t^n + (-1)^n t^{-n}\right] J_n(-x)$$

$$= (-1)^0 J_0(x) + \sum_{n=1}^{\infty}\left[t^n + (-1)^n t^{-n}\right](-1)^n J_n(x)$$

$$\left[\because \; J_n(-x) = (-1)^n J_n(x)\right]$$

$$= J_0(x) + \sum_{n=1}^{\infty}\left[(-1)^n t^n + t^{-n}\right] J_n(x) \qquad \text{...(2)}$$

Multiplying Eqs (1) and (2),

$$1 = \left[J_0(x) + \sum_{n=1}^{\infty}\left\{t^n + (-1)^n t^{-n}\right\} J_n(x)\right]\cdot\left[J_0(x) + \sum_{n=1}^{\infty}\left\{(-1)^n t^n + t^{-n}\right\} J_n(x)\right]$$

$$= J_0^2(x) + \sum_{n=1}^{\infty}\left\{(-1)^n t^{2n} + 2 + (-1)^n t^{-2n}\right\} J_n^2(x)$$

$$+ J_0(x)\left[\sum_{n=1}^{\infty}\left\{(-1)^n t^n + t^{-n}\right\} J_n(x) + \sum_{n=1}^{\infty}\left\{t^n + (-1)^n t^{-n}\right\} J_n(x)\right]$$

Equating constant terms on both the sides,

$$1 = J_0^2(x) + \sum_{n=1}^{\infty} 2\, J_n^2(x)$$

$$J_0^2(x) + 2\left[J_1^2(x) + J_2^2(x) + J_3^2(x) + \cdots\right] = 1$$

EXAMPLE 2

Show that $\cos(x\sin\theta) = J_0(x) + (2\cos 2\theta)J_2(x) + (2\cos 4\theta)J_4(x) + \cdots\cdots$

Solution

Generating function of $J_n(x)$ is

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = J_0(x) + \left(t-\frac{1}{t}\right)J_1(x) + \left(t^2+\frac{1}{t^2}\right)J_2 x + \left(t^3-\frac{1}{t^3}\right)J_3(x) + \left(t^4+\frac{1}{t^4}\right)J_4(x) + \cdots$$

Putting $t = e^i\theta$,

$$e^{\frac{x}{2}(e^{i\theta}-e^{-i\theta})} = J_0(x) + \left(e^{i\theta}-e^{-i\theta}\right)J_1(x) + \left(e^{2i\theta}+e^{-2i\theta}\right)J_2(x)$$
$$+\left(e^{3i\theta}-e^{-3i\theta}\right)J_3(x) + \left(e^{4i\theta}+e^{-4i\theta}\right)J_4(x) + \cdots$$

$$e^{\frac{x}{2}(2i\sin\theta)} = J_0(x) + (2i\sin\theta)J_1(x) + (2\cos 2\theta)J_2(x)$$
$$+(2i\sin 3\theta)J_3(x) + (2\cos 4\theta)J_4(x) + \cdots$$

$$\cos(x\sin\theta) + i\sin(x\sin\theta) = [J_0(x) + (2\cos 2\theta)J_2(x) + (2\cos 4\theta)J_4(x)]$$
$$+i[(2\sin\theta)J_1(x) + (2\sin 3\theta)J_3(x) + \cdots]$$

Comparing real part on both the sides,

$$\cos(x\sin\theta) = J_0(x) + (2\cos 2\theta)J_2(x) + (2\cos 4\theta)J_4(x) + \cdots\cdots$$

6.9 ORTHOGONALITY OF BESSEL FUNCTIONS

The sequence of Bessel functions of the first kind forms an orthogonal set on the interval $0 \le x \le R$.

$$\int_0^R x\, J_n(\alpha x)\, J_n(\beta x)\, dx \begin{cases} = 0, & \text{if } \alpha \neq \beta \\ = \dfrac{R^2}{2} J_{n+1}^2(\alpha\beta), & \text{if } \alpha = \beta \end{cases}$$

where α and β are roots of the equation $J_n(Rx) = 0$

Proof Let $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ respectively be the solutions of the equations

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \qquad ...(6.26)$$

and

$$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \qquad ...(6.27)$$

Multiplying Eq. (6.26) by $\frac{v}{x}$ and Eq. (6.27) by $\frac{u}{x}$ and subtracting,

$$x(u''v - v''u) + (u'v - v'u) + (\alpha^2 - \beta^2)xuv = 0$$

$$(\beta^2 - \alpha^2)xuv = \frac{d}{dx}\left[x(u'v - uv')\right]$$

Integrating both the sides w. r. t. x from $x = 0$ to $x = R$,

$$\int_0^R (\beta^2 - \alpha^2)xuv\,dx = \left|x(u'v - uv')\right|_0^R$$

$$(\beta^2 - \alpha^2)\int_0^R x\,J_n(\alpha x)J_n(\beta x)dx = \left|x\{\alpha J_n'(\alpha x)J_n(\beta x) - \beta J_n(\alpha x)J_n'(\beta x)\}\right|_0^R$$

[Substituting u, u', v and v']

$$\int_0^R x\,J_n(\alpha x)\,J_n(\beta x)dx = \frac{R}{\beta^2 - \alpha^2}\left[\alpha J_n'(\alpha R)J_n(\beta R) - \beta J_n(\alpha R)J_n'(\beta R)\right] \quad ...(6.28)$$

Case I $\alpha \neq \beta$

Since α and β are two distinct roots of the equation $J_n(Rx) = 0$,

$$J_n(\alpha R) = 0, \qquad J_n(\beta R) = 0$$

From Eq. (6.28),

$$\int_0^R x\,J_n(\alpha x)J_n(\beta x)dx = 0$$

Case II $\alpha = \beta$

Taking limit $\beta \to \alpha$ in Eq. (6.28),

$$\lim_{\beta\to\alpha}\int_0^R x\,J_n(\alpha x)J_n(\beta x)\,dx = \lim_{\beta\to\alpha}\frac{R\left[\alpha J_n'(\alpha R)J_n(\beta R) - \beta J_n(\alpha R)J_n'(\beta R)\right]}{\beta^2 - \alpha^2} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{\beta\to\alpha}\frac{R\left[\alpha J_n'(\alpha R)\cdot R\,J_n'(\beta R) - \beta J_n(\alpha R)\cdot RJ_n''(\beta R)\right]}{2\beta}$$

[Using L'Hospital's rule]

$$= \frac{\alpha R^2\left[J_n'(\alpha R)\right]^2}{2\alpha} \qquad [\because J_n(\alpha R) = 0]$$

$$= \frac{R^2}{2}\left[J_n'(\alpha R)\right]^2 \quad ...(6.29)$$

From recurrence formula,

$$J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$$

Putting $x = \alpha R$,

$$J_n'(\alpha R) = \frac{n}{\alpha R} J_n(\alpha R) - J_{n+1}(\alpha R)$$

$$= -J_{n+1}(\alpha R) \qquad [\because J_n(\alpha R) = 0]$$

Substituting $J_n'(\alpha R)$ in Eq. (6.29),

$$\int_0^R x\, J_n(\alpha x) J_n(\beta x)\, dx = \frac{R^2}{2} J_{n+1}^2(\alpha R)$$

Hence, $\int_0^R x\, J_n(\alpha x) J_n(\beta x)\, dx = 0, \qquad$ if $\quad \alpha \neq \beta$

$$= \frac{R^2}{2} J_{n+1}^2(\alpha R), \text{ if } \quad \alpha \neq \beta$$

EXERCISE 6.4

1. Prove that $J_{\frac{7}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{15-6x^2}{x^3}\right)\sin x + \left(\frac{15}{x^2} - 1\right)\cos x\right]$.

2. Prove that $4J_0'''(x) - 3J_1(x) + J_3(x) = 0$.

3. Prove that $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$.

4. Prove that $J_n'''(x) = \frac{1}{8}\left[J_{n-3}(x) - 3J_{n-1}(x) + 3J_{n+1}(x) - J_{n+3}(x)\right]$

5. Prove that $\frac{d}{dx}\left[x\, J_n(x)\, J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$

6. Prove that $\int J_3(x)\, dx = -\frac{2J_1(x)}{x} - J_2(x)$.

7. Prove that $\int x^3 J_0(x)\, dx = x^3 J_1(x) - 2x^2 J_2(x) + c$

8. Prove that $\frac{d}{dx}\left[x^n J_n(ax)\right] = ax^n J_{n-1}(ax)$.

9. Prove that $\int_0^{\frac{\pi}{2}} \sqrt{\pi x}\, J_{1/2}(2x)\, dx = 1$.

6.10 LEGENDRE'S EQUATION

The linear second order differential equation

$$(1-x)^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \qquad ...(6.30)$$

is called Legendre's equation, where n is a nonnegative real constant.
The solution of Legendre's equation is called Legendre function.

6.11 LEGENDRE POLYNOMIALS

In Eq. (6.30), $x = 0$ is an ordinary point. Let the series solution of Eq. (6.30) about $x = 0$ be

$$y=\sum_{m=0}^{\infty}a_m x^m = a_0+a_1x+a_2x^2+\cdots \qquad ...(6.31)$$

$$\frac{dy}{dx}=\sum_{m=0}^{\infty}a_m\, m\, x^{m-1}$$

$$\frac{d^2y}{dx^2}=\sum_{m=2}^{\infty}a_m\, m(m-1)x^{m-2}$$

Substituting in Eq. (6.30),

$$(1-x^2)\sum_{m=2}^{\infty}m(m-1)\,a_m x^{m-2}-2x\sum_{m=1}^{\infty}m\,a_m x^{m-1}+n(n+1)\sum_{m=0}^{\infty}a_m x^m=0$$

$$\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}-\sum_{m=2}^{\infty}m(m-1)\,a_m x^m-2\sum_{m=1}^{\infty}ma_m x^m+n(n+1)\sum_{m=0}^{\infty}a_m x^m=0$$

To obtain a common power of x in each term, putting $m-2=t$ in the first term,

$$\sum_{t=0}^{\infty}(t+2)\,(t+1)\,a_{t+2}x^t-\sum_{m=2}^{\infty}m(m-1)\,a_m x^m-2\sum_{m=1}^{\infty}m\,a_m x^m+n(n+1)\sum_{m=0}^{\infty}a_m x^m=0$$

Since t is a dummy variable, replacing t by m,

$$\sum_{m=0}^{\infty}(m+2)\,(m+1)\,a_{m+2}x^m-\sum_{m=2}^{\infty}m(m-1)\,a_m x^m-2\sum_{m=1}^{\infty}m\,a_m x^m+n(n+1)\sum_{m=0}^{\infty}a_m x^m=0$$

$$2a_2+6a_3x+\sum_{m=2}^{\infty}(m+2)\,(m+1)\,a_{m+2}x^m-\sum_{m=2}^{\infty}m(m-1)\,a_m x^m-2a_1x-2\sum_{m=2}^{\infty}ma_m x^m$$

$$+n(n+1)\left(a_0+a_1x\right)+n(n+1)\sum_{m=2}^{\infty}a_m x^m=0$$

$$[n(n+1)a_0 + 2a_2] + [\{n(n+1)-2\}a_1 + 6a_3]x + \sum_{m=2}^{\infty}(m+2)(m+1)\, a_{m+2}x^m$$
$$-\sum_{m=2}^{\infty}\{m(m+1)-n(n+1)\}a_m x^m = 0$$

Equating constant term, coefficient of x and coefficient of x^m to zero,

$$n(n+1)a_0 + 2a_2 = 0$$
$$\{n(n+1)-2\}a_1 + 6a_3 = 0$$

and $(m+2)(m+1)a_{m+2} - \{m(m+1)-n(n+1)\}a_m = 0$

$$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)}a_m, \quad m \ge 0 \qquad \text{...(6.32)}$$

Putting $m = 0, 1, 2, 3, \ldots\ldots$

$$a_2 = -\frac{n(n+1)}{2!}a_0,$$

$$a_3 = -\frac{(n-1)(n+2)}{3!}a_1$$

$$a_4 = -\frac{(n-2)(n+3)}{4\cdot 3}a_2 = \frac{(n-2)n(n+1)(n+3)}{4!}a_0$$

$$a_5 = -\frac{(n-3)(n+4)}{5\cdot 4}a_3 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!}a_1$$

Substituting in Eq. (6.31),

$$y = a_0 + a_1 x - \frac{n(n+1)}{2!}a_0 x^2 - \frac{(n-1)(n+2)}{3!}a_1 x^3$$
$$+\frac{(n-2)n(n+1)(n+3)}{4!}a_0 x^4 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}a_1 x^5 - \ldots\ldots$$

$$= a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)\,n(n+1)(n+3)}{4!}x^4 - \ldots\ldots\right]$$
$$+a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \ldots\ldots\right]$$

...(6.33)

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

where $y_1 = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \ldots\ldots$

and $y_2 = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \ldots\ldots$

Since y_1 and y_2 are linearly independent, Eq. (6.33) represents general solution of Legendre's equation and exists in the interval $-1 < x < 1$.

If n is an even integer, $y_1(x)$ contain terms upto x^n and the remaining terms becomes zero. Thus, $y_1(x)$ reduces to a polynomial of degree n while $y_2(x)$ remains an infinite series. Similarly if n is an odd integer, $y_2(x)$ reduces to a polynomial of degree n while $y_1(x)$ remains an infinite series.

Thus, when n is a nonnegative integer ($n \geq 0$), the general solution consists of a polynomial solution and an infinite series solution.

These polynomial solutions, with choice of a_0 and a_1 in such a manner that the polynomial is 1 for $x = 1$, are known as Legendre polynomials and is denoted by $P_n(x)$. The series which remains infinite is known as Legendre function of second kind and is denoted by $Q_n(x)$.

Thus, for a nonnegative integer n, the general solution of Legendre's equation is

$$y(x) = A\,P_n(x) + B\,Q_n(x)$$

Note: (i) $P_n(1) = 1$
(ii) $Q_n(x)$ is unbounded at $x = \pm 1$.

6.12 RODRIGUES' FORMULA

Rodrigues' formula gives a solution to Legendre's equation. It is given by

$$P_n(x) = \frac{1}{n!2^n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$$

Proof Let $v = (x^2-1)^n$

$$v_1 = \frac{\mathrm{d}v}{\mathrm{d}x} = n(x^2-1)^{n-1}\cdot 2x = \frac{2nx(x^2-1)^n}{(x^2-1)}$$

$$(x^2-1)v_1 = 2nxv$$

$$(1-x^2)v_1 + 2nxv = 0$$

Differentiating $(n+1)$ times using Leibnitz's theorem,

$$\left[(1-x^2)v_{n+2} + (n+1)(-2x)\,v_{n+1} + \frac{(n+1)n(-2)v_n}{2!}\right] + 2n\left[xv_{n+1} + (n+1)v_n\right] = 0$$

$$(1-x^2)v_{n+2} - 2x\,v_{n+1} + n(n+1)v_n = 0$$

$$(1-x^2)\frac{\mathrm{d}^2v_n}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}v_n}{\mathrm{d}x} + n(n+1)v_n = 0$$

which represents Legendre's equation of order n and has a finite series solution $P_n(x)$.

$$v_n = cP_n(x)$$

$$\frac{d^n v}{dx^n} = cP_n(x)$$

$$\frac{d^n}{dx^n}(x^2-1)^n = cP_n(x) \qquad \text{...(6.34)}$$

$$cP_n(x) = \frac{d^n}{dx^n}\left[(x-1)^n(x+1)^n\right]$$

$$= n!(x+1)^n + n\frac{n!}{1!}(x-1)\cdot n(x+1)^{n-1}$$

$$+\frac{n(n-1)}{2!}\frac{n!}{2!}(x-1)^2\cdot n(n-1)(x+1)^{n-2}+\cdots+(x-1)^n n!$$

[Using Leibnitz's theorem]

Putting $x = 1$,

$$cP_n(1) = n!\,2^n$$

$$c = n!\,2^n \qquad [\because P_n(1)=1]$$

Substituting c in Eq. (6.34),

$$\frac{d^n}{dx^n}(x^2-1)^n = n!2^n P_n(x)$$

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n$$

The Rodrigues' formula is used to find different Legendre Polynomials.

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n$$

$$= \frac{1}{n!2^n}\frac{d^n}{dx^n}\sum_{r=0}^{n} {}^nC_r (x^2)^{n-r}(-1)^r$$

$$= \frac{1}{n!2^n}\sum_{r=0}^{n}(-1)^r\, {}^nC_r \frac{d^n}{dx^n}x^{2n-2r}$$

$$= \frac{1}{n!2^n}\sum_{r=0}^{m}(-1)^r\, {}^nC_r \frac{(2n-2r)!}{(2n-2r-n)!}x^{2n-2r-n} \qquad \left[\begin{aligned}\text{where } M &= \frac{n}{2}, \text{ if } n \text{ is even}\\ &= \frac{n-1}{2}, \text{ if } n \text{ is odd}\end{aligned}\right]$$

$$= \frac{1}{n!2^n}\sum_{r=0}^{m}(-1)^r \frac{n!}{(n-r)!r!}\frac{(2n-2r)!}{(n-2r)!}x^{n-2r}$$

$$= \sum_{r=0}^{M}\frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r}$$

In particular,

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x),$$

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5) \text{ etc.}$$

Note: Any polynomial $f(x)$ of degree n can be expressed in terms of $P_n(x)$ as

$$f(x) = \sum_{r=0}^{n} a_r P_r(x)$$

EXAMPLE 1

Express the polynomial $f(x) = 4x^3 - 2x^2 - 3x + 8$ *in terms of Legendre polynomials.*

Solution

$$f(x) = \sum_{r=0}^{n} a_r\, P_r(x)$$

$$4x^3 - 2x^2 - 3x + 8 = \sum_{r=0}^{3} a_r\, P_r(x) \qquad [\because f(x) \text{ is of degree } 3]$$

$$= a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x)$$

$$= a_0 \cdot 1 + a_1 x + a_2\left(\frac{3x^2 - 1}{2}\right) + a_3\left(\frac{5x^3 - 3x}{2}\right)$$

$$= \frac{5}{2} a_3 x^3 + \frac{3}{2} a_2 x^2 + \left(a_1 - \frac{3}{2} a_3\right)x + \left(a_0 - \frac{a_2}{2}\right)$$

Equating similar powers of x on both the sides,

$$\frac{5}{2} a_3 = 4, \qquad a_3 = \frac{8}{5}$$

$$\frac{3}{2} a_2 = -2, \qquad a_2 = -\frac{4}{3}$$

$$a_1-\frac{3}{2}a_3=-3,\quad a_1-\frac{3}{2}\left(\frac{8}{5}\right)=-3,\quad a_1=-3+\frac{12}{5}=-\frac{3}{5}$$

$$a_0-\frac{a_2}{2}=8,\qquad a_0+\frac{4}{6}=8,\qquad a_0=8-\frac{2}{3}=\frac{22}{3}$$

$$\therefore\ 4x^3-2x^2-3x+8=\frac{22}{3}P_0(x)-\frac{3}{5}P_1(x)-\frac{4}{3}P_2(x)+\frac{8}{5}P_3(x)$$

EXAMPLE 2

Show that $\int_{-1}^{1}P_n(x)\,\mathrm{d}x=0$, *for* $n\neq 0$.

Solution

From Rodrigues' formula,

$$P_n(x)=\frac{1}{n!2^n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$$

$$=\frac{1}{n!2^n}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^2-1)^n\right]$$

Integrating w.r.t. x from -1 to 1,

$$\int_{-1}^{1}P_n(x)\mathrm{d}x=\frac{1}{n!2^n}\left|\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x+1)^n(x-1)^n\right|_{-1}^{1}=0$$

[$\because$ each term of R.H.S derivative has factors $(x+1)$ and $(x-1)$]

6.13 RECURRENCE FORMULAE FOR $P_n(x)$

Legendre function satisfies the following recurrence formulae. These formulae are very useful in establishing the various properties and relations of the function and are very important in applications.

(1) $(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$

Proof

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(x)t^n \qquad ...(6.35)$$

Differentiating partially w.r.t. t,

$$-\frac{1}{2}\left(1-2xt+t^2\right)^{-\frac{3}{2}}(-2x+2t)=\sum_{n=0}^{\infty}P_n(x)n\,t^{n-1}$$

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}}\left(1-2xt+t^2\right)^{-1}(x-t)=\sum_{n=0}^{\infty} n\,P_n(x)\,t^{n-1}$$

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n=\left(1-2xt+t^2\right)\sum_{n=0}^{\infty} n\,P_n(x)t^{n-1}$$

$$\sum_{n=0}^{\infty} xP_n(x)t^n-\sum_{n=0}^{\infty} P_n(x)t^{n+1}=\sum_{n=0}^{\infty} n\,P_n(x)t^{n-1}-\sum_{n=0}^{\infty} 2nxP_n(x)t^n+\sum_{n=0}^{\infty} n\,P_n(x)t^{n+1}$$

Equating coefficient of t^n on both the sides,

$$xP_n(x)-P_{n-1}(x)=(n+1)\,P_{n+1}(x)-2nxP_n(x)+(n-1)P_{n-1}(x)$$

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x) \qquad \text{...(6.36)}$$

(2) $n\,P_n(x)=xP_n'(x)-P_{n-1}'(x)$

Proof: Differentiating Eq. (6.35) partially w.r.t. x,

$$-\frac{1}{2}\left(1-2xt+t^2\right)^{-\frac{3}{2}}(-2t)=\sum_{n=0}^{\infty} P_n'(x)t^n$$

$$t\left(1-2xt+t^2\right)^{-\frac{3}{2}}=\sum_{n=0}^{\infty} P_n'(x)t^n \qquad \text{...(6.37)}$$

Differentiating Eq. (6.35) partially w.r.t. t,

$$-\frac{1}{2}\left(1-2xt+t^2\right)^{-\frac{3}{2}}(-2x+2t)=\sum_{n=0}^{\infty} P_n(x)n\,t^{n-1}$$

$$(x-t)\left(1-2xt+t^2\right)^{-\frac{3}{2}}=\sum_{n=0}^{\infty} n\,P_n(x)\,t^{n-1} \qquad \text{...(6.38)}$$

Dividing Eq. (6.38) by Eq. (6.37),

$$\frac{x-t}{t}=\frac{\sum_{n=0}^{\infty} n\,P_n(x)t^{n-1}}{\sum_{n=0}^{\infty} P_n'(x)t^n}$$

$$(x-t)\sum_{n=0}^{\infty} P_n'(x)t^n=t\sum_{n=0}^{\infty} n\,P_n(x)\,t^{n-1}$$

$$\sum_{n=0}^{\infty} xP_n'(x)t^n-\sum_{n=0}^{\infty} P_n'(x)t^{n+1}=\sum_{n=0}^{\infty} n\,P_n(x)\,t^n$$

Equating coefficient of t^n on both the sides,

$$xP_n'(x)-P_{n-1}'(x)=nP_n(x)$$

$$nP_n(x) = xP_n'(x) - P_{n-1}' \qquad ...(6.39)$$

(3) $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

Proof Differentiating Eq. (6.36) w.r.t. x,

$$(n+1)P_{n+1}'(x) = (2n+1)P_n(x) + (2n+1)xP_n'(x) - nP_{n-1}'(x) \qquad ...(6.40)$$

Putting $xP_n'(x) = nP_n(x) + P_{n-1}'(x)$ from Eq. (6.39),

$$(n+1)P_{n+1}'(x) = (2n+1)\,P_n(x) + (2n+1)\left[nP_n(x) + P_{n-1}'(x)\right] - nP_{n-1}'(x)$$

$$= (2n+1)\,(n+1)P_n(x) + (n+1)P_{n-1}'(x)$$

$$P_{n+1}'(x) = (2n+1)P_n(x) + P_{n-1}'(x)$$

$$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x) \qquad ...(6.41)$$

(4) $P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)$

Proof Rewriting Eq. (6.40),

$$(n+1)P_{n+1}'(x) = (2n+1)P_n(x) + (n+1)xP_n'(x) + nxP_n'(x) - nP_{n-1}'(x)$$

$$= (2n+1)P_n(x) + (n+1)xP_n'(x) + n\left[xP_n'(x) - P_{n-1}'(x)\right]$$

$$= (2n+1)P_n(x) + (n+1)xP_n'(x) + n\left[nP_n(x)\right] \qquad \text{[From Eq. (6.39)]}$$

$$= (2n+1+n^2)P_n(x) + (n+1)xP_n'(x)$$

$$= (n+1)^2\,P_n(x) + (n+1)xP_n'(x)$$

$$P_{n+1}'(x) = (n+1)P_n(x) + xP_n'(x)$$

$$= xP_n'(x) + (n+1)P_n(x)$$

Replacing n by $(n-1)$,

$$P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x) \qquad ...(6.42)$$

(5) $(1-x^2)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$

Proof From Eq. (6.39),

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x) \qquad ...(6.43)$$

$$x^2P_n'(x) - xP_{n-1}'(x) = xnP_n(x) \qquad ...(6.44)$$

From Eq. (6.42),

$$P_n'(x) - xP_{n-1}'(x) = nP_{n-1}(x)$$

Subtracting Eq. (6.43) from Eq. (6.44),

$$(1-x)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$$

EXAMPLE 1

Prove that $(2n+1)(1-x^2)P_n'(x) = n(n+1)\left[P_{n-1}(x) - P_{n+1}(x)\right]$

Solution

From recurrence formula (5),

$$(1-x^2)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right] \quad \text{...(1)}$$

Multiplying Eq. (1) by $(2n+1)$,

$$\begin{aligned}(2n+1)(1-x^2)P_n'(x) &= n(2n+1)\left[P_{n-1}(x) - xP_n(x)\right] \\ &= n(2n+1)P_{n-1}(x) - n(2n+1)xP_n(x)\end{aligned} \quad \text{...(2)}$$

From recurrence formula (1),

$$(2n+1)\,xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

Substituting in Eq. (2),

$$\begin{aligned}(2n+1)(1-x^2)P_n'(x) &= n(2n+1)P_{n-1}(x) - n\left[(n+1)P_{n+1}(x) + nP_{n-1}(x)\right] \\ &= n(2n+1-n)P_{n-1}(x) - n(n+1)P_{n+1}(x) \\ &= n(n+1)\left[P_{n-1}(x) - P_{n+1}(x)\right]\end{aligned}$$

6.14 GENERATING FUNCTION FOR $P_n(x)$

The Legendre polynomial $P_n(x)$ is the coefficient of t^n in the Taylor's series expansion of $(1-2xt+t^2)^{-\frac{1}{2}}$ such that

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n \cdot P_n(x)$$

Proof $(1-2xt+t^2)^{-\frac{1}{2}} = \left[1-(2xt-t^2)\right]^{-\frac{1}{2}} = (1-y)^{-\frac{1}{2}}$, where $y = 2xt - t^2 = t(2x-t)$

$$= 1 + \frac{1}{2}y + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}y^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}y^3 + \dots\dots + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{(2n-1)}{2}}{n!}y^n + \cdots$$

$$= 1 + \frac{1}{2}y + \frac{1\cdot 3}{2\cdot 4}y^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}y^3 + \cdots + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}y^n + \cdots \quad \text{...(6.45)}$$

$$\begin{aligned}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n} &= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot\frac{2\cdot 4\cdot 6\cdots 2n}{2\cdot 4\cdot 6\cdots 2n} \\ &= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots 2n}{\left[2^n(1\cdot 2\cdot 3\cdots n)\right]^2} \\ &= \frac{(2n)!}{2^{2n}(n!)^2}\end{aligned}$$

Substituting in Eq. (6.45),

$$(1-2xt+t^2)^{-\frac{1}{2}} = [1-t(2x-t)]^{-\frac{1}{2}}$$
$$=1+\frac{2!}{2^2(1!)^2}t(2x-t)+\frac{4!}{2^4(2!)^2}t^2(2x-t)^2+\dots$$
$$\dots+\frac{(2n-2r)!}{2^{2n-2r}\left[(n-r)!\right]^2}t^{n-r}(2x-t)^{n-r}+\dots+\frac{(2n)!}{2^{2n}(n!)^2}t^2(2x-t)^n+\dots$$

...(6.46)

Consider

$$t^{n-r}(2x-t)^{n-r} = t^{n-r}\begin{bmatrix}(2x)^{n-r} - {}^{n-r}C_1(2x)^{nr-1}t+\dots \\ +(-1)^r\ {}^{n-r}C_r(2x)^{n-2r}t^r+\dots+(-1)^{n-r}t^{n-r}\end{bmatrix}$$
$$=(2x)^{n-r}t^{n-r} - {}^{n-r}C_1(2x)^{n-r-1}t^{n-r+1}+\dots$$
$$+(-1)^n\ {}^{n-r}C_r(2x)^{n-2r}t^n+\dots+(-1)^{n-r}t^{2n-2r}$$

Substituting in Eq. (6.46) the coefficient of t^n in the term containing $t^{n-r}(2x-t)^{n-r}$ is

$$\text{Coefficient of } t^n = \frac{(2n-2r)!}{2^{2n-2r}\left[(n-r)!\right]^2}(-1)^r\ {}^{n-r}C_r(2x)^{n-2r}$$
$$=\frac{(2n-2r)!}{2^{2n-2r}\left[(n-r)!\right]^2}\frac{(-1)^r(n-r)!}{r!(n-2r)!}(2x)^{n-2r}$$
$$=\frac{(-1)^r(2n-2r)!}{2^n(n-r)!(n-2r)!}x^{n-2r}$$

Collecting and summing up all the coefficient of t^n from Eq. (6.46),

$$\text{Coefficient of } t^n = \sum_{r=0}^{M}\frac{(-1)^r(2n-2r)!}{2^n(n-r)!(n-2r)!}x^{n-2r} = P_n(x)$$

where $M=\frac{n}{2}$, if n is even

$=\frac{n-1}{2}$, if n is odd

using coefficient of t^n. Eq. (6.46) reduces to

$$(1-2xt+t^2)^{-\frac{1}{2}} = P_0(x)+P_1(x)t+P_2(x)t^2+\dots+P_n(x)t^n+\dots=\sum_{n=0}^{\infty}t^nP_n(x) \quad \text{...(6.47)}$$

Hence, $\left(1-2xt+t^2\right)^{-\frac{1}{2}}$ is the generating function of the Legendre Polynomial.

Some Results

(1) $P_n(1) = 1$ for all n.

Proof Putting $x = 1$ in Eq. (6.47),

$$\sum_{n=0}^{\infty} t^n P_n(1) = (1-2t+t^2)^{-\frac{1}{2}}$$
$$= \left[(1-t)^2\right]^{-\frac{1}{2}}$$
$$= (1-t)^{-1} - 1 + t + t^2 + \ldots + t^n + \ldots$$

Equating coefficient of t^n on both the sides,

$$P_n(1) = 1$$

(2) $P_n(-1) = (-1)^n$ for all n.

Proof Putting $x = -1$ in Eq. (6.47),

$$\sum_{n=0}^{\infty} t^n P_n(-1) = (1+2t+t^2)^{-\frac{1}{2}}$$
$$= \left[(1+t)^2\right]^{-\frac{1}{2}}$$
$$= (1+t)^{-1}$$
$$= 1 - t + t^2 - \cdots + (-1)^n t^n + \cdots$$

Equating coefficient of t^n on both the sides,

$$P_n(-1) = (-1)^n$$

(3) $P_n(0) = 0$, if n is odd

$$= (-1)^{\frac{n}{2}} \frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}, \text{ if } n \text{ is even}$$

Proof Putting $x = 0$ in Eq. (6.47),

$$\sum_{m=0}^{\infty} t^n P_n(x) = (1+t^2)^{-\frac{1}{2}}$$

$$= 1 - \frac{1}{2}t^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}t^4 + \cdots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left[-\frac{1}{2}-(m-1)\right]}{m!}t^{2m} + \cdots \quad \ldots(6.48)$$

Equating coefficient of t^{2m} on both the sides,

$$P_{2m}(0) = (-1)^m \frac{1\cdot 3\cdot 5\cdots(2m-1)}{2^m(1\cdot 2\cdot 3\cdots\cdots m)}$$
$$= (-1)^m \frac{1\cdot 3\cdot 5\cdots\cdots(2m-1)}{2\cdot 4\cdot 6\cdots\cdots 2m}$$

Since there is no odd term on the R.H.S., equating coefficient of t^{2m+1} on both the sides of Eq (6.48),

$$P_{2m+1}(0) = 0$$

Hence,

$$P_n(0) = 0 \quad \text{, if } n \text{ is odd}$$

$$= (-1)^{\frac{n}{2}} \frac{1\cdot3\cdot5\cdots(n-1)}{2\cdot4\cdot6\cdots n}, \text{ if } n \text{ is even}$$

(4) $P_n'(1) = \frac{1}{2}n(n+1)$

Proof Since $P_n(x)$ is a solution of Legendre's equation,

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)\,P_n(x) = 0$$

Putting $x = 1$,

$$-2P_n'(1) + n(n+1)P_n(1)$$

$$P_n'(1) = \frac{1}{2}n(n+1) \qquad [\because P_n(1) = 1]$$

6.15 ORTHOGONALITY OF LEGENDRE POLYNOMIALS

The Legendre polynomial satisfies the condition of orthogonality as

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0, \qquad m \neq n$$

$$= \frac{2}{2n+1}, \qquad m = n = 0, 1, 2, \cdots$$

Proof

(i) When $m \neq n$

Let $P_m(x)$ and $P_n(x)$ be Legendre polynomials satisfying respectively Legendre's equations

$$(1-x^2)P_m'' - 2xP_m' + m(m+1)P_m = 0 \qquad \text{...(6.49)}$$

and

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \qquad \text{...(6.50)}$$

Multiplying Eq. (6.49) by P_n and Eq. (6.50) by P_m and subtracting Eq. (6.50) from (6.49),

$$(1-x^2)\left[P_nP_m'' - P_mP_n''\right] - 2x\left(P_nP_m' - P_mP_n'\right) + P_mP_n\left[m(m+1) - n(n+1)\right] = 0$$

$$\frac{d}{dx}\left[(1-x^2)\left(P_nP_m' - P_mP_n'\right)\right] + P_mP_n\left[m(m+1) - n(n+1)\right] = 0$$

$$[m(m+1)-n(n+1)]P_m P_n = -\frac{d}{dx}\left[(1-x^2)(P_n P_m' - P_m P_n')\right]$$

Integrating both the sides w.r.t. x from -1 to 1,

$$[m(m+1)-n(n+1)]\int_{-1}^{1} P_m(x)P_n(x)\,dx = -\int_{-1}^{1}\frac{d}{dx}\left[(1-x^2)(P_n P_m' - P_m P_n')\right]dx$$

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = -\frac{1}{[m(m+1)-n(n+1)]}\left|(1-x^2)(P_n P_m' - P_m P_n')\right|_{-1}^{1}, \qquad m \neq n$$

$$= 0, \text{ if } m \neq n$$

(ii) When $m = n$

From Rodrigue's formula,

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n$$

$$2^n n! P_n(x) = D^n(x^2-1)^n \quad \text{where } D \equiv \frac{d}{dx}$$

Squaring both the sides and integrating w.r.t. x from -1 to 1,

$$2^{2n}(n!)^2\int_{-1}^{1}[P_n(x)]^2\,dx = \int_{-1}^{1} D^n(x^2-1)^n \cdot D^n(x^2-1)^n\,dx$$

$$= \left|D^n(x^2-1)^n \cdot D^{n-1}(x^2-1)^n\right|_{-1}^{1} - \int_{-1}^{1} D^{n+1}(x^2-1)^n \cdot D^{n-1}(x^2-1)^n\,dx$$

$$= 0 - \int_{-1}^{1} D^{n+1}(x^2-1)^n \cdot D^{n-1}(x^2-1)^n\,dx$$

[$\because$ each term of $D^{n-1}(x^2-1)^n$ has factor (x^2-1)]

Integrating by parts $(n-1)$ times,

$$2^{2n}(n!)^2\int_{-1}^{1}[P_n(x)]^2\,dx = (-1)^n\int_{-1}^{1} D^{2n}(x^2-1)^n \cdot (x^2-1)^n\,dx$$

$$= (-1)^n\int_{-1}^{1}(2n)!(x^2-1)^n\,dx$$

$$= (2n)!\int_{-1}^{1}(1-x^2)^n\,dx$$

$$= (2n)!2\int_{0}^{1}(1-x^2)^n\,dx \quad [\because (1-x^2)^n \text{ is an even function}]$$

Putting $x = \sin\theta$, $dx = \cos\theta\,d\theta$

When $x = 0$, $\theta = 0$

When $x = 1$, $\theta - \frac{\pi}{2}$

$$2^{2n}(n!)^2\int_{-1}^{1}[P_n(x)]^2\,dx = (2n)!2\int_{0}^{\frac{\pi}{2}}(1-\sin^2\theta)^n\cos\theta\,d\theta$$

$$= (2n)!2\int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta \, d\theta$$

$$= (2n)!\,B\left(n+1, \frac{1}{2}\right)$$

$$= (2n)!\frac{\sqrt{n+1}\,\sqrt{\frac{1}{2}}}{\sqrt{n+\frac{3}{2}}}$$

$$= (2n)!\frac{n!\sqrt{\pi}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\frac{1}{2}}}$$

$$= \frac{(2n)!\;n!\;\sqrt{\pi}\;2^{n+1}}{(2n+1)(2n-1)\cdots3\cdot1\;\sqrt{\pi}} \times \frac{2n(2n-2)\cdots4\cdot2}{2n(2n-2)\cdots4\cdot2}$$

$$= \frac{(2n)!\;n!\;2^{n+1}\cdot2^n\;n(n-1)\cdots2\cdot1}{(2n+1)\,2n\,(2n-1)\,(2n-2)\cdots-3\cdot2\cdot1}$$

$$= \frac{(2n)!\;n!\;2^{2n+1}\;n!}{(2n+1)!}$$

$$= \frac{(2n)!\;(n!)^2\;2^{2n+1}}{(2n+1)\,(2n)!}$$

$$= \frac{(n!)^2\,2^{2n+1}}{2n+1}$$

$$\int_{-1}^{1}[P_n(x)]^2\,dx = \frac{2}{2n+1}$$

Hence, $\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0, \qquad m \neq n$

$$= \frac{2}{2n+1}, \quad m = n = 0, 1, 2, \cdots$$

EXAMPLE 1

Show that $\int\limits_{-1}^{1} x^2 P_{n-1}(x)P_{n+1}(x)\,dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$

Solution

From recurrence relation (1),

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

$$xP_n(x) = \frac{1}{2n+1}\left[(n+1)P_{n+1}(x) + nP_{n-1}(x)\right] \qquad ...(1)$$

Replacing n by $(n-1)$ in Eq. (1),

$$x\,P_{n-1}(x) = \frac{1}{2n-1}[nP_n(x) + (n-1)\,P_{n-2}(x)] \qquad ...(2)$$

Replacing n by $(n+1)$ in Eq. (1),

$$x\,P_{n+1}(x) = \frac{1}{2n+3}[(n+2)\,P_{n+2}(x) + (n+1)P_n(x)] \qquad ...(3)$$

Multiplying Eq. (2) and Eq. (3),

$$x^2 P_{n-1}(x)P_{n+1}(x) = \frac{1}{(2n-1)\,(2n+3)}[n(n+2)P_n(x)\,P_{n+2}(x) + n(n+1)P_n^2(x)$$
$$+(n-1)\,(n+2)\,P_{n-2}(x)P_{n+2}(x) + (n^2-1)\,P_{n-2}(x)\,P_n(x)]$$

Integrating both the sides w.r.t. x from -1 to 1 and using orthogonality of Legendre polynomial,

$$\int_{-1}^{1} x^2 P_{n-1}(x)P_{n+1}(x)\,dx = \frac{1}{(2n-1)\,(2n+3)}\int_{-1}^{1} n(n+1)\,P_n^2(x)\,dx$$
$$= \frac{n(n+1)}{(2n-1)\,(2n+3)}\cdot\frac{2}{2n+1}$$
$$= \frac{2n(n+1)}{(2n-1)\,(2n+1)\,(2n+3)}$$

EXAMPLE 2

Prove that $\int_{-1}^{1} P_n(x)\cdot\left(1-2xt+t^2\right)^{-\frac{1}{2}}\,dx = \dfrac{2t^n}{2n+1}$

Solution

$$\int_{-1}^{1} P_n(x)\cdot\left(1-2xt+t^2\right)^{-\frac{1}{2}}\,dx = \int_{-1}^{1} P_n(x)\left[\sum_{m=0}^{\infty} t^m P_m(x)\right]dx$$

[Using generating function]

$$= \sum_{m=0}^{\infty} t^m \int_{-1}^{1} P_n(x)P_m(x)\,dx$$

$$= t^n \int_{-1}^{1} P_n^2(x)\,dx$$

[Using orthogonality of Legendre polynomial]

$$= t^n \frac{2}{2n+1}$$

$$= \frac{2t^n}{2n+1}$$

EXERCISE 6.5

1. Express x^5 in terms of Legendre Polynomials.

$$\left[\text{Ans.: } x^5 = \frac{8}{63}P_5 + \frac{28}{63}P_3 + \frac{27}{63}P_1\right]$$

2. Express $f(x) = x^3 + 2x^2 - x - 3$ in terms of Legendre polynomials.

$$\left[\text{Ans.: } f(x) = \frac{2}{5}P_3 + \frac{4}{3}P_2 - \frac{2}{5}P_1 - \frac{7}{3}P_0\right]$$

3. Prove that (i) $P_n(-x) = (-1)^n P_n(x)$ (ii) $P_n'(-x) = (-1)^{n+1} P_n'(x)$

4. Prove that (i) $P_{2n+1}(0) = 0$ (ii) $P_{2n}(0) = \dfrac{(-1)^n (2n)!}{2^{2n}(n!)^2}$

5. Prove that (i) $P_n'(1) = \dfrac{n(n+1)}{2}$ (ii) $P_n'(-1) = (-1)^n \cdot \dfrac{n(n+1)}{2}$

6. Using Rodrigues' formula, show that $P_n(x)$ satisfies the differential equation $\dfrac{d}{dx}\left[(1+x^2)\dfrac{d}{dx}\{P_n(x)\}\right] + n(n+1)\,P_n(x) = 0$.

7. Prove that $(1-x^2)P_n'(x) = (n+1)\left[xP_n(x) - P_{n+1}(x)\right]$.

8. Prove that $P_n(x)P_{n+\frac{1}{2}}(x) = \dfrac{\sqrt{\pi}}{2^{2n+1}} P_{2m}(x)$.

9. Prove that $\int_{-1}^{1} f(x)P_n(x)\,dx = \dfrac{(-1)^n}{2^n n!}\int_{-1}^{1} f(n)(x)\,(x^2-1)^n\,dx$.

Hence, deduce that $\int_{-1}^{1} x^m P_n(x)\,dx = 0$, if $m < n$

$$= \frac{2^{n+1}(n!)^2}{(2n+1)!}, \text{ if } m = n$$

10. Prove that $\int_{-1}^{1} (1-x^2)P_m'(x)\,P_n'(x)\,dx = 0$, if $m \neq n$

$$= \frac{2n(n+1)}{2n+1}, \text{ if } m = n$$

Points to Remember

Series Solution about an Ordinary Point

The power-series solution of the equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \qquad ...(1)$$

about an ordinary point x_0 be given as

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \qquad ...(2)$$

The coefficients $a_1, a_2, a_3, \ldots$ are obtained by substituting Eq.(2) and its derivatives in Eq.(1).

Frobenius Method

To obtain the solution near a regular singular point x_0, an extension of the power-series method, known as the Frobenius method (or generalised power-series method), is used.

Let x_0 be a regular singular point of the differential equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \qquad ...(3)$$

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x) = \dfrac{P_1(x)}{P_0(x)}$, $Q(x) = \dfrac{P_2(x)}{P_0(x)}$

The series solution of Eq. (3) about x_0 be

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r} = (x - x_0)^r \left[a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots\right] \qquad ...(4)$$

The general solution of Eq. (3) is given as

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x)$$

where $y_1(x)$, $y_2(x)$ are two linearly independent solutions and c_1 and c_2 are arbitrary constants.

Bessel's Equation

The linear second order differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

is called Bessel's equation, where n is a nonnegative real constant.

Bessel's Functions of the First Kind

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\overline{|n+m+1}} \left(\frac{x}{2}\right)^{2m+n}$$

Recurrence Formulae for $J_n(x)$

(1) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

(2) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

(3) $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

(4) $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

(5) $J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$

(6) $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$

Generating Function for $J_n(x)$

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=\infty}^{\infty} J_n(x)\, t^n$$

Orthogonality of Bessel Functions

The sequence of Bessel functions of the first kind forms an orthogonal set on the interval $0 \le x \le R$.

$$\int_0^R x\, J_n(\alpha x)\, J_n(\beta x)\, dx \begin{cases} = 0, & \text{if } \alpha \neq \beta \\ = \frac{R^2}{2} J_{n+1}^2(\alpha\beta), & \text{if } \alpha = \beta \end{cases}$$

where α and β are roots of the equation $J_n(Rx) = 0$

Legendre's Equation

The linear second order differential equation

$$(1-x)^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

is called Legendre's equation, where n is a nonnegative real constant.

Legendre Polynomials

$$y=\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$$

Rodrigues' Formula

$$P_n(x)=\frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n$$

Recurrence Formulae for $P_n(x)$

(1) $(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-n\,P_{n-1}(x)$

(2) $n\,P_n(x)=xP_n'(x)-P_{n-1}'(x)$

(3) $(2n+1)P_n(x)=P_{n+1}'(x)-P_{n-1}'(x)$

(4) $P_n'(x)=xP_{n-1}'(x)+nP_{n-1}(x)$

(5) $(1-x^2)P_n'(x)=n\left[P_{n-1}(x)-xP_n(x)\right]$

Generating function for $P_n(x)$

$$(1-2xt+t^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty} t^n \cdot P_n(x)$$

Some Results

(1) $P_n(1)=1$ for all n.

(2) $P_n(-1)=(-1)^n$ for all n.

(3) $P_n(0)=0$, if n is odd

$$=(-1)^{\frac{n}{2}}\frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}, \text{ if } n \text{ is even}$$

(4) $P_n'(1)=\frac{1}{2}n(n+1)$

Orthogonality of Legendre Polynomials

The Legendre polynomial satisfies the condition of orthogonality as

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0, \qquad m \neq n$$

$$= \frac{2}{2n+1}, \qquad m=n=0,1,2,\cdots$$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The singular points of the differential equation $x^3(x-1)y'' + 2(x-1)y' + y = 0$ are
   (a) 0, 1 (b) 1, 0 (c) –2, 1 (d) 1, 2
2. The regular singular point of $2x^2y'' + 3xy' + (x^2-4)y = 0$ is **[Summer 2016]**
   (a) $x = -2$ (b) $x = 1$ (c) $x = 0$ (d) $x = -1$
3. The roots of the indicial equation for the power series solution of the differential equation $2x^2y'' + xy' + (x^2-3)y = 0$ are
   (a) $\frac{3}{2}, 1$ (b) $\frac{3}{2}, -1$ (c) $\frac{2}{3}, 1$ (d) $\frac{2}{3}, -1$
4. The regular singular point of $x^3(x-2)y'' + x^3y' + 6y = 0$ is
   (a) $x = 0$ (b) $x = 1$ (c) $x = -1$ (d) $x = 2$
5. The irregular singular point of the differential equation
   $$x^2(x-2)^2y'' + 2(x-2)y' + (x+3)y = 0 \text{ is}$$
   (a) $x = 1$ (b) $x = 0$ (c) $x = -1$ (d) $x = 2$
6. The roots of the indicial equation for the power series solution of the differential equation $3x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ are
   (a) $0, \frac{1}{2}$ (b) 0, 1 (c) $0, \frac{1}{3}$ (d) $1, \frac{1}{3}$
7. The roots of the indicial equation for the power series solution of the differential equation $xy'' + 2y' + xy = 0$ are
   (a) 0, –1 (b) 0, 1 (c) 1, 2 (d) 0, –2
8. The singular point of the differential equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ is **[Summer 2017]**
   (a) $x = -1$ (b) $x = 2$ (c) $x = 1$ (d) $x = -2$
9. If J_0 and J_1 are Bessel functions then $J_1'(x)$ is given by
   (a) $-J_0$ (b) $J_0(x) - \frac{1}{x}J_1(x)$
   (c) $J_0(x) + \frac{1}{x}J_1(x)$ (d) J_0
10. If $J_n(x)$ is the Bessel function of first kind then $\int_0^{\pi}[J_{-2}(x) - J_2(x)]dx =$
    (a) 2 (b) –2 (c) 0 (d) 1
11. If $J_{n+1}(x) = \frac{2}{x}J_n(x) - J_0(x)$ then n is
    (a) 0 (b) 2 (c) (–1) (d) None of these

12. If $\int_{-1}^{1} P_n(x)dx = 2$ then n is

(a) 0 (b) 1 (c) 2 (d) 0

13. The value of $\int_{-1}^{1} (2x+1)P_3(x)dx$ where $P_3(x)$ is the third degree Legendre polynomial, is

(a) 1 (b) −1 (c) 2 (d) 0

14. The value of integral $\int_{-1}^{1} x^3 P_3(x)dx$, where $P_3(x)$ is a Legendre polynomial of degree 3, is

(a) 0 (b) $\frac{2}{35}$ (c) $\frac{4}{35}$ (d) $\frac{11}{35}$

15. The polynomial $2x^2 + x + 3$ in terms of Legendre polynomials is

(a) $\frac{1}{3}(4P_2 - 3P_1 + 11P_0)$ (b $\frac{1}{3}(4P_2 + 3P_1 - 11P_0)$

(c) $\frac{1}{3}(4P_2 + 3P_1 + 11P_0)$ (d) $\frac{1}{3}(4P_2 - 3P_1 - 11P_0)$

16. If $P_n(x)$ is the Legendre polynomial, then $P_n(x)$ is equal to

(a) $(-1)^n P_n(x)$ (b) $(1)^n P_n(x)$ (c) $(-1)^{n+1} P_n(x)$ (d) $P_n(x)$

17. Legendre polynomial $P_s(x) = \lambda(63x^5 - 70x^2 + 15x)$ where λ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{5}$ (c) $\frac{1}{8}$ (d) $\frac{1}{10}$

18. $\int_{-1}^{1} (1+x)P_n(x)dx$, $(n > 1)$, is equal to

(a) $\frac{1}{2n+1}$ (b) $\frac{2}{2n+1}$ (c) $\frac{n}{2n+1}$ (d) 0

19. $\int_{-1}^{1} x^m P_n(x)dx$, $(m < 1)$ is equal to

(a) 0 (b) $\frac{m}{2n+1}$ (c) $\frac{2}{2n+1}$ (d) $\frac{n}{2n+1}$

20. $\int_{-1}^{1} P_3(x)P_4(x)dx$ is equal to

(a) 4 (b) 3 (c) 1 (d) 0

Answers

1. (a)	2. (c)	3. (b)	4. (d)	5. (b)	6. (c)	7. (a)	8. (c)
9. (c)	10. (c)	11. (d)	12. (c)	13. (d)	14. (c)	15. (c)	16. (c)
17. (c)	18. (d)	19. (a)	20. (d)				